POINTS AND MORPHISMS, FROM THE CLASSICAL AND SCHEME-THEORETIC POINT OF VIEW

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The aim of these notes is to give a concise introduction to the classical notions of points and morphisms for affine and projective varieties over a possibly non-algebraically closed field, and to explain how these notions may be expressed in the language of schemes.

We let k denote a field throughout. We say a prime ideal $I \subseteq k[x_1, \ldots, x_n]$ is **geometrically prime** if the ideal generated by I in $\bar{k}[x_1, \ldots, x_n]$ remains prime. Since Hilbert's Nullstellensatz deals with radical ideals over an algebraically closed field, when we want to work in the classical setting we will often want to consider ideals remaining radical in $\bar{k}[x_1, \ldots, x_n]$. From there, it is reasonable to work with geometrically prime ideals, since at least over the algebraic closure, every radical ideal has a unique minimal representation as an intersection of prime ideals (corresponding to the "irreducible components" of $V(I)$). To simplify certain descriptions (particularly in the definition of a morphism of projective varieties), we will restrict to this case throughout.

1. Affine varieties

We begin by reviewing the classical notions of points and morphisms for affine varieties over general fields. We then discuss how these concepts may be translated into scheme language.

1.1. **The classical notions.**

Recall that an affine variety V inside $\mathbb{A}^n_k$ is defined as the vanishing set of a geometrically prime ideal I of polynomials inside $k[x_1, \ldots, x_n]$. We call $k[x_1, \ldots, x_n]/I$ the **coordinate ring** of V, and think of it as the ring of functions on V.

The set $V(k)$ of **k-points** of V are then simply the points of V with all coordinates in k, or more formally, the set of $\vec{c} = (c_1, \ldots, c_n) \in k^n$ such that $f(\vec{c}) = 0$ for all $f \in I$.

Recall that Hilbert's Nullstellensatz implied that if $k = \bar{k}$ is algebraically closed, then the ideal I of polynomials defining V is determined by the points of V. However, to work over a non-algebraically closed field, it is important to remember the polynomials, rather than the k-points. For instance, if $k' \supseteq k$ for any field k', we can also define the set $V(k')$ of k'-points of V in the obvious way: the points $\vec{c} = (c_1, \ldots, c_n) \in (k')^n$ such that $f(\vec{c}) = 0$ for all $f \in I$. In particular, we can define the $\bar{k}$-points in this way, and while the set of $\bar{k}$ points is determined by I, it is certainly not determined by the set of k-points, which could be empty or a single point (for instance, if we set $k = \mathbb{R}$, and $I = (x_1^2 + x_2^2 + 1)$ or $I = (x_1^2 + x_2^2)$).

Example 1.1. Consider the elliptic curve defined (in the affine plane) by $y^2 = x^3 - x$. The $\mathbb{Q}$-points of this are the solutions to this equation over $\mathbb{Q}$, and one can show that the only $\mathbb{Q}$-points are $(0, 0)$, $(1, 0)$, and $(-1, 0)$. On the other hand, the

$\mathbb{Q}(\sqrt{6})$-points also contain the point $(2, \sqrt{6})$ (among others), the $\mathbb{R}$-points contain an oval and a component going off to infinity, and the $\mathbb{C}$-points look like a torus missing one point (the point at infinity).

Now suppose we have two algebraic varieties V_1, V_2 defined over the same field k by geometrically prime ideals $I_1 \subseteq k[x_1, \ldots, x_{n_1}], I_2 \subseteq k[y_1, \ldots, y_{n_2}]$. A **morphism** $f : V_1 \to V_2$ is, roughly, a function from V_1 to V_2 defined by polynomials. More specifically, f can be defined as a function $V_1(\bar{k}) \to V_2(\bar{k})$ defined by polynomials $f_1(x_1, \ldots, x_{n_1}), \ldots, f_{n_2}(x_1, \ldots, x_{n_1})$ with coefficients in k.

Let's recast this definition by looking at the ring of functions. Given a morphism f, and an element $\varphi \in k[y_1, \ldots, y_{n_2}]$, if we consider $\varphi(f)$ we get an element of $k[x_1, \ldots, x_{n_1}]$. Conceptually, f defines a morphism $\mathbb{A}^{n_1}_k \to \mathbb{A}^{n_2}_k$, and the map we obtain $k[y_1, \ldots, y_{n_2}] \to k[x_1, \ldots, x_{n_1}]$ is just composition of functions with the map f.

Moreover, we have:

Proposition 1.2. *A collection f of n_2 polynomials in n_1 variables defines a morphism $f : V_1 \to V_2$ if and only if the induced map $k[y_1, \ldots, y_{n_2}] \to k[x_1, \ldots, x_{n_1}]$ sends I_2 to I_1; i.e., if and only if it induces a well-defined map from the coordinate ring of V_2 to the coordinate ring of V_1.*

Example 1.3. As a first example of a morphism of algebraic varieties, we give a map (in fact an isomorphism) f from the affine line $\mathbb{A}^1_k$, with coordinate t, to the parabola C defined in the xy-plane by $y - x^2 = 0$. The map is simply $(t) \mapsto (t, t^2)$.

We can check directly that this defines a map of k'-points for any $k' \supseteq k$, but we also see that the induced map $k[x, y] \to k[t]$ is given by $x \mapsto t$, $y \mapsto t^2$, so we have that $y - x^2 \mapsto 0$, and we get an induced map $k[x, y]/(y - x^2) \to k[t]$ of the rings of functions.

Note that this map is in fact an isomorphism, with inverse given by the projection $(x, y) \mapsto (x)$.

Example 1.4. Consider the curves C_1 and C_2, defined in the plane by the single equations $y^2 = x^3 + ax + b$ and $ny^2 = x^3 + ax + b$ respectively, where $a, b, n \in \mathbb{Q}$, with n non-zero.

If $n = m^2$ for some $m \in \mathbb{Q}$, we have the map (in fact an isomorphism) $f : C_1 \to C_2$ obtained by $(x, y) \mapsto (x, \frac{y}{m})$. Again, one can check both on points and on rings that this gives a map from C_1 to C_2, and it is defined over $\mathbb{Q}$.

On the other hand, if n is not a perfect square in $\mathbb{Q}$, we see that as long as we consider C_1 and C_2 as curves over $\mathbb{Q}$, it is not possible to define the isomorphism f between them. On the other hand, if we consider them as curves over $\mathbb{Q}(\sqrt{n})$, we see that we can define f as before, and C_1 and C_2 are isomorphic. We say that f is **defined over** $\mathbb{Q}(\sqrt{n})$, but not over $\mathbb{Q}$.

In particular, f will give a natural bijection between the $\mathbb{Q}(\sqrt{n})$ points of C_1 and C_2, but not the $\mathbb{Q}$-points (which might look quite different). Thus, when dealing with non-algebraically closed fields, we see that it is important to consider not only the fields over which the points are defined, but also the fields over which the morphisms are defined.

1.2. **The scheme-theoretic point of view.** We fix affine schemes $X_1 = \operatorname{Spec} A_1$, $X_2 = \operatorname{Spec} A_2$, with $A_1 = k[x_1, \ldots, x_{n_1}]/I_1$, and $A_2 = k[y_1, \ldots, y_{n_2}]/I_2$ for geometrically prime ideals I_1, I_2. When we are dealing with only a single scheme at a time (e.g., when discussing points), we will drop the subscripts. At first, we see that the situation seems rather far-removed from the classical point of view.

First considering the points of X, Hilbert's Nullstellensatz says that if k is algebraically closed, we will get what we want (at least for points over k itself) simply by considering the closed points of the underlying topological space $\mathrm{sp}(X)$, so this isn't too bad. However, if k is an arbitrary field, we have seen this relationship is far more complicated.

Example 1.5. Consider the case of $\mathbb{A}^1_k = \operatorname{Spec} k[t]$. The closed points correspond to maximal ideals of $k[t]$, which is a PID, so we see that we get a closed point for every monic irreducible polynomial $p(t) \in k[t]$.

In particular, for any k-point of $\mathbb{A}^1_k$, which is simply some $c \in k$, we have the irreducible polynomial $t - c$, so the k-points are naturally contained among the closed points of $\mathbb{A}^1_k$.

On the other hand, if k is not algebraically closed, we also have irreducible polynomials of higher degree. We can relate these to points of $\mathbb{A}^1_{\bar{k}}$ by considering the roots of the polynomials $p(t)$. If $c \in \bar{k}$ is any element of the algebraic closure, we could consider it as corresponding to the closed point given by $p(t)$, the minimal polynomial of c over k. On the other hand, given any $p(t)$, there is some $c \in \bar{k}$ a root of $p(t)$. But this is not a unique correspondence, so we can summarize as follows:

There is a map from $\bar{k}$-points of $\mathbb{A}^1_k$ to closed points of $\mathbb{A}^1_k$ which is surjective, and such that any $c, c' \in \bar{k}$ map to the same point if and only if they have the same irreducible polynomial over k.

Moreover, under this relationship, if $c \in \bar{k}$ maps to $p(t)$ the residue field at $p(t)$ (which is simply $k[t]/(p(t))$) can also be considered as the smallest extension of k containing c.

We therefore see that we can recover the k-points as the closed points which have residue field k, and this in fact holds for any X. However, for k' finite over k, this description clearly becomes somewhat less useful, and for k' transcendental over k, the k'-points seem completely unrelated to the points of $\mathrm{sp}(X)$.

Before discussing points any further, we take an apparent detour into the topic of morphisms.

At first glance, the situation with morphisms seem even worse than with points.

Example 1.6. Let's consider $X_1 = X_2 = \operatorname{Spec} k$ a single point. Classically, there should only be one morphism from a point to a point: the identity map. We see that this is not the case with schemes.

The map on underlying topological spaces is of course unique, but we also have the additional data of the map on structure sheaves, which in this case is just a homomorphism $f^\sharp : k \to k$. This is necessarily injective, but that's about all one can say: there could be many such maps, and in general they need not be surjective.

For instance, we see that $\operatorname{Aut}(\operatorname{Spec} k) = \operatorname{Aut}(k)$, which could be extremely large (if $k = \bar{\mathbb{Q}}$, then $\operatorname{Aut}(k) = \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$).

On the other hand, if $k = k'(t)$ for some smaller field k', then we have the non-surjective map corresponding to $t \mapsto t^2$.

Which is stranger? That there can be many maps from a point to itself, or that there exist non-invertible maps from a point to itself?

What can I say, other than "it's not a bug, it's a feature!"

Allow me to explain. First, the feature part: the fact that we see $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, an object of great interest to number-theorists, popping up in the context of schemes, is no coincidence, but rather fits into Grothendieck's vision of using scheme theory to unify number theory and algebraic geometry. This has turned out be extremely powerful, leading to many breakthroughs in number theory, from Deligne's proof of the Weil conjectures on the number of points of algebraic varieties over finite fields, to Falting's proof of the Mordell conjecture on rational points of higher genus curves, to the proof by Wiles, Taylor, Ribet, Serre, et al. of Fermat's Last Theorem. Similarly, the map $t \mapsto t^2$ mentioned above is nothing more than the map induced on generic points by the map $\mathbb{A}^1_k \to \mathbb{A}^1_k$ given by $t \mapsto t^2$. Grothendieck's idea that maps of algebraic varieties could be systematically studied by looking at generic points has also had a substantial impact, making systematic and rigorous study of behavior for "general" points substantially more straightforward.

Second, why it's not a bug: in short, we can express the classical notion of a morphism fairly easily in scheme language. The first observation is that both A_1 and A_2 have the natural structure of k-algebras, i.e., we have natural maps $X_1 \to \operatorname{Spec} k$ and $X_2 \to \operatorname{Spec} k$ (these are called the **structure maps** of X_1 and X_2). The basic observation is that if we consider X_1 and X_2 not as abstract schemes, but as schemes over $\operatorname{Spec} k$, we obtain the right notion of morphism:

Proposition 1.7. *The set of morphisms $X_1 \to X_2$ over $\operatorname{Spec} k$ (i.e., which commute with the structure maps to $\operatorname{Spec} k$), coincides with the classical notion of maps from X_1 to X_2.*

This is not hard to check: the basic idea is that we are now restricted to looking at morphisms of k-algebras, and any morphisms *of k-algebras* $k[y_1, \dots, y_{n_2}] \to k[x_1, \dots, x_{n_1}]$ is necessarily given by polynomials.

We now return to the topic of points. Having seen that looking at the points of the underlying topological space isn't as helpful as it might be, we observe that another natural definition of the points of a topological space X might be the set of maps from a single point to X. One might then hope that the k'-points of a scheme X could be described as the maps from $\operatorname{Spec} k'$ to X. Assuming that one works again over $\operatorname{Spec} k$, it turns out that this gives the right answer:

Proposition 1.8. *For any field $k' \supseteq k$, the classical k'-points of X can be described in scheme language as the morphisms $\operatorname{Spec} k' \to X$ over $\operatorname{Spec} k$.*

At first, it looks like there's a problem when k' is not algebraic over X: we see that $\operatorname{Spec} k'$ could map to a generic point of X, which does not look like it should occur in the classical picture. However, this is in fact precisely what is necessary to get the correct classical description: for instance, if we consider $\operatorname{Spec} \mathbb{A}^1_{\mathbb{Q}}$, every complex point not defined over $\bar{\mathbb{Q}}$ will necessarily arise from a map from $\operatorname{Spec} \mathbb{C}$ to the generic point.

We thus see that although the most naive translation of points and morphisms into scheme language doesn't coincide with the classical notions, it is nonetheless not difficult to express the classical notions in the language of schemes. The method we have presented for doing so is in line with Grothendieck's general philosophy on schemes, which is that the natural way to express their properties is not in terms of the underlying sets and topological spaces (which are nothing more than technical tools), but in terms of morphisms.

Remark 1.9. It follows easily from Exercise 6 of Problem Set 1 that the k'-valued points of X can be described as pairs (x, φ) with x a point of X, and $\varphi : \kappa(x) \to k'$ a map of fields commuting with the inclusion of k into each.

In the case that k' is finite over k, one checks that x has to be a closed point, so one obtains another description of the k'-points of X, in terms of the closed points of X, together with the additional data of φ.

2. Projective varieties

We move on to address the same ideas of points and morphisms in the context of projective varieties.

2.1. **Basic definitions in the classical setting.** We begin by discussing projective space $\mathbb{P}^n_k$. While $\mathbb{A}^n_k$ had points described simply as n-tuples of elements of k, the points of $\mathbb{P}^n_k$ are slightly more complicated. One usually defines them to be $(n+1)$-tuples $(C_0, \ldots, C_n)$ with $C_i \in k$ and not all 0, subject to the relation that for any $\lambda \in k^*$, we consider the point $(C_0, \ldots, C_n)$ to be the same as $(\lambda C_0, \ldots, \lambda C_n)$. Another way to say this is that the points of $\mathbb{P}^n_k$ correspond to lines through the origin in k^n.

However, we can also consider $\mathbb{P}^n_k$ as a compactification of $\mathbb{A}^n_k$: the points of $\mathbb{P}^n_k$ with $C_0 \neq 0$ can all be written uniquely as $(1, c_1, \ldots, c_n)$ (where $c_i = C_i/C_0$), so we see that a copy of $\mathbb{A}^n_k$ is contained in $\mathbb{P}^n_k$ as the open subset $C_0 \neq 0$; the complement can be thought of as "points at infinity". In fact, considering the loci where $C_i \neq 0$ for each i, we obtain an open cover of $\mathbb{P}^n_k$ by copies of $\mathbb{A}^n_k$, which justifies thinking of $\mathbb{P}^n_k$ as an algebraic variety.

Because of the equivalence relation, an arbitrary polynomial in $n+1$ variables can't be used to define an algebraic variety in $\mathbb{P}^n_k$. However, the situation is different if we have a polynomial f which is **homogeneous of degree** d, i.e., with each monomial having total degree d. In that case, $f(\lambda C_0, \ldots, \lambda C_n) = \lambda^d f(C_0, \ldots, C_n)$, so even though f can't be used to define a function on projective space, its zero set is well-defined.

Definition 2.1. We say that $V \subseteq \mathbb{P}^n_k$ is a **projective algebraic set** if it is the zero set of a collection of homogeneous polynomials f_i in $n+1$ variables (not necessarily of equal degree).

In the case of affine algebraic sets over an algebraically closed field, we observed that the zero set of a collection of polynomials was the same as the zero set of the ideal they generate, and Hilbert's Nullstellensatz implied that this in turn was the same as the vanishing set of the radical of the ideal. We thus restricted our attention to radical ideals. The situation is the same here:

Definition 2.2. We say that an ideal I of $k[X_0, \ldots, X_d]$ is **homogeneous** if it can be generated by homogeneous elements. For a homogeneous ideal I, we define $V(I)$ to be the set of points in $\mathbb{P}^n_k$ on which the homogeneous elements of I vanish. For a set $V \subseteq \mathbb{P}^n_k$, we define $I(V)$ to be the (homogeneous) ideal generated by the set of homogeneous polynomials which vanish on V.

Proposition 2.3. *Suppose k is algebraically closed. For any homogeneous ideal $I \subseteq k[X_0, \ldots, X_d]$, we have that $I(V(I))$ is the radical of I.*

One can prove this by forgetting homoegeneity issues and considering $k[X_0, \ldots, X_d]$ simply as the ring of functions on $\mathbb{A}^{n+1}_k$, and then using Hilbert's Nullstellensatz.

Definition 2.4. $V \subseteq \mathbb{P}^n_k$ is a **projective variety** if it is the vanishing set of a geometrically prime ideal I. Given a projective variety $V \subseteq \mathbb{P}^n_k$, the ring $k[X_0, \ldots, X_n]/I$ is the **homogeneous coordinate ring** of V.

Using the imbedding of $\mathbb{A}^n_k$ in $\mathbb{P}^n_k$, we can construct the closure of an affine variety inside $\mathbb{P}^n_k$:

Definition 2.5. Suppose that $V \subseteq \mathbb{A}^n_k$ is defined by polynomials

$$f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n),$$

of degrees $d_1, \ldots, d_m$. The **homogenization** of f_i is the polynomial $\tilde{f}_i(X_0, \ldots, X_n)$ homogeneous of degree d_i, obtained by adding enough powers of X_0 to each monomial to make it have degree d_i. The **projective closure** $\tilde{V} \subseteq \mathbb{P}^n_k$ of V is the projective variety defined by the $\tilde{f}_i$.

It is easy to check that on the locus $X_0 \neq 0$, if we set $x_i := X_i/X_0$, we have that $\tilde{V}$ is the same as V. One can also check that (in the Zariski topology) V is dense in $\tilde{V}$, justifying the terminology closure.

2.2. **Points and morphisms in the classical setting.** The definitions of points and morphisms of projective varieties are similar to the affine case.

Definition 2.6. Given $k' \supseteq k$, and V a projective variety over k, the set of k'**-points** of V is defined to be the set of points $(C_0, \ldots, C_n) \in (k')^n$ which lie on V; i.e., with $f(C_0, \ldots, C_n) = 0$ for all f in the ideal defining V.

Note that a point $(C_0, \ldots, C_n)$ may still be a k'-point even if the C_i are not in k', if there exists some non-zero λ with $(\lambda C_0, \ldots, \lambda C_n) \in (k')^{n+1}$.

Definition 2.7. Given projective varieties $V_1 \subseteq \mathbb{P}^{n_1}_k, V_2 \subseteq \mathbb{P}^{n_2}_k$, a **morphism** from V_1 to V_2 is a function f from the $\bar{k}$-points of V_1 to the $\bar{k}$-points of V_2 such for any $\bar{k}$-point $P \in V_1$, there exists a tuple

$$(F_0(X_0, \ldots, X_{n_1}), \ldots, F_{n_2}(X_0, \ldots, X_{n_1}))$$

of homogeneous polynomials with coefficients in k, all having the same degree, such that not all the F_i vanish at P, and such that on the open subset of V_1 on which not all the F_i vanish, the tuple $(F_0, \ldots, F_{n_2})$ induces the function f on $\bar{k}$-points.

Note that the condition that the F_i all be homogeneous of the same degree is necessary for the map to be well-defined modulo scaling, while the non-vanishing condition is necessary to ensure that the morphism doesn't map any points to $(\underbrace{0, \ldots, 0}_{n_2+1})$, which is not a valid point of $\mathbb{P}^{n_2}_k$.

Suppose we want to describe explicitly when a collection of tuples of forms defines a morphism from V_1 to V_2. We can describe this in two steps: first, we describe morphisms $V_1 \to \mathbb{P}^{n_2}_k$, and then we describe which such morphisms induce morphisms $V_1 \to V_2$.

For the first part, suppose we have tuples of homogeneous polynomials $(F_0, ..., F_{n_2})$ and $(G_0, ..., G_{n_2})$. It is clear that if there are homogeneous F', G' such that for all i, $F'F_i - G'G_i$ is a homogeneous element of I_1, then both tuples define the same function on the open subset of V where at least one F_i and at least one G_i are non-vanishing. Thus, a collection of such tuples gives a well-defined morphism to $\mathbb{P}^{n_2}_k$.

Next, if V_1 and V_2 are defined by homogeneous ideals I_1, I_2, then if we have a function $V_1 \to \mathbb{P}^{n_2}$ defined as above, the condition that V_1 gives a morphism to V_2 is equivalent to saying that for any $(n+1)$-tuple of forms F_i as above, the homomorphism $k[y_1, ..., y_{n_2}] \to k[x_1, ..., x_{n_1}]$ induced by the forms maps I_1 to I_2, i.e. descends to the homogeneous coordinate rings.

To summarize, although morphisms of projective varieties can be described to some extent in terms of morphisms of the homogeneous coordinate rings, they are not characterized as cleanly as in the affine case. In particular, unlike the affine case, we can have isomorphic projective varieties with non-isomorphic homogeneous coordinate rings. After we have introduced line bundles, we will revisit the classical way of thinking about maps to projective spaces, described in terms of "linear series".

Example 2.8. We return to Example 1.3, and see that the map of affine curves can be extended to a map of the projective closures. Of course, the projective closure of $\mathbb{A}^1_k$ is simply $\mathbb{P}^1_k$, say with coordinates T_0, T_1. For the parabola, we replace the coordinates x, y by x_1, x_2, and for the projective closure, we get that the curve $\tilde{C}$ is defined inside $\mathbb{P}^1_k$ by the equation $X_2X_0 - X_1^2 = 0$.

In projective coordinates, our earlier map becomes (by homogenizing the coordinates) $(T_0, T_1) \mapsto (T_0^2, T_0T_1, T_1^2)$, which we see gives a well-defined morphism $\mathbb{P}^1_k \to \mathbb{P}^2_k$. Moreover, the induced map on rings is given by $X_0 \mapsto T_0^2$, $X_1 \mapsto T_0T_1$, $X_2 \mapsto T_1^2$, so we see that $X_2X_0 - X_1^2 \mapsto 0$, and we have a well-defined map $\mathbb{P}^1_k \to \tilde{C}$.

This morphism is an isomorphism, but we now see that the inverse map cannot be given by a single tuple of polynomials. We can try to define an inverse by $(X_0, X_1, X_2) \mapsto (X_0, X_1)$. This works away from the point $(0, 0, 1)$, but the polynomials X_0, X_1 both vanish at $(0, 0, 1)$. Similarly, if we consider $(X_0, X_1, X_2) \mapsto (X_1, X_2)$, everything is fine away from $(1, 0, 0)$. We note that these define the same two functions on the intersection: (X_0, X_1) defines the same map as (X_0X_2, X_1X_2), which is equivalent modulo $X_2X_0 - X_1^2$ to (X_1^2, X_1X_2), which defines the same map as (X_1, X_2), as desired. Thus, we have defined a morphism $C \to \mathbb{P}^1_k$, but were not able to do so with a single pair of polynomials that worked simultaneously on all of C.

We also check that this is an inverse to the original function, so the two projective curves are in fact isomorphic. However, we note that the homogeneous coordinate rings $k[T_0, T_1]$ and $k[X_0, X_1, X_2]/(X_2X_0 - X_1^2)$ are visibly not isomorphic.

What this example says is that the projective imbedding ends up giving extra data beyond the data intrinsic to the variety. In some sense, it is a miracle that this does not occur with affine varieties. That is, if we define a morphism of affine varieties as a function locally described by polynomials, then there is always a single tuple of polynomials which suffices to describe the affine variety. This fact is hidden in the result stating that a morphism of affine schemes is always induced by a homomorphism of the corresponding rings.

We make a further remark on the subtlety of morphisms of projective varieties, which requires two more definitions:

Definition 2.9. A projective variety $V \subseteq \mathbb{P}^n_k$ is a **hypersurface of degree** d if it is the zero set of a single homogeneous polynomial F of degree d. In this case, V is **smooth** if there is no point of V at which $\partial F/\partial X_i$ vanishes for all $i = 0, \ldots, n$.

We then conclude with the following open question:

Question 2.10. Fix k an algebraically closed field of characteristic 0. Suppose we have hypersurfaces $V_1, V_2 \subseteq \mathbb{P}^n_k$ of degrees d_1, d_2, with $n \geq 4$. Suppose further that V_2 is smooth, that $d_2 \geq 2$, and that there is a non-constant morphism $f : V_1 \to V_2$. Then is it necessary to have $d_2 | d_1$, and f given by polynomials of degree $\frac{d_1}{d_2}$?

2.3. **The scheme-theoretic point of view.** In fact, there is little to add here beyond the picture discussed for affine varieties. Once again, if we consider the points of the underlying space of the scheme, or morphisms of abstract schemes, we find a picture only loosely related to the classical one, but the classical picture can be expressed in much the same way as in the affine case.

We let $A_1 = k[X_0, \ldots, X_{n_1}]/I_1$ and $A_2 = k[Y_0, \ldots, X_{n_2}]/I_2$, where I_1, I_2 are geometrically prime homogeneous ideals. We consider the schemes $X_1 = \operatorname{Proj} A_1$, and $X_2 = \operatorname{Proj} A_2$, which as in the affine case, we can naturally consider as schemes over $\operatorname{Spec} k$.

Proposition 2.11. *The morphisms $X_1 \to X_2$ as schemes over* $\operatorname{Spec} k$ *correspond to the classical maps of projective varieties. For any $k' \supseteq k$, the classical k'-points of X correspond to maps* $\operatorname{Spec} k' \to X$ *over* $\operatorname{Spec} k$.

The fact that these definitions work so universally will lead us to define an abstract algebraic variety as a certain type of scheme over $\operatorname{Spec} k$ (again, the structure map to $\operatorname{Spec} k$ is part of the data of the variety), including simultaneously affine varieties, projective varieties, open subsets of projective varieties, and even some varieties which can't be imbedded in any projective space. Motivated by the classical picture in the affine and projective cases, we will then define k'-valued points of an algebraic variety X to be maps $\operatorname{Spec} k' \to X$ over $\operatorname{Spec} k$.

Indeed, we will ultimately go further, defining "T-valued points of X", where T and X are any schemes (possibly over a base scheme S), to be morphisms $T \to X$. These roughly correspond to family of points of X which are parametrized by T, and play an important role when we discuss moduli spaces and the functor of points.