FIXING THE ZARISKI TOPOLOGY: SEPARATED AND PROPER MORPHISMS

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1. What's wrong with the Zariski topology

Classical algebraic geometry usually studied varieties over $\mathbb{C}$, and was therefore able to make use of the complex topology. Indeed, any affine or projective variety V gets an induced topology V^{an} from the complex topology on $\mathbb{A}^n_{\mathbb{C}}$ or $\mathbb{P}^n_{\mathbb{C}}$. In this situation, one can easily apply our topological intuition: for instance, $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ is not (quasi)-compact, while $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}}$ is, and both are Hausdorff. This complex topology allowed algebraic geometers to apply the entire theory of complex manifolds to the study of algebraic geometry, leading to many important breakthroughs.

Unfortunately, for varieties over more general fields, let alone schemes, we have only the Zariski topology, and its relation to our intuitive picture is tenuous at best. For instance, while $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ is Hausdorff but not quasi-compact, the Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ is quasi-compact but not Hausdorff. A major motivating force in modern algebraic geometry has been the attempt to understand how one can get around this issue. Indeed, much of the technical machinery of algebraic geometry can be thought of as being guided by the following question:

Question 1.1. Given a notion in topology, differentiable topology, or algebraic topology, how can this notion be translated into the context of schemes in such a way that it behaves formally as expected, and in the case of schemes over $\mathbb{C}$, is (at least roughly) equivalent to the standard notion as applied to X^{an}?

We will focus here on the first part of this, namely properties which are purely topological in nature. More specifically, we focus on the notions of Hausdorff and quasi-compactness, which translate to separatedness and properness in the context of schemes. The standard notions are useless in the Zariski topology, because essentially nothing is Hausdorff, and essentially everything is quasi-compact. As a first approximation, we would like definitions under which an affine scheme is Hausdorff but not quasi-compact (unless it consists of a finite set of points), and a projective scheme is both, but a line with a doubled origin is not Hausdorff.

2. The Analytic Topology and "Good" Definitions

We now discuss what it means to have a "good" definition for a topological property in the scheme setting. One basic trick is to come up with a definition which is equivalent in the topological setting, but not in the scheme-theoretic setting, for instance by making use of the fact that the topological space of a product is not simply the product of the topological spaces. A second point of view is obtained by restricting to schemes of finite type over $\mathbb{C}$, and considering the analytic topology:

Definition 2.1. Let X be a scheme of finite type over $\mathbb{C}$. Let X_{an} be the topological space whose points are the $\mathbb{C}$-valued points (equivalently, the closed points) of X, and with base consisting of sets U obtained as follows: every element U of the base is the intersection of (the $\mathbb{C}$-valued points of) an affine open subscheme $X_U(\mathbb{C}) \subset \mathbb{C}^n$ with an open subset $\tilde{U}$ in the usual topology on $\mathbb{C}^n$. This is called the **analytic topology** for X.

Proposition 2.2. *The analytic topology is a topology on $X(\mathbb{C})$. If $U \subseteq X$ is affine, then $X_{\text{an}} \cap U(\mathbb{C}) = U_{\text{an}}$, and the topology is simply the subset topology inherited from affine space.*

The proof is left as an exercise.

The topological space X_{an} is then the intuitive topology for X; for schemes of finite type over $\mathbb{C}$, we could simply require that X_{an} be Hausdorff, or quasi-compact, and get the result that we expect. However, we need definitions that will work more generally. To summarize, for each topological property P, we would like to find a definition of a property P', which will be the scheme-theoretic equivalent of P, with the following good behavior:

(I) The definition makes sense in the generality of schemes.
(II) If the definition is applied to the category of topological spaces, it is equivalent to the usual one.
(III) If we restrict to schemes X of finite type over $\mathbb{C}$, then X has P' if and only if X_{an} has P.

Remark 2.3. Condition (III) serves as a check that we have really come up with the right definition: without it, we could just impose P on the underlying topological space of X, and we would end up with a definition satisfying properties (I) and (II), but with behavior completely unrelated to what we want!

There is one caveat: there will be cases (as with properness, discussed below), where the topological property is converted into a property of a morphism. In this case, the correct statement of (II) will be that if the definition is applied in the topological category, to the constant map from a space to a point, then it is equivalent to the usual definition for that space. The correct statement of (III) will be that the morphism $X \to \operatorname{Spec} \mathbb{C}$ has P' if and only if X_{an} has P.

Although it is not the case that condition (III) follows formally from (II), the below propositions help to relate the two.

Definition 2.4. A subset of X is **constructible** if it is a finite union of intersections of closed and open subsets (in the Zariski topology).

We leave as exercises the following results:

Proposition 2.5. *There is a natural continuous map $X_{\text{an}} \to X$, inducing a bijection on closed points.*

A morphism $X \to Y$ over $\operatorname{Spec} \mathbb{C}$ *induces a continuous map $X_{\text{an}} \to Y_{\text{an}}$.*

Furthermore, given morphisms $X \to Z$, $Y \to Z$, we have $(X \times_Z Y)_{\text{an}} = X_{\text{an}} \times_{Z_{\text{an}}} Y_{\text{an}}$ as topological spaces (where the topology on the right is the usual product topology).

The following is more substantial:

Theorem 2.6. *The closed subsets of X are precisely the constructible subsets whose preimages in X_{an} are closed.*

Remark 2.7. Note that if we work with absolute products, if X and Y are of finite type over $\mathbb{C}$, then $X \times Y$ is not of finite type over $\mathbb{C}$! What we really want is $X \times_{\operatorname{Spec} \mathbb{C}} Y$; this is the first hint that in scheme land, we will want to work with properties of morphisms (e.g., of a scheme *over* $\operatorname{Spec} \mathbb{C}$), rather than properties of the absolute scheme.

3. Connectedness, Hausdorfficity, and Compactness

We now discuss three concrete examples of topological properties translated to the scheme setting, in increasing order of technical difficulty. The easiest topological property is connectedness: in fact, one can use the same definition as before:

Definition 3.1. A scheme X is **connected** if its underlying topological space is connected.

One direction of the following is not too hard, but the converse is surprisingly non-trivial:

Theorem 3.2. *This definition satisfies condition (III) above; i.e., X is connected in the Zariski topology if and only if X_{an} is connected.*

The fact that this theorem isn't a mere formality is underlined by the observation that it is false for the real topology: for instance, an elliptic curve is always irreducible in the Zariski topology, but can have two connected components over the real numbers.

Next, we want a property analogous to a space being Hausdorff. Here, the translation to a equivalent topological property is fairly straightforward:

Definition 3.3. A scheme X is **separated** if the diagonal map $\Delta : X \to X \times X$ is closed.

Here we can define $X \times X := X \times_{\operatorname{Spec} \mathbb{Z}} X$. Maps from T to $X \times X$ correspond to arbitrary pairs of maps $T \to X$.

We leave the following as exercises:

Proposition 3.4. *This definition satisfies condition (II) above, i.e., in the topological category it is equivalent to the usual definition.*

Theorem 3.5. *This definition satisfies condition (III) above, i.e., if X is of finite type over $\mathbb{C}$, then X_{an} is Hausdorff if and only if X is separated.*

In the case of separatedness, the absolute definition is formally sufficient, as the above result demonstrates. However, we note that the definition naturally lends itself to a relative version:

Definition 3.6. Given a morphism $f : X \to Y$ of schemes, we say that f is **separated**, or X is **separated over** Y if the diagonal map $\Delta : X \to X \times_Y X$ is closed.

Moreover, we see that in examples, the relative version is much more tractable.

Example 3.7. It is easy to check that $\mathbb{A}^1_k$ is separated over $\operatorname{Spec} k$. In fact, $\mathbb{P}^1_k$ is also separated over $\operatorname{Spec} k$, although it is slightly more annoying to check this.

However, if X is the affine line with the doubled origin, X is not separated over $\operatorname{Spec} k$: $X \times_{\operatorname{Spec} k} X$ is the affine plane with a quadruple origin, and the diagonal map only hits two of the four origins, so is not closed.

If we tried to analyze directly whether or not these spaces are separated in the absolute sense, it becomes a mess: for instance, if k is relatively large, $\mathbb{A}^1_k \times \mathbb{A}^1_k$ is some huge scheme, no longer of finite type over k.

Remark 3.8. Since $X \times X = X \times_{\operatorname{Spec} \mathbb{Z}} X$, we see X is separated if and only if X is separated over $\operatorname{Spec} \mathbb{Z}$. Thus the relative definition is strictly more general.

The relative definition of separatedness work with the relative version of condition (III): we have that for X of finite type over $\mathbb{C}$, X is separated over $\operatorname{Spec} \mathbb{C}$ if and only if X_{an} is Hausdorff: i.e., X is separated over $\operatorname{Spec} \mathbb{C}$ if and only if X is separated. This is because of several facts: first, if X is separated, then X is separated over anything; second, $\operatorname{Spec} \mathbb{C}$ is separated over $\operatorname{Spec} \mathbb{Z}$; and finally, the composition of separated maps is separated.

Finally, we move on to quasi-compactness. Our first pass will be based on the following fact, which we leave as an exercise:

Proposition 3.9. *A topological space X is quasi-compact if and only if for every topological space Z, the projection map $X \times Z \to Z$ is closed.*

Definition 3.10. We say that X is **universally closed** if for all schemes Z, the map $X \times Z \to Z$ is closed.

While this definition is reasonable given the topological setting, and we will use it as stated, this turns out to be one of those times when an absolute definition is inadequate for application to the classical situation over $\mathbb{C}$. Indeed, although a point should certainly be quasi-compact, we see that it is not, if one attempts to use the above definition directly: in fact, one can check that $\operatorname{Spec} \mathbb{C}$ itself is not universally closed.

The fix is to switch to a relative definition; in fact, an obvious possibility presents itself, and is the source of the terminology used above:

Definition 3.11. Given $f : X \to Y$, we say that f is **universally closed**, or X is **universally closed over** Y, if for all morphisms $g : Z \to Y$, the map $X \times_Y Z \to Z$ is closed.

Example 3.12. $\mathbb{A}^1_k$ is not universally closed over $\operatorname{Spec} k$: indeed, if we set $Z = \mathbb{A}^1_k$ also, the map $X \times_Y Z \to Z$ is the projection map $\mathbb{A}^2_k \to \mathbb{A}^1_k$, and this is not closed because, for instance, the hyperbola defined by $xy = 1$ in $\mathbb{A}^2_k$ is closed, but its image is the open subset $\mathbb{A}^1_k \smallsetminus (0)$.

It turns out that $\mathbb{P}^1_k$ is universally closed over $\operatorname{Spec} k$, but this is harder to show, and we leave it until after we have proved the valuative criterion for properness.

Once again, X is universally closed if and only if it is universally closed over $\operatorname{Spec} \mathbb{Z}$. In fact, a concept closely related to this already exists in topology:

Definition 3.13. A continuous map of topological spaces $f : X \to Y$ is said to be **proper** if first, f is closed; and second, $f^{-1}(K)$ is quasi-compact for any quasi-compact subset K of Y. In particular, X is quasi-compact if and only if the map $X \to P$ for P a point is proper.

Proposition 3.14. *f is proper if and only if for all topological spaces Z, the map $X \times Z \to Y \times Z$ is closed.*

If X is Hausdorff and Y is locally quasi-compact, f is proper if and only if $f^{-1}(K)$ is quasi-compact for any quasi-compact subset K of Y (i.e., this condition implies f is closed).

If Y is Hausdorff, f is proper if and only if for all continuous maps $f : Z \to Y$, the map $X \times_Y Z \to Z$ is closed.

Motivated by the French inclusion of a Hausdorff hypothesis in the definition of compactness, and including a hypothesis ensuring algebraic finiteness, we finally define:

Definition 3.15. A morphism $f : X \to Y$ is **proper** if it is separated, universally closed, and of finite type.

These conditions, and particularly the algebraic finiteness, are calibrated as the most general under which a number of important and general theorems can be proved: for instance, that the pushforward of a coherent sheaf will remain coherent, as we will discuss later.

One direction of the following theorem is now straightforward, while the other will require Chow's lemma:

Theorem 3.16. *If X is of finite type over* $\operatorname{Spec} \mathbb{C}$*, then X is proper over* $\operatorname{Spec} \mathbb{C}$ *if and only if X_{an} is compact.*

In fact, one can show the following more general statement:

Theorem 3.17. *If X and Y are of finite type over $\mathbb{C}$, and $f : X \to Y$ a morphism, then f is proper if and only if the induced map $f_{\mathrm{an}} : X_{\mathrm{an}} \to Y_{\mathrm{an}}$ is proper, if and only if $f_{\mathrm{an}}^{-1}(K)$ is compact for every compact $K \subseteq Y_{\mathrm{an}}$.*

4. A Look Ahead

This same philosophy will guide much of what we do later, as we look at properties like smoothness and étaleness. However, such properties involve going past the elementary topology and into issues of differential topology, as one must be able to distinguish, for instance, a rational cuspidal curve from the affine line. However, many of the same ideas apply: we can put a sheaf of functions on X_{an} which tells us what the differentiable functions are. We will then see that for X of finite type over $\mathbb{C}$, we will have X smooth over $\operatorname{Spec} \mathbb{C}$ if and only if X_{an} is a complex manifold. Similarly, if $f : X \to Y$ is a morphism of schemes of finite type over $\operatorname{Spec} \mathbb{C}$, then f is étale if and only if f_{an} is locally an isomorphism at every point of X.