CHEAT SHEET: PROPERTIES OF SCHEMES

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The purpose of this cheat sheet is to provide an easy reference for definitions of various properties of schemes, and basic results about them. The organization is more or less according to Hartshorne.

1. §2.3, DEFINITIONS

Definition 1.1. A scheme X is **connected** (respectively, **irreducible**, respectively **quasi-compact**) if its underlying topological space $\mathrm{sp}(X)$ is.

(Recall: a topological space is connected if it cannot be written as the union of two disjoint proper closed subsets. It is irreducible if it cannot be written as the union of two proper closed subsets. It is quasi-compact if every open cover has a finite subcover.)

Definition 1.2. A scheme X is **reduced** if for every open subset $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

Definition 1.3. A scheme X is **integral** if for every open subset $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Definition 1.4. A scheme is **locally Noetherian** if it can be covered by affine open subschemes $\operatorname{Spec} A_i$ with each A_i a Noetherian ring. A scheme is **Noetherian** if further finitely many of the $\operatorname{Spec} A_i$ suffice to cover it.

Definition 1.5. A scheme X over $\operatorname{Spec} k$ is **geometrically reduced** (resp., **geometrically irreducible**, **geometrically integral**) if for every field $k' \supseteq k$, we have that $X \times_k k'$ is reduced (resp., irreducible, integral).

2. §2.4, DEFINITIONS

Definition 2.1. A scheme X is **separated** if and only if the canonical morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated.

Although the following is defined in terms of properties of morphisms, I just can't make myself put it into the morphisms cheat sheet.

Definition 2.2. A **variety** X over a field k is a geometrically integral separated scheme of finite type over $\operatorname{Spec} k$.

3. §2.6, DEFINITIONS

Definition 3.1. A scheme X is **regular** if every local ring $\mathcal{O}_{X,x}$ is regular. X is **regular in codimension one** if every local ring $\mathcal{O}_{X,x}$ having dimension one is regular.

Definition 3.2. A scheme X is **locally factorial** if every local ring $\mathcal{O}_{X,x}$ is a UFD.

4. Miscellaneous definitions

Definition 4.1. Let X be a Noetherian scheme. We say X is **catenary** if for all irreducible closed subsets $Z_1 \subseteq Z_2$ of X, any two maximal chains of irreducible closed subsets from Z_1 to Z_2 have the same length. X is **universally catenary** if for all X' of finite type over X, we have that X' is catenary.

Definition 4.2. A scheme X is **Cohen-Macaulay** if [censored].

Definition 4.3. A scheme X is a local complete intersection if every local ring $\mathcal{O}_{X,x}$ can be written as R/I for R a regular local ring, such that $\dim \mathcal{O}_{X,x} = \dim R - n$, where n is the minimal number of generators of I.

5. §2.3, results

The following is clear from the definition.

Lemma 5.1. *An irreducible scheme is connected.*

Being reduced is local in a strong sense:

Proposition 5.2. *Let X be a scheme. Then the following are equivalent:*

a) *X is reduced;*
b) *every stalk $\mathcal{O}_{X,x}$ has no nilpotents;*
c) *X has an cover by open affines subschemes U_i such that each $\mathcal{O}_X(U_i)$ has no nilpotents.*

The equivalence of a) and b) is Exercise II.2.3 (a) of [1]. (a) is visibly stronger than (c), so it is enough to check that (c) implies (b), which is Example II.3.0.1 of [1] together with the implication (a) implies (b) for the U_i.

Proposition 5.3. *A scheme is integral if and only if it is irreducible and reduced.*

See Proposition II.3.1 of [1].

Warning 5.4. Being irreducible is not a local property, and hence neither is being integral. For instance, the disjoint union of two integral schemes is not irreducible. It is however true that a connected scheme is irreducible if and only if it has a cover by open irreducible subsets; it follows that a connected scheme is integral if and only if it has a cover by open integral subschemes. As was pointed out to me by Georges Elencwajg, it is not true in general that a connected scheme is integral if and only if every stalk is integral. However, this is true if the scheme has locally finitely many irreducible components, and in particular if it is locally Noetherian.

We have:

Lemma 5.5. *An affine scheme is quasi-compact.*

This is Exercise II.2.13 (b) of [1].

The following follows trivially from the above lemma and the definitions.

Lemma 5.6. *A scheme is Noetherian if and only if it is locally Noetherian and quasi-compact.*

Being locally Noetherian is local, but in a weaker (!) sense than being reduced:

Proposition 5.7. *A scheme X is locally Noetherian if and only if for every open affine subscheme* $\operatorname{Spec} A$*, we have that A is a Noetherian ring.*

This is Proposition II.3.2 of [1].

Warning 5.8. It is not the case that if every local ring is Noetherian, then the scheme is locally Noetherian (although there is a correct version of this statement – see Exercise 2 (a) of Problem Set 11). It is also not the case that for a Noetherian scheme X, and U an arbitrary open subscheme, $\mathcal{O}_X(U)$ is necessarily Noetherian. Finally, although it is true that every Noetherian scheme has Noetherian underlying topological space (i.e., every descending chain of closed subsets stabilizes), the converse is not true in general.

6. §2.4, results

Theorem 6.1. *X is separated if and only if the diagonal morphism $X \to X \times X$ is quasi-compact, and for every valuation ring R, with field of fractions K, any morphism $\operatorname{Spec} K \to X$ factors through at most one morphism $\operatorname{Spec} R \to X$.*

This is the valuative criterion of properness applied over $\operatorname{Spec} \mathbb{Z}$; see the cheat sheet on morphisms.

7. §2.6, results

Proposition 7.1. *A regular scheme is locally factorial.*

See Remark II.6.11.1A of [1].

8. Miscellaneous results

Theorem 8.1. *Any Cohen-Macaulay scheme is universally catenary.*

See Theorem 17.9 of [2].

Proposition 8.2. *A regular scheme is a local complete intersection.*

See Exercise 3 of Problem Set 11.

Proposition 8.3. *A local complete intersection is Cohen-Macaulay.*

This follows from Theorem II.8.21A (a) and (d) of [1].

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
2. Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.