CHEAT SHEET: COHOMOLOGY OF SHEAVES

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Cohomology of sheaves was introduced to algebraic geometry by Serre, in the form of Cech cohomology. While topological cohomology theories may be thought of as primarily intended to provide invariants of spaces, the cohomology of sheaves in algebraic geometry is used more often as a computational tool. It is useful for computing the spaces of global sections of sheaves, but also is sometimes used directly to describe objects of interest, as in deformation theory, and Picard groups.

Note that following current convention, we often use "line bundle" to refer to an "invertible sheaf".

1. Sheaf cohomology: the basics

We do not include here the definitions of sheaf cohomology, but rather collect its properties as follows:

Theorem 1.1. *Given a topological space X, there is a collection of functors H^i indexed by $i \geq 0$, from the category of sheaves of abelian groups on X to the category of abelian groups. Notation is $\mathcal{F} \mapsto H^i(X, \mathcal{F})$. These functors satisfy the following properties:*

(i) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

(ii) *Given a short exact sequence $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0$, there exist maps $\delta^i : H^i(X, \mathcal{H}) \to H^{i+1}(X, \mathcal{G})$ giving a long exact sequence*

$$\dots H^i(X, \mathcal{G}) \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{H}) \xrightarrow{\delta^i} H^{i+1}(X, \mathcal{G}) \to H^{i+1}(X, \mathcal{F}) \dots .$$

(iii) *For any $\mathcal{F}, \mathcal{G}$ the natural map $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}(H^i(X, \mathcal{F}), H^i(X, \mathcal{G}))$ is additive.*

(iv) *The δ^i maps are functorial: i.e., given a morphism of short exact sequences, the induced maps on the H^i and H^{i+1} commute with the δ^i maps.*

This is Theorem III.1.1A of [1], applied to the Definition of §III.2.

The following two vanishing theorems play a key role in the cohomology of sheaves on schemes.

Theorem 1.2. *Let X be a Noetherian topological space of dimension n. Then for any $i > n$, and any $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.*

See Theorem III.2.7 of [1].

Theorem 1.3. *Let X be an affine scheme. Then for any $i > 0$, and any $\mathcal{F}$ a quasicoherent $\mathcal{O}_X$-module, $H^i(X, \mathcal{F}) = 0$.*

See Theorem III.3.5 of [1] for the Noetherian case, and Theorem 1.3.1 of [2] for the general case.

Remark 1.4. Note that when $\mathcal{F}$ is an $\mathcal{O}_X$-module, we can still define its cohomology by considering only the underlying structure of a sheaf of abelian groups. While Theorem 1.2 applies to any sheaf of abelian groups, Theorem 1.3 is special to the case of quasicoherent sheaves (indeed, we will see that if it held for arbitrary sheaves of abelian groups, every ring of integers would be a PID!).

In fact, it is a theorem of Serre that Theorem 1.3 has a strong converse:

Theorem 1.5. *Let X be a Noetherian scheme, and suppose that $H^1(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on X. Then X is affine.*

See Theorem III.3.7 of [1].

We conclude with several important statements on dimensions of cohomology groups. The following is a special case of a much more general theorem.

Theorem 1.6. *Let X be a proper scheme of finite type over* $\operatorname{Spec} k$ *for some field k. Then $H^i(X, \mathcal{F})$ is a finite-dimensional k-vector space for any coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

See Theorem III.5.2(a) of [1] for the case that X is projective, and Theorem 3.2.1 of [2] for the general statement (after appropriate definition-chasing).

Definition 1.7. If X is a proper scheme of finite type over $\operatorname{Spec} k$, and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, we write $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$.

The following result is a generalization of the classical fact that on a projective variety, the only global regular functions are the constants:

Theorem 1.8. *Let X be a proper, geometrically integral scheme of finite type over a field k. Then $H^0(X, \mathcal{O}_X) = k$.*

See Exercise II.4.5 of [1] for the case that $k = \bar{k}$; the general case follows from the following (which is itself a special case of a much more general statement):

Proposition 1.9. *Let X be separated and of finite type over* $\operatorname{Spec} k$*, and $\mathcal{F}$ a quasicoherent $\mathcal{O}_X$-module. Suppose that k' is any field containing k. Write $X' := X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ and $\mathcal{F}' = \mathcal{F} \otimes_k k'$. Then for all $i \geq 0$, we have:*

$$h^i(X, \mathcal{F}) = h^i(X', \mathcal{F}').$$

This is a special case of Proposition III.9.3 of [1].

2. Cech cohomology and comparison

While the derived functor cohomology introduced by Grothendieck has a number of good theoretical properties, it is typically completely impossible to compute directly. In contrast, Cech cohomology is often both computable and equivalent.

Given a cover $\{U_j\}$ of X, and a sheaf $\mathcal{F}$, we define a complex of **Cech cochains** by taking the ith term of the complex to be $\prod_{j_0 < \cdots < j_i} \mathcal{F}(U_{j_0,\dots,j_i})$, where $U_{j_0,\dots,j_i}$ denotes $U_{j_0} \cap \cdots \cap U_{j_i}$. We define the complex maps in terms of alternating sums; the details are given at the beginning of §III.4 of [1]. We then define the **Cech i-cocycles** to be the kernel of the complex map at the ith place, and **Cech i-coboundaries** to be the image of the complex map. We then define $\check{H}^i(\{U_j\}, \mathcal{F})$ to be the ith cohomology group of the complex, which is to say the i-cocycles modulo the i-coboundaries.

Next, if $\{U'_j\}$ is a refinement of $\{U_j\}$, we see that we have a natural map $\check{H}^i(\{U_j\},\mathcal{F}) \to \check{H}^i(\{U'_j\},\mathcal{F})$ induced by restriction (see Exercise III.4.4 of Hartshorne). We then define

$$\check{H}^i(X,\mathcal{F}) := \varinjlim_{\{U_j\}} \check{H}^i(\{U_j\},\mathcal{F}).$$

The basic theorem is the following:

Theorem 2.1. *Suppose that $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module, with X a separated scheme. Then $H^i(X,\mathcal{F}) = \check{H}^i(X,\mathcal{F})$. In fact, if $\{U_i\}$ is any affine open cover of X, then we have*

$$H^i(X,\mathcal{F}) = \check{H}^i(\{U_j\},\mathcal{F}) = \check{H}^i(X,\mathcal{F}).$$

The first equality is Theorem III.4.5 of [1] in the case that X is Noetherian, and Proposition 1.4.1 of [2] for the general case. The second equality follows formally.

Although the above theorem fails if $\mathcal{F}$ is an arbitrary sheaf of abelian groups, we do have the following:

Proposition 2.2. *Let X be any topological space, and $\mathcal{F}$ a sheaf of abelian groups on X. Then $H^i(X,\mathcal{F}) = \check{H}^i(X,\mathcal{F})$ for $i = 0,1$.*

See Lemma III.4.1 and Exercise III.4.4 of [1]. In fact, it is also true that there is a natural injection $\check{H}^2(X,\mathcal{F}) \hookrightarrow H^2(X,\mathcal{F})$, but this is slightly harder to see.

3. Curves: Riemann-Roch and first applications

The single most important theorem in cohomology of curves is undoubtedly the following.

Theorem 3.1. *(Riemann-Roch) Let C be a smooth, proper curve over $\operatorname{Spec} k$, and $\mathcal{L}$ a line bundle on C. Let $g := h^0(C,\Omega^1_{C/k})$ be the genus of C, and $d := \deg \mathcal{L}$. Then we have*

$$h^0(C,\mathcal{L}) - h^0(C,\mathcal{L}^\vee \otimes \Omega^1_{C/k}) = d + 1 - g.$$

More sharply, we have

$$h^0(C,\mathcal{L}) - h^1(C,\mathcal{L}) = d + 1 - g,$$

and a natural isomorphism

$$H^1(C,\mathcal{L}) \xrightarrow{\sim} H^0(C,\mathcal{L}^\vee \otimes \Omega^1_{C/k})^\vee.$$

The last isomorphism is known as **Serre duality**. Note that because every line bundle is isomorphic to $\mathcal{O}_C(D)$ for some Weil divisor D, the theorem may be restated equivalently in terms of a relation between the dimensions of spaces of functions and differential forms with prescribed vanishing and/or allowed poles along a given divisor.

See Theorem IV.1.3 of [1] (rephrased in slightly more classical language) for the case that $k = \bar{k}$. The general case follows by several applications of Proposition 1.9, which we use in particular to conclude:

Lemma 3.2. *The genus of a curve is invariant under extension of the base field.*

Our first application of the theorem is to compute the degree of $\Omega^1_{C/k}$:

Corollary 3.3. *In the situation of the theorem, we have $\deg \Omega^1_{C/k} = 2g - 2$.*

Proof. Indeed, if we set $\mathcal{L} = \Omega^1_{C/k}$, applying the theorem we find

$$d + 1 - g = h^0(C, \Omega^1_{C/k}) - h^0(C, \mathcal{O}_C) = g - 1,$$

giving the desired identity. □

We need the following easy lemma:

Lemma 3.4. *Let $\mathcal{L}$ have negative degree on a smooth proper curve C over* $\operatorname{Spec} k$*. Then $h^0(C, \mathcal{L}) = 0$.*

Proof. Suppose not. Then we have a non-zero section $\sigma \in H^0(C, \mathcal{L})$, whose associated divisor of zeroes $D(\sigma)$ is an effective Weil divisor, and hence has non-negative degree. But $\mathcal{L} \cong \mathcal{O}_C(D(\sigma))$, so $\mathcal{L}$ must also have non-negative degree. □

Combining the lemma and the theorem, we can immediately conclude:

Corollary 3.5. *Let $\mathcal{L}$ be a line bundle with* $\deg \mathcal{L} > 2g - 2$*. Then* $h^0(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$*.*

We give two applications to deformation theory:

Corollary 3.6. *The space of first-order deformations of a curve of genus $g \geq 2$ has dimension $3g - 3$.*

The space of first-order deformations of $\mathcal{O}_C$ on a curve C of genus g has dimension g.

Proof. The first assertion is an immediate consequence of Serre duality and the two previous corollaries, since we had shown earlier that the deformation space is described by

$$H^1(C, T_C) = H^1(C, (\Omega^1_{C/k})^\vee) \xrightarrow{\sim} H^0(C, (\Omega^1_{C/k})^{\otimes 2}).$$

The second assertion is a consequence of Serre duality, since we had described the deformation space as

$$H^1(C, \mathcal{E}nd(\mathcal{O}_C)) = H^1(C, \mathcal{O}_C) \xrightarrow{\sim} H^0(C, \Omega^1_{X/k}).$$

□

Remark 3.7. Note that in each part of the corollary, if we had C affine, both deformation spaces would be 0, but we see that without the affineness, we obtain non-trivial first-order deformations in both cases. We also note that Theorem 1.3 implies that there are no non-trivial first-order deformations of a vector bundle $\mathcal{E}$ on an affine scheme, whether or not $\mathcal{E}$ is trivial.

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
2. Alexander Grothendieck with Jean Dieudonné, *Éléments de géométrie algébrique: III. Étude cohomologique des faisceaux cohérents, premiére partie*, vol. 11, Publications mathématiques de l'I.H.É.S., no. 2, Institut des Hautes Études Scientifiques, 1961.