CHEAT SHEET: PROPERTIES OF MORPHISMS OF SCHEMES

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The purpose of this cheat sheet is to provide an easy reference for definitions of various properties of morphisms of schemes, and basic results about them. The organization is more or less according to Hartshorne.

1. §2.3, definitions

Definition 1.1. A morphism $f : X \to Y$ is **quasi-compact** if there exists a covering of Y by open affine subschemes $V_i = \operatorname{Spec} B_i$, such that each $f^{-1}(V_i)$ is quasi-compact.

Definition 1.2. A morphism $f : X \to Y$ is **locally of finite type** if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, we have an open affine cover $\{U_{i,j} = \operatorname{Spec} A_{i,j}\}$ of $f^{-1}(V_i)$, with each $A_{i,j}$ finitely generated as a B_i-algebra.

f is of **finite type** if further for each i, the cover $\{U_{i,j}\}$ can be chosen to be finite.

Definition 1.3. A morphism $f : X \to Y$ is **finite** if there exists a covering of Y by open affine subschemes $V_i = \operatorname{Spec} B_i$, such that each $f^{-1}(V_i)$ is affine, say, $\operatorname{Spec} A_i$, and each A_i is finitely generated as a B_i-module.

Definition 1.4. A morphism $f : X \to Y$ is **quasi-finite** if it is of finite type, and if for each $y \in Y$, the fiber $f^{-1}(y)$ consists of a finite set of points.

Remark 1.5. Note that this differs from the definition given in [2] in that we require a quasi-finite morphism to be of finite type (equivalently, locally of finite type). Without this hypothesis, one would have that $\operatorname{Spec} k[[t]] \to \operatorname{Spec} k$ is quasi-finite (i.e., the fibers need not consist of a discrete set), and also, that although $\operatorname{Spec} \bar{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ would be quasi-finite, $\operatorname{Spec} \bar{\mathbb{Q}} \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} \bar{\mathbb{Q}} \to \operatorname{Spec} \bar{\mathbb{Q}}$ is not (i.e., being quasi-finite is not preserved under base change).

Definition 1.6. A morphism $f : X \to Y$ is an **open immersion** if it induces an isomorphism of X onto an open subscheme of Y.

Definition 1.7. A **closed subscheme** $Z \subseteq Y$ is a scheme whose underlying topological space $\operatorname{sp}(Z)$ is a closed subset of $\operatorname{sp}(Y)$, and such that the sheaf of rings $\mathcal{O}_Z$ is obtained as a quotient of $\iota^{-1}\mathcal{O}_Y$, where $\iota : \operatorname{sp}(Z) \to \operatorname{sp}(Y)$ is the inclusion map.

Definition 1.8. A morphism $f : X \to Y$ is a **closed immersion** if f induces an isomorphism onto a closed subscheme of Y.

Remark 1.9. Note that this definitional convention is different from, but equivalent to, the one in [2].

Definition 1.10. A morphism $f : X \to Y$ is an **immersion** if f can be written as a composition of a closed immersion followed by an open immersion.

2. §2.4, definitions

Definition 2.1. A morphism $f : X \to Y$ is **separated** if the diagonal morphism $X \to X \times_Y X$ is closed.

Definition 2.2. A morphism $f : X \to Y$ is **universally closed** if for all $Z \to Y$, the morphism $X \times_Y Z \to Z$ is closed.

Definition 2.3. A morphism $f : X \to Y$ is **proper** if it is of finite type, separated, and universally closed.

Definition 2.4. A morphism $f : X \to Y$ is **projective** if it factors as a closed immersion into $\mathbb{P}^n_Y$, followed by the projection map $\mathbb{P}^n_Y \to Y$, for some n. f is **quasi-projective** if it factors as an open immersion followed by a projective morphism.

3. Miscellaneous definitions

Definition 3.1. A morphism $f : X \to Y$ is **flat** at a point $x \in X$ if $\mathcal{O}_{X,x}$ is flat when viewed as an $\mathcal{O}_{Y,f(x)}$-module via $f^\sharp$. f is **flat** if it is flat at every $x \in X$.

Definition 3.2. A morphism $f : X \to Y$ is **formally smooth** (respectively, **unramified**, respectively **étale**) if for every commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A' & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

with $\operatorname{Spec} A' \to \operatorname{Spec} A$ a closed immersion induced by a map $A \twoheadrightarrow A'$ with kernel I such that $I^2 = 0$, the dotted arrow exists (respectively, is unique, respectively exists and is unique).

Definition 3.3. A morphism is **smooth** (respectively, **unramified**, respectively **étale**) if it is locally of finite presentation and formally smooth (respectively, unramified, respectively étale).

Definition 3.4. A morphism $f : X \to Y$ is **smooth** (respectively, **unramified**, respectively **étale**) at a point $x \in X$ if there exists an open neighborhood U of x such that $f|_U$ is smooth (respectively, unramified, respectively étale).

4. §2.3, results

The following is clear from the definition, together with the fact that an affine scheme is quasi-compact.

Lemma 4.1. *Let $f : X \to Y$ be a morphism. Then f is of finite type if and only if it is quasi-compact and locally of finite type.*

The following is frequently useful, as it says that checking various properties on an open affine cover is enough to imply that they hold for every affine open:

Proposition 4.2. *Let $f : X \to Y$ be a morphism of schemes, and let P be one of the following properties of a morphism of schemes: locally of finite type, quasi-compact, of finite type, or finite. Then P was defined in terms of the existence of an affine open cover $\{V_i\}$ of Y, such that each $f^{-1}(V_i)$ had a certain property P'. In fact, f has property P if and only if for every open affine subscheme V of Y, the preimage $f^{-1}(V)$ has property P'.*

This is exercises II.3.1, II.3.2, II.3.3 (a) and (b), and II.3.4 of [2].

Proposition 4.3. *Let $f : X \to Y$ be a morphism of schemes. Then f is of finite type if and only if for every open affine $V = \operatorname{Spec} B$ of Y, the preimage $f^{-1}(V)$ is quasi-compact, and for every $U \subseteq f^{-1}(V)$ open and affine, say $U = \operatorname{Spec} A$, we have A a finitely generated B-algebra.*

This is exercise II.3.3 (c) of [2].

Proposition 4.4. *A finite morphism is quasi-finite.*

This is exercise II.3.5 of [2].

Proposition 4.5. *Some properties of morphisms of finite type:*

(i) *a closed immersion is of finite type;*
(ii) *a composition of two morphisms of finite type is of finite type;*
(iii) *a base change of a morphism of finite type is of finite type;*
(iv) *if $f : X \to Y$ and $g : Y \to Z$ are morphisms, with f quasi-compact and $g \circ f$ of finite type, then f is of finite type;*
(v) *a quasi-compact open immersion is of finite type;*

This is Exercise II.3.13 of [2].

Remark 4.6. After defining separatedness, we will give a general statement which implies that that parts (i)-(iii) of the above proposition imply also that a product of morphisms of finite type is of finite type, and that $f : X \to Y$ is of finite type if and only if $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ is of finite type.

5. §2.4, RESULTS

Proposition 5.1. *If $f : X \to Y$ is a morphism of affine schemes, then f is separated, and in fact the diagonal morphism is a closed immersion.*

This is Proposition II.4.1 of [2].

Corollary 5.2. *$f : X \to Y$ is separated if and only if the diagonal morphism is a closed immersion.*

This is Corollary II.4.2 of [2].

Theorem 5.3. *Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if the diagonal map is quasi-compact, and furthermore for any valuation ring R, with field of fractions K, any two morphisms $\operatorname{Spec} R \to X$ which agree on $\operatorname{Spec} K$, and agree after composition with f, must be the same.*

Note that the hypothesis on the diagonal is satisfied if, for instance, X is Noetherian. This result follows from the argument for Theorem II.4.3 of [2], but see also Proposition 7.2.3 of [4].

Theorem 5.4. *Let $f : X \to Y$ be a morphism of schemes. Then f is proper if and only if f is of finite type, the diagonal map is quasi-compact, and for any valuation ring R, with field of fractions K, and any commutative diagram containing the solid arrows of*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array},$$

there exists a unique way to fill in the dotted arrow so that the entire diagram commutes.

If Y is locally Noetherian, we may restrict the above R to discrete valuation rings, and do not need to assume that the diagonal map is quasi-compact.

Note that if Y is locally Noetherian, the condition on the diagonal follows automatically from the finite type hypothesis. Once again, this follows from the argument for Theorem II.4.7 of [2]; see also Theorem 7.3.8 of [4] for a statement from which one can conclude this version directly.

Theorem 5.5. *A projective morphism is proper. A quasi-projective morphism of locally Noetherian schemes is separated and of finite type.*

Proposition 5.6. *Suppose that P is a property of morphisms such that:*

(i) *A closed immersion has P;*
(ii) *P is closed under composition;*
(iii) *P is stable under base change.*

Then it follows that

(iv) *A product of morphisms having P has P;*
(v) *if $f : X \to Y$ and $G : Y \to Z$ are morphisms such that $g \circ f$ has P and g is separated, then f has P;*
(vi) *if $f : X \to Y$ has P, then $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ has P.*

Furthermore, if instead of (i), it is the case that every immersion has P, then the separatedness condition on g is unnecessary in (v).

The first part is Exercise II.4.8 of [2]; the second part follows from the same argument.

Proposition 5.7. *The following properties of morphisms satisfy (i)-(iii) above: closed immersions, quasi-compact, locally of finite type, (hence finite type), finite, quasi-finite, separated, universally closed (hence proper), and projective.*

Furthermore, every immersion satisfies (ii) and (iii), and also has the properties locally of finite type, and separated.

Finally, locally of finite presentation satisfies (ii)-(vi) (without any separatedness hypothesis in (v)). Hence, of finite presentation satisfies (ii)-(vi).

Note that applying this to locally of finite type and using that finite type is equivalent to locally of finite type and quasi-compact, we recover Proposition 4.5 above. Hartshorne includes finite type in Exercise II.3.13 of [2], and projective in Exercise II.4.9 of [2]. He also gives separated and proper in Corollary II.4.6 and Corollary II.4.8 of [2], but under unnecessary Noetherian hypotheses; see Proposition 5.5.1 of [3] and Proposition 5.4.2 of [4] for the general statements. For the others, see Proposition 4.2.5 and Corollary 4.3.2 of [3] (immersions), Proposition 6.6.1 of [3] (quasi-compact), Proposition 6.6.6 of [3] (locally of finite type), Remark 5.4.9 of [4] (universally closed), Proposition 6.1.5 of [4] (finite), Proposition 6.2.4 of [4] (quasi-finite), and Proposition 1.4.3 of [5] (locally of finite presentation). Note that even a closed immersion is not necessarily locally of finite presentation.

Proposition 5.8. *A finite morphism is proper.*

This is Exercise II.4.1 of [2].

6. Miscellaneous results

Theorem 6.1. *If f is flat and locally of finite presentation, then f is open (hence universally open).*

The case that Y is Noetherian is Exercise III.9.1 of [2]. The general statement is Theorem 2.4.6 of [6].

Theorem 6.2. *Suppose that $f : X \to Y$ is locally of finite type, with Y locally Noetherian. Suppose further that Y is reduced, and the fibers of f are geometrically reduced. Then if f is universally open, it is flat.*

See Corollary 15.2.3 of [7].

Proposition 6.3. *Given $f : X \to Y$ with Y integral, and regular of dimension 1, we have that f is flat if and only if every associated point of X maps to the generic point of Y. In particular, if X is reduced, f is flat if and only if every component of X maps dominantly to Y.*

This is Proposition III.9.7 of [2].

Theorem 6.4. *Let $f : X \to Y$ be locally of finite presentation, with Y reduced and locally Noetherian. Then f is flat if and only if for every morphism $\operatorname{Spec} A \to Y$, with A a DVR, the base change of f is flat.*

This is Theorem 11.8.1 of [7].

Theorem 6.5. *Let X and Y be locally of finite type over an algebraically closed field k, and assume that X is non-singular. Suppose we have a morphism $f : X \to Y$. Then f is smooth (respectively, étale) if and only if the induced maps on tangent spaces of closed points are all surjective (respectively, bijective).*

This is Theorem 17.11.1 and Corollary 17.11.2 of [8].

Proposition 6.6. *Let X and Y be locally of finite type over an algebraically closed field k, and suppose we are given a morphism $f : X \to Y$. Then f is unramified if and only if the induced maps on tangent spaces of closed points are all injective.*

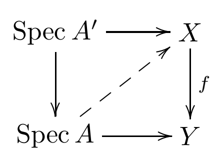
See Theorem 17.4.1 of [8], together with the exact sequence of Proposition 8.11 of [2].

Theorem 6.7. *A morphism f which is locally of finite presentation is smooth if and only if it is flat and has geometrically regular fibers.*

This is Theorem 17.5.1 of [8].

Corollary 6.8. *A variety X over an algebraically closed k is smooth if and only if X is regular.*

Theorem 6.9. *Let $f : X \to Y$ be locally of finite type, with Y locally Noetherian. Given $x \in X$, to check whether f is smooth (respectively, unramified, respectively étale) at x, it is enough to consider diagrams*



where we not only have $\operatorname{Spec} A' \to \operatorname{Spec} A$ *a closed immersion induced by a map* $A \twoheadrightarrow A'$ *with kernel* I *such that* $I^2 = 0$*, but we can further assume that* A' *and* A *are both local Artinian rings.*

If also $k(x)$ *is finite over* $k(f(x))$*, we can restrict to the case that* A' *and* A *are finite over* $\mathcal{O}_{Y,f(x)}$*.*

See Proposition 17.14.2 of [8], as well as Remark 17.14.3.

The following result says that, although (for instance) a smooth variety is not Zariski-locally isomorphic to an open subset of affine space, it is at least Zariski-locally étale over affine space.

Theorem 6.10. *A morphism* $f : X \to Y$ *is smooth at* $x \in X$ *if and only if there exists a neighborhood* U *of* x*, and an étale map* $U \to \mathbb{A}^n_Y$ *for some* n*, such that* $f|_U$ *factors as the given map* $U \to \mathbb{A}^n_Y$ *followed by the projection* $\mathbb{A}^n_Y \to Y$*.*

See [1], Proposition 2.2.11.

Theorem 6.11. *Let* $f : X \to Y$ *be a smooth morphism, and suppose* Y *is regular. Then* X *is regular.*

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