THE FUNCTOR OF POINTS, YONEDA'S LEMMA, MODULI SPACES, AND UNIVERSAL PROPERTIES

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1. The functor of points and Yoneda's lemma

Our discussion is motivated by two questions. The first arises as a matter of aesthetics. Whenever we define an object by a universal property (as in the case of sheafification), we find that if there is an object with the described property, it is unique up to unique isomorphism. Furthermore, the argument is always purely formal nonsense. We therefore ask:

Question 1.1. How can one formulate a rigorous framework for a "universal property" such that one can prove that any object satisfying a universal property is unique up to unique isomorphism?

The second question is far more substantive, and deals with moduli spaces. These are algebraic varieties (or schemes) which are supposed to naturally parametrize certain objects, as the Grassmannian $G(r,d)$ parametrizes r-dimensional subspaces of a fixed d-dimensional space. The problem is that usually such a description only describes the points as a set, and doesn't explain what the geometric structure should be. Frequently (as with the case of the Grassmannian), there is a clear "natural" geometric structure, but it's not clear how to formalize what the "right" structure is.

Question 1.2. How does one formalize a notion of moduli space so that the space is uniquely determined as a scheme (or variety), and not simply as a set?

These questions are both addressed by the machinery of the functor of points, and Yoneda's lemma.

Recall that a **category** $\mathcal{C}$ consists of a collection of **objects** $\mathrm{Obj}(\mathcal{C})$ and **morphisms** between objects: i.e., for each ordered pair (a,b) of (not necessarily distinct) objects of $\mathcal{C}$, we have a set $\mathrm{Mor}(a,b)$ of morphisms from a to b. We are also given the data of **composition** of morphisms: for any (a,b,c) of objects on $\mathcal{C}$, a map of sets $\mathrm{Mor}(a,b) \times \mathrm{Mor}(b,c) \to \mathrm{Mor}(a,c)$; given $f \in \mathrm{Mor}(a,b)$ and $g \in \mathrm{Mor}(b,c)$, we write the resulting element of $\mathrm{Mor}(a,c)$ with the usual notation $g \circ f$.

The two conditions for such a collection of data to form a category are:

(i) that composition is **associative**, i.e. that for all (a,b,c,d) and all f,g,h in $\mathrm{Mor}(a,b)$, $\mathrm{Mor}(b,c)$ and $\mathrm{Mor}(c,d)$ respectively, that $h \circ (g \circ f) = (h \circ g) \circ f$;

(ii) that for any object $a \in \mathrm{Obj}(\mathcal{C})$, there is an **identity** element $1_a \in \mathrm{Mor}(a,a)$ such that for any b and any morphism $f \in \mathrm{Mor}(a,b)$, we have $f \circ 1_a = f$, and for any $f \in \mathrm{Mor}(b,a)$ we have $1_a \circ f = 1_a$.

We also have the notion of a **functor**. A functor can be either **covariant** or **contravariant**. A covariant functor is a mapping between categories; specifically, $F : \mathcal{C}_1 \to \mathcal{C}_2$ associates to each object a of $\mathcal{C}_1$ an object $F(a)$ of $\mathcal{C}_2$, and associates, for any pair (a, b) of objects of $\mathcal{C}_1$ and any morphism $f \in \mathrm{Mor}(a, b)$, a morphism $F(f) \in \mathrm{Mor}(f(a), f(b))$. A functor must have the properties that $F(1_a) = 1_{F(a)}$ for any $a \in \mathrm{Obj}(\mathcal{C}_1)$, and that for any (a, b, c) and $f \in \mathrm{Mor}(a, b), g \in \mathrm{Mor}(b, c)$, we have $F(g \circ f) = F(g) \circ F(f)$. A contravariant functor is the same, except that it reverses directions of morphisms, so that if $f \in \mathrm{Mor}(a, b)$, then $F(f) \in \mathrm{Mor}(F(b), F(a))$.

Now, let $\mathcal{C}$ be a category, and $X \in \mathrm{Obj}(\mathcal{C})$. We observe that we have associated to X a natural contravariant functor $h_X : \mathcal{C} \to \mathrm{Set}$ defined by $h_X(T) = \mathrm{Mor}(T, X)$ (with h_X acting on morphisms by composition). Motivated by geometric categories, we call h_X the **functor of points** of X.

We will also sometimes use the covariant version of h_X, which we denote by h^o_X, with $h^o_X(T) = \mathrm{Mor}(X, T)$.

Yoneda's lemma says that h_X (and similarly, h^o_X) determines X uniquely. To state it more precisely, let's suppose we have two contravariant functors $F_1, F_2 : \mathcal{C} \to \mathrm{Set}$. Then a **morphism** $\varphi : F_1 \to F_2$ consists of the data, for all objects $X \in \mathcal{C}$, of a function $\varphi(X) : F_1(X) \to F_2(X)$, such that for any morphism $f : Y \to X$ in $\mathcal{C}$, we have that the diagram

$$\begin{array}{ccc} F_1(X) & \xrightarrow{\varphi(X)} & F_2(X) \\ \downarrow{\scriptstyle F_1(f)} & & \downarrow{\scriptstyle F_2(f)} \\ F_1(Y) & \xrightarrow{\varphi(Y)} & F_2(Y) \end{array}$$

commutes. φ is **isomorphism** if it has an inverse, or as one can check is equivalent, if every $\varphi(X)$ is a bijection.

Note that if we have a morphism $X \to Y$ in $\mathcal{C}$, we obtain a morphism $h_X \to h_Y$ by composition, and the same holds for isomorphisms. We also note that everything applies also to covariant functors and h^o_X, if we switch the direction of the arrows.

Lemma 1.3. *(Yoneda's lemma) Given objects $X, Y \in \mathcal{C}$, suppose we have a morphism $\varphi : h_X \to h_Y$ of the associated functor of points. Then φ is induced by a unique morphism $f : X \to Y$. In particular, any isomorphism $h_X \xrightarrow{\sim} h_Y$ is induced by an isomorphism $X \xrightarrow{\sim} Y$.*

The same holds for h^o_X and h^o_Y.

Proof. We consider the map $\varphi(X) : h_X(X) \to h_Y(X)$. In $h_X(X)$ we have the identity element, and we set $f = (\varphi(X))(\mathrm{id}) \in \mathrm{Mor}(X, Y)$. It is then a simple exercise to check that this induces a bijection between morphisms $h_X \to h_Y$ and morphisms $X \to Y$.

The same argument works for h^o_X and h^o_Y. □

Let's notice that if we have a contravariant functor $F : \mathcal{C} \to \mathrm{Set}$, then an object $X \in \mathrm{Obj}(\mathcal{C})$ together with an $\eta \in F(X)$ induces a morphism $h_X \to F$. Indeed, given a morphism $\varphi : Y \to X$, we obtain a map $F(\varphi) : F(X) \to F(Y)$, and declaring $(F(\varphi))(\eta)$ to be the image of φ defines the morphism $h_X \to F$.

Definition 1.4. Given a functor $F : \mathcal{C} \to \mathrm{Set}$, we say that a pair $X \in \mathrm{Obj}(\mathcal{C})$, $\eta \in F(X)$ **represents** F if the induced morphism $h_X \to F$ is an isomorphism.

Yoneda's lemma immediately implies:

Corollary 1.5. *If (X, η) represents a functor F, then the pair is unique up to unique isomorphism; that is, if (Y, η') also represents F, there is a unique isomorphism $\varphi : X \xrightarrow{\sim} Y$ sending η' to η.*

We remark that the object η is an important part of the data for representing a functor. However, we will frequently omit it from notation. We also have the same situation for covariant functors F, saying that (X, η) **corepresents** F if the induced map $h_X^o \to F$ is an isomorphism. Again, (X, η) is unique up to unique isomorphism.

2. Universal properties

We first discuss how to interpret Yoneda's lemma to rigorously conclude that anything satisfying a universal property is unique up to unique isomorphism. To give a simple example, we define:

Definition 2.1. Let $\mathcal{C}$ be a category, and X, Y, Z objects of $\mathcal{C}$. Fix also morphisms $\pi_X : X \to Z$, $\pi_Y : Y \to Z$. Given this data, we say that an object P of $\mathcal{C}$, together with morphisms $p_1 : P \to X$, $p_2 : P \to Y$ is a **fiber product** of X with Y over Z if it satisfies the following universal property:

For every object $T \in \mathrm{Obj}(\mathcal{C})$, and every pair of morphisms $f : T \to X$, $g : T \to Y$ such that $\pi_X \circ f = \pi_Y \circ g$, there exists a unique morphism $h : T \to P$ such that $f = p_1 \circ h$ and $g = p_2 \circ h$.

In this case, we write P as $X \times_Z Y$.

Thus, a fiber product is defined to represent the functor of pairs of maps to X and Y which give the same map to Z. We therefore see immediately that if it exists, it is unique up to unique isomorphism (note that in this case, the data of η consists of the maps p_1, p_2).

However, some universal properties are slightly less obvious, in that interpreting them as representing (or corepresenting) a functor can involve careful choice of the correct category. For instance, we defined the sheafification of a presheaf $\mathcal{F}$ in terms of morphisms from $\mathcal{F}$ to sheaves; we therefore see that we have to work in the category of sheaves, rather than presheaves. Specifically, we see:

Proposition 2.2. *Given a presheaf $\mathcal{F}$, let $\mathcal{F}^+$ be the sheafification, and recall that we have a natural map $\mathcal{F} \to \mathcal{F}^+$. Then the pair $(\mathcal{F}^+, \mathcal{F} \to \mathcal{F}^+)$ corepresents the covariant functor* Sheaf $\to$ Set *which associates to a sheaf $\mathcal{G}$ the set of morphisms (as presheaves!) $\mathcal{F} \to \mathcal{G}$. Therefore, it is unique up to unique isomorphism.*

In fact, situations like this arise frequently in algebraic geometry (for instance, with normalization, and with the reduced scheme associated to a scheme), so we take a moment to give a more general statement.

Definition 2.3. Let $\mathcal{C}$ be a category. Somewhat informally, a category $\mathcal{C}'$ is a **half-full subcategory** of $\mathcal{C}$ if each object of $\mathcal{C}'$ is identified with a unique object of $\mathcal{C}$, in such a way that for objects $X, Y \in \mathrm{Obj}(\mathcal{C}')$, we have $\mathrm{Mor}_{\mathcal{C}'}(X, Y) \subseteq \mathrm{Mor}_{\mathcal{C}}(X, Y)$, and every isomorphism in $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is also in $\mathrm{Mor}_{\mathcal{C}'}(X, Y)$. $\mathcal{C}'$ is a **full subcategory** if for each $X, Y \in \mathrm{Obj}(\mathcal{C}')$, we in fact have $\mathrm{Mor}_{\mathcal{C}'}(X, Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$.

We then have the following consequence of Yoneda's lemma:

Corollary 2.4. *Let $\mathcal{C}'$ be a half-full subcategory of $\mathcal{C}$, and $X \in \mathrm{Obj}(\mathcal{C})$ an object. An object $X' \in \mathrm{Obj}(\mathcal{C}')$ with a morphism $\varphi : X' \to X$ is* **universal for morphisms from** $\mathcal{C}'$ *if for any $Y \in \mathrm{Obj}(\mathcal{C}')$ with $f : Y \to X$, there exists a unique morphism $f' : Y \to X'$ such that $f = \varphi \circ f'$. Then (X', φ) is unique up to unique isomorphism, in either $\mathcal{C}'$ or $\mathcal{C}$.*

The hypothesis that $\mathcal{C}'$ is a half-full subcategory of $\mathcal{C}$ is necessary to pass freely between isomorphisms in $\mathcal{C}'$ and in $\mathcal{C}$. Specifically, it allows us to conclude that (X', φ) is unique up to unique isomorphism not only in $\mathcal{C}'$, but also considered in the larger category $\mathcal{C}$.

We note that the same statement also holds for morphisms to objects of $\mathcal{C}'$, and it is this version which contains sheafification as a special case, with $\mathcal{C}$ being the category of presheaves, and $\mathcal{C}'$ the category of sheaves.

3. Moduli spaces and schemes representing functors

We now turn to the question of how to describe moduli spaces rigorously, and discuss representability of functors in more detail when our category is the category of schemes. We have already seen some examples of schemes representing functors: for instance, the fiber product mentioned above will be an important construction in the context of schemes; also, we have seen in Exercise 1 in Problem Set 3 that $\operatorname{Spec} \mathbb{Z}[t]$ represents the functor sending a scheme X to $\mathcal{O}_X(X)$, the ring of global sections of the structure sheaf of X, and that $\operatorname{Spec} k[t]$ represents the same functor in the category of schemes over $\operatorname{Spec} k$.

However, representability of functors plays an important role as well in the theory of moduli spaces. The basic idea is quite simple. We return to the example of a Grassmannian $G(r, d)$, and suppose we have fixed a base field k. We have specified that $G(r, d)$ is supposed to parametrize r-dimensional subspaces of a d-dimensional k-vector space V. Without saying anything more, this only specifies $G(r, d)$ as a set, and doesn't say anything about its geometry.

Before proceeding, we briefly digress into how to think about a morphism. Let's consider a simple example: say, the curve in $\mathbb{A}^d_k$ parametrized by $(t, t^2, \dots, t^d)$. This is an algebraic curve in $\mathbb{A}^d_k$, but it's not described as such. At its most elementary, this is a collection of points in $\mathbb{A}^d_k$ parametrized by t. We can also think of it as a morphism $\mathbb{A}^1_k \to \mathbb{A}^d_k$ where the coordinate on $\mathbb{A}^1_k$ is t, since it is visibly described by polynomials in t. More generally, we can picture a morphism $T \to X$ as a collection of points of X, which are parametrized nicely by T.

Let's return to the situation that X is a moduli space, and specifically the Grassmannian. We then see that, if the points of $G(r, d)$ are supposed to correspond to subspaces of V, then given a scheme T, it makes sense to ask that morphisms $T \to G(r, d)$ should correspond to families of subspaces of V, with the subspaces in the family parametrized by T. We will not give a precise definition until we have discussed sheaves of modules. However, we now know that given such a description, we have asked that $G(r, d)$ represents a functor, and this determines it uniquely as a scheme, if it exists. We will ultimately show that $G(r, d)$ does exist, and is indeed covered by open subschemes isomorphic to $\mathbb{A}^{r(d-r)}_k$, as suggested by Exercise 1 of Problem Set 1. In fact, we will use the open cover to prove that $G(r, d)$ exists.

Before moving on to a more general study of schemes representing functors, we would be remiss if we did not explain that when studying moduli spaces, it is often the case that the functors are not representable. This is typically due to the presence of automorphisms of the objects being parametrized, which can prevent a functor from being a Zariski sheaf. We will see an example of this when we define the Picard functor. However, in such situations, we can sometimes use the concept of corepresentability to give a weaker notion of a moduli space, which exists more often. When a functor associated to a moduli problem is representable, we say that the resulting moduli space is a **fine moduli space**. In some cases where no fine moduli space exists, we nonetheless have a **coarse moduli space**, which is typically defined in terms of a corepresenting condition, and some additional conditions.

We now pull various concepts together with the notion of corepresentability for moduli spaces.

Definition 3.1. Let $F : \mathrm{Sch}_S \to \mathrm{Set}$ be a contravariant functor. Suppose we have (X, η), with $X \in \mathrm{Obj}(\mathrm{Sch}_S)$, and $\eta \in \mathrm{Mor}(F, h_X)$. We say that (X, η) **corepresents** F if for every $Y \in \mathrm{Obj}(\mathrm{Sch}_S)$, every morphism $F \to h_Y$ factors uniquely through η.

Proposition 3.2. *Suppose that (X, η) corepresents a functor F. Then (X, η) is unique up to unique isomorphism.*

Proof. Indeed, we note that by Yoneda's lemma, the category Sch_S is imbedded as a full subcategory of the category of contravariant functors $F : \mathrm{Sch}_S \to \mathrm{Set}$. The proposition is then simply Corollary 2.4. □

Note that the terminology, while slightly sloppy, makes sense: the definition implies that (X, η) corepresents F if and only (X, η) corepresents the covariant functor $\mathrm{Sch}_S \to \mathrm{Set}$ given by $Y \mapsto \mathrm{Mor}(F, Y)$.

We now move on to see what we can say about schemes representing functors. Our first observation is on the usefulness of Yoneda's lemma in constructing morphisms of schemes which represent functors. Recall that a **T-valued point** of a scheme X is a morphism $T \to X$ (over S, when we are working in Sch_S). We can directly rephrase Yoneda's lemma as follows.

Corollary 3.3. *Given $X, Y \in \mathrm{Obj}(\mathrm{Sch}_S)$, a morphism $f : X \to Y$ is equivalent to the following:*

For each $T \in \mathrm{Obj}(\mathrm{Sch}_S)$, a function f_T from the T-valued points of X to the T-valued points of Y, such that for any $g : T' \to T$ in Sch_S, we have $f_{T'} \circ g_X = g_Y \circ f_T$, where $g_X : \mathrm{Mor}_S(T, X) \to \mathrm{Mor}_S(T', X)$ is induced by composition with g, and similarly for g_Y.

Indeed, the collection of f_T is equivalent to the data of a morphism $h_X \to h_Y$, so this follows immediately from Yoneda's lemma. However, the corollary is very useful in dealing with schemes representing functors, as it means that we can construct morphisms between them solely in terms of the functors, even if we don't have the slightest idea how to understand the geometry of the schemes!

Finally, we examine a certain property that a contravariant functor $F : \mathrm{Sch} \to \mathrm{Set}$ must have in order to be representable, the property of being a "Zariski sheaf". In fact, we will fix a base scheme S, and work with the category Sch_S of schemes over S. There is no loss of generality here, since if we want to work with the category Sch of all schemes, we can simply set $S = \mathrm{Spec}\,\mathbb{Z}$.

The basic observation is the following:

Proposition 3.4. *(Gluing of morphisms of schemes) Let X and Y be schemes over S, and $\{U_i\}$ an open covering of X. Then morphisms $f : X \to Y$ over S are in one-to-one correspondence with collections of morphisms $f_i : U_i \to Y$ over S, such that for all i, j we have $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ as morphisms $U_i \cap U_j \to Y$ over S.*

Proof. Left as an exercise. □

This means that in order for a functor $F : \mathrm{Sch}_S \to \mathrm{Set}$ to be representable, it needs to satisfy a certain property, analogous to the condition for a presheaf to be a sheaf:

Definition 3.5. A contravariant functor $F : \mathrm{Sch}_S \to \mathrm{Set}$ is a **Zariski sheaf** if it satisfies the following condition:

For every $X \in \mathrm{Obj}(\mathrm{Sch}_S)$, and every open cover $\{U_i\}$ of X, the natural map

$$\{\eta \in F(X)\} \to \{\{\eta_i \in F(U_i)\} : \forall i, j, \eta_i|_{U_i \cap U_j} = \eta_j|_{U_i \cap U_j}\}$$

is a bijection.

It is simply a matter of definition-chasing to see that the above proposition can be rephrased to say the following:

Corollary 3.6. *Let* $F : \mathrm{Sch}_S \to \mathrm{Set}$ *be a contravariant functor. In order for* F *to be representable, it is necessary that* F *be a Zariski sheaf.*

This is by no means a sufficient condition, but we will see that if F is a Zariski sheaf, and we want to construct an X representing F, then it is enough to carry out the construction locally. We will use this to prove that fiber products exist in the category of schemes, and later we will apply the same technique to Grassmannians.