FIBER PRODUCTS AND ZARISKI SHEAVES

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1. Fiber products and Zariski sheaves

We recall the definition of a fiber product:

Definition 1.1. Let $\mathcal{C}$ be a category, and X, Y, Z objects of $\mathcal{C}$. Fix also morphisms $\pi_X : X \to Z$, $\pi_Y : Y \to Z$. Given this data, we say that an object P of $\mathcal{C}$, together with morphisms $p_1 : P \to X$, $p_2 : P \to Y$ is a **fiber product** of X with Y over Z if it satisfies the following universal property:

For every object $T \in \text{Obj}(\mathcal{C})$, and every pair of morphisms $f : T \to X$, $g : T \to Y$ such that $\pi_X \circ f = \pi_Y \circ g$, there exists a unique morphism $h : T \to P$ such that $f = p_1 \circ h$ and $g = p_2 \circ h$.

In this case, we write P as $X \times_Z Y$.

Thus, a fiber product represents a functor, which we will denote by $\underline{X \times_Z Y}$, and is unique if it exists. This much is true in any category, but existence is a question with much more substance. In particular, there are categories in which fiber products do not exist.

Our immediate goal is to prove:

Theorem 1.2. *Fiber products exist in the category of schemes, and therefore also in the category of schemes over S for any fixed scheme S.*

Let's observe that the fiber product $X \times_Z Y$ has the same universal property in the category of schemes as in the category of schemes over some fixed scheme S. Thus, it is enough to work in the full category of schemes.

Recall that by studying gluing of morphisms, we had a necessary condition for a functor to be representable.

Proposition 1.3. *(Gluing of morphisms of schemes) Let X and Y be schemes over S, and $\{U_i\}$ an open covering of X. Then morphisms $f : X \to Y$ over S are in one-to-one correspondence with collections of morphisms $f_i : U_i \to Y$ over S, such that for all i, j we have $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ as morphisms $U_i \cap U_j \to Y$ over S.*

Proof. Left as an exercise. □

This means that in order for a functor $F : \text{Sch}_S \to \text{Set}$ to be representable, it needs to satisfy a certain property, analogous to the condition for a presheaf to be a sheaf:

Definition 1.4. A contravariant functor $F : \text{Sch}_S \to \text{Set}$ is a **Zariski sheaf** if it satisfies the following condition:

For every $X \in \text{Obj}(\text{Sch}_S)$, and every open cover $\{U_i\}$ of X, the natural map

$$\{\eta \in F(X)\} \to \{\{\eta_i \in F(U_i)\} : \forall i, j, \eta_i|_{U_i \cap U_j} = \eta_j|_{U_i \cap U_j}\}$$

is a bijection.

It is simply a matter of definition-chasing to see that the above proposition can be rephrased to say the following:

Corollary 1.5. *Let* $F : \mathrm{Sch}_S \to \mathrm{Set}$ *be a contravariant functor. In order for* F *to be representable, it is necessary that* F *be a Zariski sheaf.*

This is by no means a sufficient condition, but it is nonetheless an important condition to check. Basically, the idea is that if we have a functor F, and want to construct an X representing F, then if we first check that F is a Zariski sheaf, we will be able to construct X "locally", in a sense we will make precise shortly.

With this motivation, let's check:

Proposition 1.6. *Given schemes* X, Y, Z *and morphisms* $\pi_X : X \to Z$, $\pi_Y : Y \to Z$, *the functor* $\underline{X \times_Z Y}$ *is a Zariski sheaf.*

Proof. Indeed, this is just a formality, since the functor is defined in terms of morphisms to X and Y. Specifically, let T be a scheme; by definition, $\underline{X \times_Z Y}(T)$ is defined to be the set of pairs of morphisms $f : T \to X$, $g : T \to Y$ such that $\pi_X \circ f = \pi_Y \circ g$. If we have an open cover $\{U_i\}$ of T, because we know that h_X and h_Y are Zariski sheaves, we have that the morphism $f : T \to X$ is uniquely determined by a collection of $f_i : U_i \to X$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, and similarly for $g : T \to Y$. Moreover, because morphisms $T \to Z$ are uniquely determined by collections of morphisms $U_i \to Z$, we also see that $\pi_X \circ f = \pi_Y \circ g$ if and only if $\pi_X \circ f_i = \pi_Y \circ g_i$ for all i. Thus, an element of $\underline{X \times_Z Y}(T)$ is determined uniquely by a family of elements of $\underline{X \times_Z Y}(U_i)$ which agree on the $U_i \cap U_j$, and we see that $\underline{X \times_Z Y}$ is a Zariski sheaf. □

We also observe that it is easy to see that fiber products of affine schemes exist.

Proposition 1.7. *Suppose that* $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$ *are affine schemes. Then* $\underline{X \times_Z Y}$ *is represented by* $\operatorname{Spec}(A \otimes_C B)$, *with the natural maps to* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$.

Proof. By Exercise 1 of Problem Set 2, we have $\mathrm{Mor}(T, X) = \mathrm{Hom}(A, \mathcal{O}_T(T))$ (where Hom denotes the set of ring homomorphisms), and similarly $\mathrm{Mor}(T, Y) = \mathrm{Hom}(B, \mathcal{O}_T(T))$ and $\mathrm{Mor}(T, \operatorname{Spec}(A \otimes_C B)) = \mathrm{Hom}(A \otimes_C B, \mathcal{O}_T(T))$. Also, the maps π_X and π_Y correspond to ring homomorphisms $\pi_X^\sharp : C \to A$ and $\pi_Y^\sharp : C \to B$ (which is what we use to define the tensor product). Let's write $p_1^\sharp : A \to A \otimes_C B$ and $\pi_2^\sharp : B \to B \otimes_C B$ for the maps $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$. We see that we want to check that composition with $\pi_1^\sharp$ and $\pi_2^\sharp$ induces a bijection between $\mathrm{Mor}(A \otimes_C B, \mathcal{O}_T(T))$ and $\{(f^\sharp, g^\sharp) \in \mathrm{Hom}(A, \mathcal{O}_T(T)) \times \mathrm{Hom}(B, \mathcal{O}_T(T)) : \pi_X^\sharp \circ f^\sharp = \pi_Y^\sharp \circ g^\sharp\}$.

This is now a statement entirely in terms of rings, and is in fact the universal property of the tensor product. However, it is not hard to check directly: given the pair of morphisms $(f^\sharp, g^\sharp)$, we define a morphism $A \otimes_C B \to \mathcal{O}_T(T)$ by $a \otimes b \mapsto f^\sharp(a) g^\sharp(b)$, checking that the composition condition on $f^\sharp$ and $g^\sharp$ means that the map is well-defined. □

2. Representability of Zariski sheaves and the fiber product

We have now verified that the functor $\underline{X \times_Z Y}$ is a Zariski sheaf, so in principle, it could be representable. Moreover, the functor is representable in the affine case, and since every scheme has an open cover by affines, one might well speculate that it should be possible to glue together affine fiber products to get the fiber product in general. In fact, this is true, and is a special case of a rather general statement about Zariski sheaves. In order to proceed further, we need to go back and consider what it would mean to show locally that a functor is representable.

Thus, let $F : \mathrm{Sch}_S \to \mathrm{Set}$ be a representable functor, and suppose that F is represented by (X, η), where X is a scheme over S, and η an element of $F(X)$. Now suppose that U is an open subscheme of X (i.e., a scheme obtained from X be restricting to an open subset), with $\iota : U \to X$ the inclusion map. We then get an $\eta' \in F(U)$ as $(F(\iota))(\eta)$. We want to say that h_U is an "open subfunctor" of h_X under the map induced by ι, which then would mean that (U, η') represents a certain open subfunctor of F.

How can we formulate this precisely? The first statement is quite simple: for any scheme T over S, the morphisms $T \to U$ are a subset of the morphisms $T \to X$. The more subtle idea is as follows: for any morphism $f : T \to X$ over S, taking the preimage of U gives us an open subscheme T_U of T, and a map $f_U : T_U \to U$ (still over S). Moreover, we see that if $T' \to T$ is any morphism such that the composition $T' \to X$ has image contained in U, then $T' \to T$ factors uniquely through the inclusion $T_U \to T$. More formally, for any element $f \in h_X(T)$, we have an open subscheme $\iota : T_U \to T$ such that: for all T', and $g \in h_T(T')$, if $f \circ g \in h_X(T')$ lies in the subset $h_U(T') \subseteq h_X(T')$, then there is a unique $h \in h_{T_U}(T')$ such that $g = \iota \circ h$.

We can now formalize this situation.

Definition 2.1. Let F, G be contravariant functors $\mathrm{Sch}_S \to \mathrm{Set}$, and $G \to F$ a morphism of functors. We say that G is an **open subfunctor** of F if:

(i) for every $T \in \mathrm{Obj}(\mathrm{Sch}_S)$, the map $G(T) \to F(T)$ is injective.
(ii) for every $T \in \mathrm{Obj}(\mathrm{Sch}_S)$, and every $\eta \in F(T)$, there exists an open subscheme U of T and an $\eta' \in G(U)$ such that η and η' agree under the induced maps to $F(U)$, and furthermore, for any $f : T' \to T$, if $(F(f))(\eta) \in G(T') \subseteq F(T')$, then there exists a unique $g : T' \to U$ such that $f = \iota \circ g$, where $\iota : U \to T$ is the inclusion map.

Note that the condition on the (U, η') means that it represents a functor (specifically, the functor of morphisms to T whose image in X is contained in U), so is unique if it exists.

We can then easily talk about a cover of a functor by open subfunctors:

Definition 2.2. A collection $F_i \to F$ of open subfunctors of F are said to **cover** F if for every T and $\eta \in F(T)$, if U_i is the open subscheme of T associated to η by F_i, then the U_i cover T.

We observe that if $U \subseteq X$ is an open subscheme, then h_U is an open subfunctor of h_X, and if $\{U_i\}$ is an open cover of X, then h_{U_i} is a cover by open subfunctors of h_X.

Now, suppose we have a cover of F by open subfunctors F_i, and each F_i is represented by (U_i, η_i). We want to glue the (U_i, η_i) together to give an (X, η) representing F. We will see that in order to do this, we will need that F is a Zariski sheaf. However, the statement itself is surprisingly simple.

Proposition 2.3. *Suppose that* $F : \mathrm{Sch}_S \to \mathrm{Set}$ *is a Zariski sheaf, and that* F *has a cover by open subfunctors* $F_i \to F$. *Suppose further that each* F_i *is represented by* (U_i, η_i), *with* $U_i \in \mathrm{Obj}(\mathrm{Sch}_S)$ *and* $\eta_i \in F_i(U_i)$.

Then there exists (X, η) *representing* F, *with maps* $\psi_i : U_i \to X$ *such that:*

(i) *each* U_i *maps isomorphically to an open subscheme of* X;
(ii) *the* $\psi_i(U_i)$ *cover* X;
(iii) $(F(\psi_i))(\eta) = \eta_i$ *for each* i.

Before we give the proof, we need to understand what is necessary in general to glue together schemes, without worrying about functors. We already discussed how to glue together a pair of schemes along an open subscheme when we discussed the examples $\mathbb{P}^1_k$ and the affine line with the doubled origin, but the general situation is a bit more complicated. We have:

Proposition 2.4. *Let* $\{U_i\}$ *be a collection of schemes, and for each* $i \neq j$, *suppose we have an open subscheme* $U_{i,j} \subseteq U_i$. *Suppose we also have for every* $i \neq j$ *an isomorphism* $\varphi_{i,j} : U_{i,j} \xrightarrow{\sim} U_{j,i}$, *satisfying:*

(i) *for each* $i \neq j$, *we have* $\varphi_{i,j} = \varphi_{j,i}^{-1}$;
(ii) *for each* i, j, k *pairwise distinct,* $\varphi_{i,j}(U_{i,j} \cap U_{i,k}) = U_{j,i} \cap U_{j,k}$, *and* $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ *on* $U_{i,j} \cap U_{i,k}$.

Then we can glue together the U_i *along the* $(U_{i,j}, \varphi_{i,j})$ *to obtain a scheme* X. *More precisely, there is a scheme* X, *and morphisms* $\psi_i : U_i \to X$, *such that:*

(i) *each* ψ_i *is an isomorphism onto an open subscheme of* X;
(ii) *the* $\psi_i(U_i)$ *cover* X;
(iii) *for each* $i \neq j$, *we have* $\psi_i(U_{i,j}) = \psi_i(U_i) \cap \psi_j(U_j)$;
(iv) *for each* $i \neq j$, *we have* $\psi_i = \psi_j \circ \varphi_{i,j}$ *on* $U_{i,j}$.

Proof. Left as an exercise (Exercise 2.12 of Chapter II of [1]). □

Before giving the proof of Proposition 2.3, we observe that we can "intersect" two open subfunctors to obtain a new open subfunctor, in the obvious way: given G, G' two open subfunctors of F, we consider $G(T), G'(T) \subseteq F(T)$ under the maps provided, and get a new functor by intersecting inside $F(T)$ for every T. If $U, U' \subseteq T$ are the open subsets provided by (ii) of the definition, we check that $U \cap U'$ works for the intersected functor. Also, to reduce the amount of notation necessary, if we have a functor F, object T, and $\eta \in F(T)$, together with a morphism $T' \to T$, we will write $\eta|_{T'}$ to denote the element of $F(T')$ obtained as the image of η under the induced map $F(T) \to F(T')$.

Proof of Proposition 2.3. We first use the η_i to glue together the U_i into a single scheme X. According to the previous proposition, we need open subschemes $U_{i,j} \subseteq U_i$ and isomorphisms $\varphi_{i,j} : U_{i,j} \xrightarrow{\sim} U_{j,i}$ satisfying the stated compatibility conditions. The idea is simple: by the definition of an open subfunctor, if we are given $i \neq j$, then associated to $\eta_i \in F(U_i)$, the open subfunctor F_j gives us an open subscheme, which we denote $U_{i,j}$, of U_i, and an element $\eta_{i,j} \in F_j(U_{i,j})$ such that η_i agrees with $\eta_{i,j}$ in $F(U_{i,j})$, and such that $(U_{i,j}, \eta_{i,j})$ represents the functor of morphisms $T' \to U_i$ such that $\eta_i|_{T'} \in F_j(T') \subseteq F_(T')$.

One checks that in fact $(U_{i,j}, \eta_{i,j})$ represents $F_i \cap F_j$. It follows that $(U_{i,j}, \eta_{i,j})$ and $(U_{j,i}, \eta_{j,i})$ represent the same functor, so by Yoneda's lemma we obtain a unique isomorphism $\varphi_{i,j} : U_{i,j} \xrightarrow{\sim} U_{j,i}$ sending $\eta_{j,i}$ to $\eta_{i,j}$. We then claim that the $U_{i,j}$

and $\varphi_{i,j}$ satisfy the conditions of Proposition 2.4. Indeed, the first condition is immediate, while the second follows from writing everything in terms of the functor $F_i \cap F_j \cap F_k$, and following through the definitions, again using Yoneda's lemma.

It thus follows that we obtain a scheme X and maps $\psi_i : U_i \to X$ as in the proposition. It is then enough to show that we have an $\eta \in F(X)$ such that (X, η) represents F, and $(F(\psi_i))(\eta) = \eta_i$ for each i. Here we finally use that F is a Zariski sheaf: for each i, we can consider $\eta_i \in F_i(U_i)$ as an element of $F(U_i)$, and considering the U_i as open subsets of X, following through the definitions we find that $\eta_i|_{U_i \cap U_j} = \eta_j|_{U_i \cap U_j}$ for every i, j. Thus, by the Zariski sheaf condition, the η_i glue uniquely to give an element $\eta \in F(X)$ with $\eta|_{U_i} = \eta_i$ for all i. We claim that (X, η) represents F. Indeed, given T and $\zeta \in F(T)$, the F_i give us an open cover $T_{U,i}$ of T, and because (U_i, η_i) represent F_i, we get morphisms $T_{U,i} \to U_i$, with $\zeta|_{T_{U_i}} = \eta_i|_{T_{U_i}}$, and which agree on the intersections. We can therefore glue to obtain a morphism $T \to X$, with $\zeta = \eta|_T$, and it follows by definition that (X, η) represent F. □

We have already checked that the fiber product functor $\underline{X \times_Z Y}$ is a Zariski sheaf, and that if X, Y, Z are affine, the fiber product exists. To prove that fiber products exist in general, we will show that $\underline{X \times_Z Y}$ can be covered by open subfunctors of the form $\underline{U \times_W V}$, where U, V, W are affine open subschemes of X, Y, Z respectively. We will then be able to apply Proposition 2.3 to conclude that the fiber product exists.

Proposition 2.5. *Given $\pi_X : X \to Z$, $\pi_Y : Y \to Z$, let U, V, W be open subschemes of X, Y, Z respectively such that $U \subseteq \pi_X^{-1}(W)$, $V \subseteq \pi_Y^{-1}(W)$. Then $\underline{U \times_W V}$ is naturally an open subfunctor of $\underline{X \times_Z Y}$.*

Furthermore, if $\{(U_i, V_i, W_i)\}$ are a family such that for any $P \in X, Q \in Y$ with the same image in Z, there is some i with $P \in U_i$ and $Q \in V_i$, the subfunctors $\underline{U \times_W V}$ cover $\underline{X \times_Z Y}$.

Proof. This is really just a formality. We certainly have a morphism $\underline{U \times_W V} \to \underline{X \times_Z Y}$ by composing with the inclusions $U \to X$, $V \to Y$. Moreover, the map $\underline{U \times_W V}(T) \to \underline{X \times_Z Y}(T)$ is clearly injective for any T. Now, given morphisms $f : T \to X$, $g : T \to Y$ agreeing on Z, the open subscheme $f^{-1}(U) \cap g^{-1}(V) \subseteq T$ clearly has the property that for any $T' \to T$ with images contained in $U \subseteq X$ and $V \subseteq Y$, we must have $T' \to T$ factoring uniquely through $f^{-1}(U) \cap g^{-1}(V)$. Thus, $\underline{U \times_W V}$ is an open subfunctor of $\underline{X \times_Z Y}$.

Finally, given a collection $\{(U_i, V_i, W_i)\}$, and a pair of morphisms $f : T \to X$, $g : T \to Y$ agreeing on Z, let $t \in T$ be any point, with images $P \in X$ and $Q \in Y$; then P and Q have the same image in Z. By assumption, there is an i such that $P \in U_i$, and $Q \in V_i$, so it follows that $t \in f^{-1}(U_i) \cap g^{-1}(V_i)$. Thus, our open subsets cover T, and our opensubfunctors cover $\underline{X \times_Z Y}$, as desired. □

We can now prove our theorem.

Proof of Theorem 1.2. By Proposition 1.6, the functor $\underline{X \times_Z Y}$ is a Zariski sheaf. By Proposition 2.3, it therefore suffices to produce a cover by open subfunctors which are each representable. By Propositions 2.5 and 1.7, we are done if we can produce a family of affine open subsets $\{(U_i, V_i, W_i)\}$ of X, Y, Z respectively such that: U_i and V_i map into $W_i \subseteq Z$ for all i; and, for any $P \in X, Q \in Y$ with the same image in Z, there is some i with $P \in U_i$ and $Q \in V_i$. But if we simply let $\{(U_i, V_i, W_i)\}$ be the family of all such affine open subsets, we see this has the desired property, as there is some W open affine containing $\pi_X(P) = \pi_Y(Q)$, and then some affine open U in $\pi_X^{-1}(W)$ containing P, and some affine open V in $\pi_Y^{-1}(W)$ containing Q. Thus, we conclude that $\underline{X \times_Z Y}$ is representable, as desired. □

While this may seem like a lot of machinery to prove that fiber products exist, the advantage is that we have done most of our work as part of a more general framework, which will also apply, for instance, to proving that Grassmannians exist. Hopefully, it also illuminates more clearly the concepts behind the standard proof.

3. The fiber product: comments and examples

We discuss now the fiber product: how to think about it, some examples, and some applications.

Roughly speaking, one should think of the fiber product $X \times_Z Y$ as parametrizing $(x, y) \in X \times Y$ which map to the same thing in Z. This can easily be made precise on the level of k-valued points, but also tends to make the fiber product sound simpler than it is.

We first consider the case of very simple varieties over a field k.

Example 3.1. Perhaps the most basic example of a fiber product is $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k \cong \mathbb{A}^2_k$. Indeed, this follows immediately from the isomorphism $k[x] \otimes_k k[y] \cong k[x, y]$. We see however that if we look at the underlying topological spaces, the points of the product are not the product of the points, and the topology on the product is far finer than the product topology.

This is true even if k is algebraically closed. In this case, the points of $\mathbb{A}^2_k$ aren't so far from the product of the points of $\mathbb{A}^1_k$; indeed, the closed points of $\mathbb{A}^2_k$ are in bijection with pairs of closed points of $\mathbb{A}^1_k$. The problem comes from non-closed points. If η is the generic point of $\mathbb{A}^1_k$, the product of the points of $\mathbb{A}^1_k$ consists of pairs (x, y) where each of x and y is either a closed point of $\mathbb{A}^1_k$ (i.e., an element of k) or η. On the other hand, the points of $\mathbb{A}^2_k$ also include generic points corresponding to every irreducible curve in the plane (i.e., every irreducible polynomial in x, y), as well as a generic point for the whole plane (corresponding to the 0 ideal). We can think of the points (x_0, η) for $x_0 \in k$ as corresponding to the line $x = x_0$ in $\mathbb{A}^2_k$, and (η, y_0) corresponds to $y = y_0$. Furthermore, it is reasonable for (η, η) to correspond to the generic point of $\mathbb{A}^2_k$. However, this means that for any irreducible curve $C \subseteq \mathbb{A}^2_k$ which is not a vertical or horizontal line, there is a point of $\mathbb{A}^2_k$ not corresponding to any pair of points of $\mathbb{A}^1_k$.

If we restrict to closed points, we at least have a bijection between points of $\mathbb{A}^2_k$ and pairs of points of $\mathbb{A}^1_k$, but the topology on $\mathbb{A}^2_k$ is finer than the product topology, and for the same reasons that the generic points differ: in the product topology, closed sets are intersections of products of closed sets of $\mathbb{A}^1_k$, and the only closed sets of $\mathbb{A}^1_k$ (other than the empty set and the whole space) are finite sets of points, so in the product topology, the only closed sets (other than the empty set and whole space) are finite collections of vertical and horizontal lines, and finite collections of points. On the other hand, in the Zariski topology in $\mathbb{A}^2_k$, any curve in the plane gives a closed set, so we see that the Zariski topology is considerably finer than the product topology.

The situation gets even worse if we generalize slightly.

Example 3.2. Consider $\operatorname{Spec}\mathbb{Q}(i)\otimes_{\operatorname{Spec}\mathbb{Q}}\operatorname{Spec}\mathbb{Q}(i)$. This is the product of a point with a point over a point, and yet $\mathbb{Q}(i)\otimes_{\mathbb{Q}}\mathbb{Q}(i)\cong\mathbb{Q}(i)\times\mathbb{Q}(i)$, which means that we have

$$\operatorname{Spec}\mathbb{Q}(i)\times_{\operatorname{Spec}\mathbb{Q}}\operatorname{Spec}\mathbb{Q}(i)\cong\operatorname{Spec}\mathbb{Q}(i)\amalg\operatorname{Spec}\mathbb{Q}(i),$$

i.e., the fiber product is the disjoint union of two copies of $\operatorname{Spec}\mathbb{Q}(i)$.

However, we can make a clean statement about the points of a fiber product, if we work with k-valued points instead of points of the underlying topological space. Specifically, we see immediately from the definition that the k-valued points of $X\times_Z Y$ are precisely pairs of k-valued points of X and Y which map to the same k-valued point of Z. This is true regardless of whether or not X,Y,Z are over a field, but we also see that if X,Y,Z are over $\operatorname{Spec}k$, and we have some field $k'\supseteq k$, then we have the same situation: the k'-valued points of $X\times_Z Y$ over $\operatorname{Spec}k$ are precisely pairs of k'-valued points of X and Y over $\operatorname{Spec}k$ mapping to the same point of Z. Thus, if X,Y,Z are classical (affine or projective) varieties, the classical k'-points of the fiber product are precisely pairs of k'-points of X and Y mapping to the same k'-point of Z.

As far as the topology is concerned, it turns out that it will actually be to our benefit that the topology on the product is different from the product topology, as it will give us additional flexibility in trying to adapt notions such as Hausdorff to the case of schemes.

We now discuss some applications of the fiber product.

Recall that if $f:X\to Y$ is a continuous map of topological spaces, and $y\in Y$, the **fiber** of f over y is the set $f^{-1}(y)$, equipped with the topology inherited from X. The fiber product allows us to define a scheme-theoretic version of fibers, as follows: suppose $f:X\to Y$ is a morphism of schemes, and $y\in Y$ a point. The **fiber** X_y of f over y is defined to be $\operatorname{Spec}\kappa(y)\times_Y X$, where $\operatorname{Spec}\kappa(y)\to Y$ is the unique morphism with image y. This immediately gives us a scheme structure, and it is left as an exercise that the topological space of X_y is homeomorphic to the fiber of f over y when f is considered as a map of topological spaces.

More generally, if $f:X\to Y$ is a morphism, and $Y'\to Y$ is any other morphism, the natural map $X\times_Y Y'\to Y'$ is called the **base change** (base extension, in [1]) of f under $Y'\to Y$. This is an important concept in general, but we mention one special case which has a simple classical meaning: if $Y=\operatorname{Spec}k$, and $Y'=\operatorname{Spec}k'$ for some $k'\supseteq k$, then the base change $X':=X\times_{\operatorname{Spec}k}\operatorname{Spec}k'\to\operatorname{Spec}k'$ gives us a scheme over $\operatorname{Spec}k'$ starting from one over $\operatorname{Spec}k$. If X were an affine variety, given by polynomials with coefficients in k, then X' is simply the variety we get by considering the polynomials as having coefficients in the larger field k'. In particular, if $k'=\bar{k}$, then X' is a variety over an algebraically closed field, and it is frequently useful to take what we know in this case and see what we can conclude about our original X.

We conclude with one example of fiber products of schemes lending some geometric intuition to an odd algebraic fact.

Example 3.3. A common example of the strangeness of tensor products is the fact that $\mathbb{Z}/2\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/3\mathbb{Z}$ is the zero ring. However, we now have a very geometric way of thinking about this fact: $\mathbb{Z}/2\mathbb{Z}\otimes_Z Z\mathbb{Z}/3\mathbb{Z}$ being zero is equivalent to $\operatorname{Spec}(\mathbb{Z}/2\mathbb{Z})\times_{\operatorname{Spec}\mathbb{Z}}\operatorname{Spec}(\mathbb{Z}/3\mathbb{Z})$ being empty.

But for any p, $\mathrm{Spec}(\mathbb{Z}/p\mathbb{Z})$ is a point lying over the point of $\mathbb{Z}$ corresponding to the prime ideal (p), and it is clear that if we have $X \to Z$ and $Y \to Z$ with disjoint images, then $X \times_Z Y$ can't have any points, and must be empty!

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.