CHEAT SHEET: ADDITIONAL RESULTS

BRIAN OSSERMAN

The purpose of this cheat sheet is to provide an easy reference for additional results not fitting neatly into properties of schemes, or properties of morphisms. Many of these results will involve deriving properties of schemes from properties of morphisms. The organization is more or less according to Hartshorne.

1. §2.3

Proposition 1.1. *Let $f : X \to Y$ be a morphism of finite type, and Y a Noetherian scheme. Then X is Noetherian.*

This is Exercise II.3.13 (g) of [1].

Definition 1.2. A subset of a topological space is **locally closed** if it is the intersection of an open set and a closed set. A subset is **constructible** if it is a finite union of locally closed subsets.

Theorem 1.3. *(Chevalley) Let $f : X \to Y$ be a morphism of finite type, with X and Y Noetherian schemes. Then $f(X)$ is a constructible subset of* $\mathrm{sp}(Y)$*.*

This is Exercise II.3.19 of [1].

Remark 1.4. More generally, one has the notion of a subset being **locally constructible**, which means that every point has a neighborhood inside the subset which is constructible. A Noetherian topological space is quasi-compact, so every locally constructible subset is constructible. One also has a notion of a morphism being **of finite presentation**, which is the same as being of finite type, except that the algebras are required to be not only of finite type, but also of finite presentation. A morphism of Noetherian schemes which is of finite type is automatically of finite presentation.

Then one can state Chevalley's theorem as saying that if $f : X \to Y$ is of finite presentation, and $Z \subseteq X$ is any locally constructible subset, then $f(Z)$ is locally constructible in $\mathrm{sp}(Y)$. In particular, for $Z = X$, we see that $f(X)$ is locally constructible. See Theorem 1.8.4 of [3].

2. §2.4

Theorem 2.1. *(Chow's lemma) let $f : X \to S$ be proper, with S Noetherian. Then there exists a scheme X' and a morphism $g : X' \to X$ such that X is projective over S, and for some dense open subset $U \subset X$, we have that g induces an isomorphism $g^{-1}(U) \overset{\sim}{\to} Ui$.*

This is Exercise II.4.10 of [1].

3. Miscellaneous

We begin with a criterion for flatness in terms of fiber dimension. For a scheme X and point $x \in X$, write $\dim_x X$ to mean $\dim \mathcal{O}_{X,x}$ (note that this differs from EGA notation).

Theorem 3.1. *Let $f : X \to Y$ be a morphism locally of finite type of locally Noetherian schemes, and suppose further that Y is regular, and X is Cohen-Macaulay. Given a point $x \in X$, set $y = f(x)$, and suppose that*

$$\dim_x X = \dim_y Y + \dim_x f^{-1}(y).$$

Then f is flat at x.

See Theorem 23.1 of [2].

We next turn to the relation between the sheaf of relative differentials, and smoothness and unramifiedness.

Theorem 3.2. *Given $f : X \to \operatorname{Spec} k$ of finite type, then f is smooth if and only if $\Omega^1_{X/k}$ is locally free of rank equal to $\dim X$.*

If $\operatorname{char} k = 0$, then f is smooth if and only if $\Omega^1_{X/k}$ is locally free.

Theorem 3.3. *A morphism $f : X \to Y$ is unramified if and only if $\Omega^1_{X/Y} = 0$.*

Theorem 3.4. *(Jacobian criterion) Let $X \subseteq \mathbb{A}^n_k$ be affine and of finite type over k, and the ideal of X is generated by polynomials $f_1, \ldots, f_t$, and X has pure dimension r. Then the points at which X is not smooth over k are precisely the points of X at which the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$ has rank strictly less than $n - r$.*

Note that it follows that the non-smooth locus is cut out explicitly by the $(n - r) \times (n - r)$ minors of the matrix, and is therefore a Zariski-closed subset of X.

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
2. Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
3. Alexander Grothendieck with Jean Dieudonné, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, premiére partie*, vol. 20, Publications mathématiques de l'I.H.É.S., no. 2, Institut des Hautes Études Scientifiques, 1964.