ASSOCIATED POINTS AND APPLICATIONS

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1. Associated points and support of sections

A basic tool in algebraic geometry is the decomposition of the underlying topological space of a Noetherian scheme into its irreducible components. However, there is a finer scheme-theoretic version of this decomposition, which is frequently important even in classical situations. We will study this via the language of imbedded points.

Definition 1.1. Let $\mathscr{F}$ be a sheaf of modules on a scheme X. We say that $x \in X$ (not necessarily closed) is an **associated point** of $\mathscr{F}$ if for all $f \in \mathfrak{m}_x \subseteq \mathscr{O}_{X,x}$, we have that $\mathscr{F}_x \overset{\times f}{\to} \mathscr{F}_x$ is not injective. In particular, we say $x \in X$ is an associated point of X if it is an associated point of $\mathscr{O}_X$, or equivalently if $\mathfrak{m}_x$ consists entirely of zero divisors.

The definition of associated points is slightly opaque, in order to make it obviously intrinsic and local. However, at least on a Noetherian scheme it is closely tied to supports of sections of $\mathscr{F}$:

Proposition 1.2. *If X is Noetherian and $\mathscr{F}$ is coherent, for any open subset U and any section $s \in \mathscr{F}(U)$, the support of s is a finite union of closures of associated points of $\mathscr{F}$ inside U. If U is affine, then conversely every such set arises as the support of a section $s \in \mathscr{F}(U)$.*

We will first need a lemma from commutative algebra:

Lemma 1.3. *Let A be a Noetherian ring, and M a finite-generated A-module. Then the set of prime ideals of the form* $\operatorname{Ann}(m)$ *for some $m \in M$ is finite.*

Proof. This is standard: one shows that in a short exact sequence $0 \to M' \to M \to M'' \to 0$, the possible supports of sections of M are unions of supports of sections of M' and M'', and then shows that any finite-generated A-module M has a finite filtration with successive quotients of the form $A/\mathfrak{p}$; see Theorem 6.5 (i) of [2] for details. $\square$

Proof of proposition. We know quite generally that $\operatorname{Supp} s$ is a closed set (indeed, for any sheaf of abelian groups on any topological space; see Exercise II.1.14 of [1]). On a Noetherian scheme, it is then a finite union of irreducible components, and we simply need to show that each generic point of each component is an associated point of $\mathscr{F}$. But this is clear, since if x is such a generic point, locally around x we have $\operatorname{Supp} s = \bar{x}$, so s annihilates any element of $\mathfrak{m}_x \subseteq \mathscr{O}_{X,x}$.

The converse is a bit subtler. We begin by showing that if $x \in X$ is an associated point of $\mathscr{F}$, then there is a section $s \in \mathscr{F}_x$ such that $\operatorname{Ann}(s) = \mathfrak{m}_x$; i.e., $fs = 0$ if and only if $f \in \mathfrak{m}_x$. We are given that for any $f \in \mathfrak{m}_x$, multiplication by f is not injective in $\mathscr{F}_x$. We first claim that if we choose $s \in \mathscr{F}_x$ such that $\mathrm{Ann}(s) \subseteq \mathscr{O}_{X,x}$ is maximal among non-zero s (this is always possible since $\mathscr{O}_{X,x}$ is Noetherian), then $\mathrm{Ann}(s)$ is prime. Indeed, if $fg \in \mathrm{Ann}(s)$, we have $fgs = 0$. If $g \notin \mathrm{Ann}(s)$, then $\mathrm{Ann}(gs) \supseteq \mathrm{Ann}(s)$ and we must have equality by the maximality hypothesis, so we conclude that $f \in \mathrm{Ann}(gs) = \mathrm{Ann}(s)$. Thus $\mathrm{Ann}(s)$ is prime, as desired. We next note that every element $f \in \mathscr{O}_{X,x}$ such that multiplication by f is not injective on $\mathscr{F}_x$ is contained in $\mathrm{Ann}(s)$ for some $s \in \mathscr{F}_x$, and hence contained in some maximal $\mathrm{Ann}(s)$. But by hypothesis, this means that every element of $\mathfrak{m}_x$ is contained in some maximal $\mathrm{Ann}(s)$, and by the lemma, there are only finitely many such ideals, so their union can be equal to $\mathfrak{m}_x$ if and only if $\mathrm{Ann}(s) = \mathfrak{m}_x$ for some s.

We next show that there is some $s' \in \mathscr{F}(U)$ with support $\bar{x}$. If $\mathfrak{p}_x$ is the prime ideal of $\mathscr{F}(U)$ corresponding to x, this is equivalent to $\mathrm{Ann}(s') = \mathfrak{p}_x$. Now, write $s = \frac{s''}{f}$ for some $s'' \in \mathscr{F}(U)$ and $f \notin \mathfrak{p}_x$; we still have $\mathrm{Ann}(s'') = \mathfrak{m}_x$ in $\mathscr{O}_{X,x}$. For any $t \in \mathfrak{p}_x$, we have $ts'' = 0$ in $\mathscr{O}_{X,x}$, so $ts''f' = 0$ in $\mathscr{F}(U)$ for some $f' \notin \mathfrak{p}_x$. Because $\mathfrak{p}_x$ is finitely generated, if we choose $f'_i \notin \mathfrak{p}_x$ so that $t_i s'' f'_i = 0$ for some finite generating set $\{t_i\}$ of $\mathfrak{p}_x$, we find that $s' = s'' \prod_i f'_i$ has $\mathrm{Ann}(s') = \mathfrak{p}_x$, as desired.

Finally, if we have a finite union of sets $\bar{x}$, we can choose an $s \in \mathscr{F}(U)$ as above for each x, and their sum will have the desired support. □

Remark 1.4. In some references, such as [2], the definition of associated prime is closer to the condition given in the proposition; however, we follow the definitional convention from [1].

We now obtain two fundamental corollaries of the proposition.

Corollary 1.5. *If X is Noetherian and $\mathscr{F}$ is coherent, the set of associated points of $\mathscr{F}$ is finite.*

Proof. We can cover X by a finite collection of affine open sets $\{U_i\}$; on each U_i, by the proposition we have that the associated primes of $\mathscr{F}|_{U_i}$ are precisely generic points of $\mathrm{Supp}(s)$ for $s \in \mathscr{F}(U_i)$ with $\mathrm{Supp}(s)$ irreducible, and this is finite by the lemma. □

Corollary 1.6. *Suppose X is Noetherian and $\mathscr{F}$ is coherent. For any invertible sheaf $\mathscr{L}$ and $f \in \Gamma(X, \mathscr{L})$, we have $\mathscr{F} \stackrel{\times f}{\to} \mathscr{F} \otimes \mathscr{L}$ is injective if and only if f does not vanish at any associated point of $\mathscr{F}$.*

Proof. It is enough to work locally, so we may assume $\mathscr{L} = \mathscr{O}_X$. If f vanishes at an associated point x of $\mathscr{F}$, by definition we have $f \in \mathfrak{m}_x \subseteq \mathscr{O}_{X,x}$, and then that the multiplication by f map is not injective on $\mathscr{F}_x$, by definition of an associated point. On the other hand, suppose multiplication by f is not injective on $\mathscr{F}(U)$ for some U; then there is some $s \in \mathscr{F}(U)$ with $fs = 0$; in particular, f vanishes on $\mathrm{Supp}(s)$, and by the proposition $\mathrm{Supp}(s)$ contains associated points of $\mathscr{F}$. □

2. Associated points of schemes

We now focus on the associated points of a scheme itself, proving some basic facts which are frequently useful.

Lemma 2.1. *Suppose X is Noetherian. The generic points of each irreducible component of X are associated points of X.*

Proof. Indeed, if x is a generic point of an irreducible component of X, then $\mathscr{O}_{X,x}$ is a Noetherian local ring of dimension 0, hence an Artinian local ring, and $\mathfrak{m}_x$ consists entirely of nilpotents. □

Definition 2.2. We say that $x \in X$ is an **imbedded point** if it is an associated point which is not the generic point of an irreducible component of X.

There are two situations in which we need not worry about imbedded points, so we have agreement between the notions of associated points and generic points of irreducible components:

Theorem 2.3. *Let X be Noetherian. If X is reduced, or Cohen-Macaulay, then X has no imbedded points.*

Proof. First suppose that X is reduced and $x \in X$ is not a generic point of a component: then in $\mathscr{O}_{X,x}$ we have that the minimal primes $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are not equal to $\mathfrak{m}_x$, and hence have union distinct from $\mathfrak{m}_x$. If $f \in \mathfrak{m}_x \smallsetminus (\mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r)$, we claim that f is not a zero divisor in $\mathscr{O}_{X,x}$, so that x is not an associated point of X. Suppose that $fg = 0$ for some $g \in \mathscr{O}_{X,x}$. We then have in particular that $g \in \mathfrak{p}_i$ for each i, so g is in every minimal prime, and must be nilpotent, so we conclude $g = 0$, and f is not a zero divisor, as desired.

Next suppose that X is Cohen-Macaulay. This case is essentially Theorem 17.3 (i) of [2], which says that for every $x \in X$, the local ring $\mathscr{O}_{X,x}$ has no non-minimal associated primes. Thus if x is an associated point of X, we know there is a section $s \in \mathscr{O}_X(U)$ for some neighborhood of x whose support is precisely $\bar{x}$, and hence the annihilator of s in $\mathscr{O}_{X,x}$ is the maximal ideal. Thus the maximal ideal is an associated prime of $\mathscr{O}_{X,x}$, which is a contradiction unless it is also minimal, i.e. unless x is the generic point of a component. □

Corollary 2.4. *If X is a Noetherian local complete intersection, it has no imbedded points.*

Proof. Indeed, a local complete intersection is Cohen-Macaulay by Theorem 21.3 of [2]. □

3. Multiplicities of components

We now come to the definition of the multiplicity of a coherent sheaf along a component of its support. We first recall the notion of length:

Definition 3.1. Let M be an A-module. The **length** of M is the length of any filtration $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = (0)$ with each M_i/M_{i+1} isomorphism to $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ (which is allowed to depend on i).

Remark 3.2. Some general comments on length:

If there is such a filtration, its length is independent of the filtration. Moreover, the existence of such a filtration is equivalent to M being both Noetherian and Artinian.

We note that if A is a field, then the length of M is simply its dimension over A.

We also note that the length of M over A has little to do with A: for instance, the length of A over itself is rarely equal to 1. The length of M over A is the same as the length of M over $A/\operatorname{Ann}(M)$. If A is local, we may think of the length roughly as the "dimension" of M over $A/\mathfrak{m}$.

However, this is only roughly true, because M need not be a vector space over $A/\mathfrak{m}$, if $A/\mathfrak{m}$ is not contained in A: for instance, the length of $\mathbb{Z}/p^m\mathbb{Z}$ over itself is equal to m.

We leave the following as a simple exercise for the reader.

Lemma 3.3. *Length is additive in short exact sequences. In particular, if M be an A-module of finite length, and $m \in M$ non-zero, then any non-trivial quotient of M has length strictly less than the length of M.*

Definition 3.4. Let X be a Noetherian scheme, and $\mathscr{F}$ a coherent sheaf. Suppose $x \in X$ is the generic point of a component of $\operatorname{Supp} \mathscr{F}$. We define $\mu_x(\mathscr{F})$, the **multiplicity** of $\mathscr{F}$ at x (or along $\bar{x}$) to be the length of $\mathscr{F}_x$ over $\mathscr{O}_{X,x}$. If $\mathscr{F} = \mathscr{O}_X$, we write $\mu_x := \mu_x(\mathscr{O}_X)$, and call this simply the multiplicity of X at x.

Note that this will always be finite by the argument for Lemma 1.3. Note also that this is only defined for generic points of topological components: the length will not be finite at an associated point.

Conceptually, we will focus our attention on the case that $\mathscr{F} = \mathscr{O}_X$.

Example 3.5. Let X be 0-dimensional over a field k, and $x \in X$. Then $\mu_x = \dim_{k(x)} \mathscr{O}_{X,x}$.

Example 3.6. Let X be a regular Noetherian scheme, and D an effective divisor. Write $D = \sum_i c_i P_i$ for prime divisors P_i. Then if x_i is the generic point of P_i, we have $\mu_{x_i}(\mathscr{O}_D) = c_i$. Indeed, at x_i, we have P_i cut out by some $f \in \mathscr{O}_{X,x}$, which necessarily generates a prime ideal, and in fact maximal ideal. Then D is cut out at x_i by f^{c_i}, so $\mathscr{O}_{D,x_i} = \mathscr{O}_{X,x_i}/(f^{c_i})$, and the filtration $\mathscr{O}_{D,x_i} \supsetneq (f) \supsetneq \cdots \supsetneq (f^{c_i} - 1) \supsetneq (f^{c_i}) = (0)$ gives that the length is c_i.

We next give a basic result on existence of certain short exact sequences:

Proposition 3.7. *Let X be a Noetherian scheme, and $\mathscr{F}$ a coherent sheaf.*

Let Z be the union of the closures of the embedded points of $\mathscr{F}$. Then there is an exact sequence

$$0 \to \mathscr{F}_Z \to \mathscr{F} \to \mathscr{G} \to 0,$$

where $\mathscr{F}_Z$ is the subsheaf of $\mathscr{F}$ with supports in Z, and $\mathscr{G}$ has the same support as $\mathscr{F}$, the same multiplicity along every component of their support, and no imbedded points.

Next, let $x \in X$ be a generic point of a component of $\operatorname{Supp} \mathscr{F}$*. Then there is an exact sequence*

$$0 \to \mathscr{R} \to \mathscr{F} \to \mathscr{G} \to 0,$$

where $\operatorname{Supp}(\mathscr{G}) = \bar{x}$*, with $\mu_x(\mathscr{G}) = 1$ and $\mathscr{G}$ having no imbedded points; then we also have $\mu_x(\mathscr{R}) = \mu_x(\mathscr{F}) - 1$ and $\mu_{x'}(\mathscr{R}) = \mu_{x'}(\mathscr{F})$ for the generic point x' of any other component of* $\operatorname{Supp}(\mathscr{F})$*.*

Proof. For the first sequence, let $\mathscr{G}$ be the cokernel of $\mathscr{F}_Z \hookrightarrow \mathscr{F}$. It's then clear that $\operatorname{Supp}(\mathscr{F}) = \operatorname{Supp}(\mathscr{G})$ with the same multiplicities along components, since $\mathscr{F} \to \mathscr{G}$ is an isomorphism away from Z. Moreover, from looking at stalks it is clear that if we have $s \in \mathscr{F}(U)$ for some U, with image $t \in \mathscr{G}(U)$, then $\operatorname{Supp}(s) \supseteq \operatorname{Supp}(t)$, and the supports must agree on any component of $\operatorname{Supp}(\mathscr{F})$. Thus we see that if t has support not containing any component of $\operatorname{Supp}(\mathscr{F})$, then so must s, and in particular, $\mathrm{Supp}(s) \subseteq Z$, so we conclude $t = 0$. Thus we see that $\mathscr{G}$ has no imbedded points, as desired.

For the second sequence, let $\mathscr{R}$ be the subsheaf of $\mathscr{F}$ generated by sections of $\mathscr{F}$ whose image in $\mathscr{F}_x$ is contained inside $\mathfrak{m}_x\mathscr{F}_x$, and let $\mathscr{G}$ be the cokernel. Then away from $\bar{x}$, $\mathscr{R} = \mathscr{F}$, so $\mathrm{Supp}(\mathscr{G}) \subseteq \bar{x}$. On the other hand, $\mathscr{G}_x \cong \mathscr{O}_{X,x}/\mathfrak{m}_x$, so $\mu_x(\mathscr{G}) = 1$, and $\mathrm{Supp}(\mathscr{G}) = \bar{x}$. As with the first sequence, suppose we have $s \in \mathscr{F}(U)$ for some U, with image $t \in \mathscr{G}(U)$, and suppose that $\mathrm{Supp}(t)$ is strictly contained in $\bar{x}$. Then $t = 0$ in $\mathscr{G}_x$, so $s \in \mathfrak{m}_x\mathscr{F}_x$, and $t = 0$, showing that $\mathscr{G}$ has no imbedded points, as desired. □

We conclude this discussion by demonstrating the usefulness of not having imbedded points when trying to check equality of a closed subscheme, or reducedness.

Proposition 3.8. *Suppose that $X \subseteq Y$ is a closed subscheme, and Y is without imbedded points. Then $X = Y$ if and only if X and Y have the same multiplicity along every component of Y.*

Proof. We claim that if X and Y have the same multiplicity along every component of Y, then the map $\mathscr{O}_Y \to \mathscr{O}_X$ must be injective, hence an isomorphism. Indeed, let $\mathscr{R}$ be the kernel, and suppose for some U we have a non-zero $s \in \mathscr{R}(U)$. Then since Y has no imbedded points, we have that $\mathrm{Supp}(s)$ must contain a component of Y, say with generic point y. It suffices to show that this implies that $\mu_y(\mathscr{O}_Y) > \mu_y(\mathscr{O}_X)$, but this is clear from the earlier lemma, since s will be non-zero in $\mathscr{O}_{Y,y}$. □

Proposition 3.9. *A Noetherian scheme X is reduced if and only if it has multiplicity 1 along each component, and no imbedded points.*

Proof. Supposed X is reduced. Since $\mathscr{O}_{X,x}$ is Artinian for any x a generic point of a component of X, the only way that $\mathscr{O}_{X,x}$ can be without nilpotents is if it is a field, which implies the multiplicity of X at x is 1. We already saw in Theorem 2.3 that if X is reduced it has no imbedded points.

Conversely, suppose that X has multiplicity 1 along each component and no imbedded points, and suppose we have $s \in \mathscr{O}_X(U)$ for some U. Since X has no imbedded points, the support of S must contain some component of X, say with generic point $x \in U$, so we have that s is non-zero in $\mathscr{O}_{X,x}$. But since $\mathscr{O}_{X,x}$ has length 1 over itself, it must be isomorphic to $\mathscr{O}_{X,x}/\mathfrak{m}_x$, and we conclude that s cannot be nilpotent in $\mathscr{O}_{X,x}$, hence not in U. Thus X is reduced. □

When we discuss Hilbert polynomials and degrees, we will gives a stronger version of the first proposition in the case of projective varieties, showing that it is enough to compare degrees.

4. Existence of good hyperplanes

Corollary 4.1. *Let k be an infinite field, and $\mathscr{F}$ a coherent sheaf on $\mathbb{P}^n_k$. Then there exists a hyperplane $H = V(F)$ for some linear form $F \in \Gamma(\mathbb{P}^n_k, \mathscr{O}(1))$ such that the map $\mathscr{F}(-1) \stackrel{\times F}{\to} \mathscr{F}$ is injective, with cokernel having support of dimension precisely one smaller than $\mathscr{F}$.*

Proof. First note that the cokernel will be supported precisely on $H \cap \mathrm{Supp}\,\mathscr{F}$, since this is the locus on which $\mathscr{F}_x \neq 0$ and the image of multiplication by F is contained in $\mathfrak{m}_x\mathscr{F}_x$. As long as H does not contain any component of $\mathrm{Supp}\,\mathscr{F}$, we then have that the cokernel is supported in dimension one less than $\mathscr{F}$, and in particular nonempty as long as the support of F is positive-dimensional. Next, by Corollary 1.6, we have the desired injectivity condition if and only if H does not contain any associated point of $\mathscr{F}$. We have thus reduced our assertion to the following lemma. $\square$

Lemma 4.2. *Let k be an infinite field, and $Z_1, \dots, Z_r$ a finite collection of closed subsets of $\mathbb{P}^n_k$. Then there exists a hyperplane $H \subseteq \mathbb{P}^n_k$ not containing any Z_i.*

Proof. By choosing a closed point on each Z_i, we immediately reduce to the case that the Z_i are are all closed points. For each Z_i, the map $\Gamma(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(1)) \to \mathscr{O}_{\mathbb{P}^n_k}(1)|_{Z_i}$ is a k-linear map (even if $\kappa(Z_i)$ strictly contains k), and its kernel is a proper subspace, since we know there is at least one linear form (for instance, an appropriate coordinate form) not vanishing at Z_i. Since k is infinite, the union of these kernels cannot be all of $\Gamma(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(1))$, so we see there exists a linear form not vanishing at any of the Z_i, as desired. $\square$

Remark 4.3. The corollary implies that one can modify the argument of Exercise III.5.2 of [1] to show that the Hilbert polynomial of $\mathscr{F}$ has degree equal to $\dim \operatorname{Supp} \mathscr{F}$. Since the lemma and hence the corollary might fail if k were finite, one must first check that the Hilbert polynomial is invariant under extension of base field, so that it suffices to work over an infinite field.

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
2. Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.