A CRITERION FOR CLOSED IMMERSIONS AND APPLICATIONS

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We describe a criterion for closed immersions in terms of injectivity on points and on tangent spaces. This includes a more general version of the criterion in Hartshorne [1], from which we deduce his version. Finally, we state an even simpler version in the special case of curves, and use it to give some applications of the Riemann-Roch theorem.

1. The criterion

The general idea is that we wish to produce a criterion for immersions in terms of injectivity of points and of tangent spaces. However, we immediately see that this is not quite possible:

Example 1.1. Let $C \subseteq \mathbb{A}^2_k$ be the nodal cubic plane curve given by $y^2 = x^3 + x^2$, where k is some algebraically closed field. Let $\tilde{C}$ be its normalization; then $\tilde{C} \cong \mathbb{A}^1$. Finally, let D be the open subcurve of $\tilde{C}$ obtained by removing one of the two points lying over the node of C. We consider the morphism $D \to \mathbb{A}^2_k$. Via an explicit parametric form, it is easy to check that this morphism is injective both on points and on tangent spaces. However, we also see that the map is not an immersion: for instance, an immersion would give an isomorphism of D onto its image, but while D is non-singular, its image has a node.

It turns out that the way to salvage such a criterion is to restrict our attention to closed immersions. However, we see immediately that it is not enough to simply require that the morphism in question be closed: indeed, the morphism of the above example is also closed, as the only closed subsets of D are finite collections of closed points, which maps to closed subsets of $\mathbb{A}^2_k$. We will remedy this by restricting our attention to morphisms which are not just closed, but universally closed, and in fact proper. Since every closed immersion is proper, we will still be able to obtain an "if and only if" criterion, under the mild hypothesis that the schemes in question are locally Noetherian.

Proposition 1.2. *Let $f : X \to Y$ be a morphism, with Y locally Noetherian. Then f is a closed immersion if and only if f is proper, and both of the following conditions hold:*

- (i) *(Strong injectivity on points) For each $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$, and furthermore $k(y) \xrightarrow{\sim} k(x)$ under the induced map.*
- (ii) *(Injectivity on tangent spaces) For each $x \in X$, the induced map $T_x(X) \to T_{f(x)}(Y)$ is injective.*

Note that the last part of (i) justifies the existence of the map on tangent spaces in (ii).

The proof of the proposition rests on the following three ingredients: a trivial topological lemma, an easy lemma in commutative algebra, and a theorem which is quite non-trivial in general, but not so difficult in the case which we will mainly use it, which is morphisms of projective varieties.

Lemma 1.3. *Let $f : X \to Y$ be a continuous map of topological spaces, and suppose that f is injective and closed. Then f is a homeomorphism onto a closed subset of Y.*

Lemma 1.4. *Let $f : A \to B$ be a local homomorphism of Noetherian local rings, and suppose that the following conditions hold:*

(i) *f induces an isomorphism $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$;*
(ii) *f induces a surjection $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$;*
(iii) *f makes B into a finite-generated A-module.*

Then f is surjective.

Proof. This is an easy application of Nakayama's lemma: see Lemma II.7.4 of [1]. □

Theorem 1.5. *Let $f : X \to Y$ be a proper morphism, with Y locally Noetherian, and $\mathcal{F}$ a coherent sheaf on X. Then $f_*\mathcal{F}$ is coherent on Y.*

See Corollary II.5.20 for the case of projective varieties over a field. The general statement is the $i = 0$ case of Theorem 3.2.1 of [3].

Proof of proposition. We leave as an exercise the statement that if f is a closed immersion, then f is proper and satisfies both the asserted conditions. For the converse, the first observation is that by the topological lemma above, properness and (i) imply that on underlying topological spaces f is a homeomorphism onto a closed subset of Y. It therefore suffices to check that $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, which may be checked on stalks. Fix $x \in X$, with $y = f(x)$. Because f is a homeomorphism onto its image, $(f_*\mathcal{O}_X)_y = \mathcal{O}_{X,x}$, so we just need to check that the induced map on stalks $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective. But the hypotheses (i) and (ii) imply that conditions (i) and (ii) of the above algebra lemma are satisfied, and the theorem above implies that $f_*\mathcal{O}_X$ is coherent on Y, and hence $\mathcal{O}_{X,x}$ is finitely generated as a $\mathcal{O}_{Y,y}$-module. Therefore the lemma implies the desired surjectivity, and we are done. □

If we work with schemes of finite type over a field, we can strengthen Proposition 1.2 as follows:

Proposition 1.6. *In the situation of Proposition 1.2, suppose that f is a morphism of schemes of finite type over* $\operatorname{Spec} k$*, for some field k. Then in (i) and (ii) it is enough to consider only the closed points of X and Y.*

In particular, if further $k = \bar{k}$, we have that f is a closed immersion if and only if it is proper, injective on closed points, and injective on tangent spaces at closed points.

The proof is left as an exercise.

2. Linear series and morphisms to projective space

Recall that global sections of line bundles are closely related to morphisms of a variety to projective space:

Theorem 2.1. *Let X be a scheme over* $\operatorname{Spec} k$*. Then morphisms $X \to \mathbb{P}^n_k$ over* $\operatorname{Spec} k$ *are in bijection with equivalence classes of tuples $(\mathcal{L}, (s_0, \dots, s_n))$, where $\mathcal{L}$ is a line bundle on X, and $s_i \in H^0(X, \mathcal{L})$, such that for each $x \in X$, there is some i with $s_i \notin \mathfrak{m}_x \mathcal{L}_x$.*

$$(\mathcal{L}, (s_0, \dots, s_n)) \sim (\mathcal{L}', (s'_0, \dots, s'_n))$$

if there exists an isomorphism of sheaves $\alpha : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that $\alpha(s_i) = s'_i$ for all i.

The correspondence is induced by associating to each morphism $f : X \to \mathbb{P}^n_k$ the line bundle $f^*\mathcal{O}(1)$, with global sections $f^*(X_i)$, where $X_0, \dots, X_n$ are the coordinate forms on $\mathbb{P}^n_k$.

Recall that if X is a scheme, and $\mathcal{L}$ a line bundle on X, with $s \in H^0(X, \mathcal{L})$, then we have an associated effective Cartier divisor on X, and hence a closed subscheme of X, which we will denote by $Z(s)$. This should be thought of as the closed subscheme on which s vanishes; its points are the points $x \in X$ with $s \in \mathfrak{m}_x \mathcal{L}_x$.

We make the following basic observations:

Proposition 2.2. *Let $f : X \to Y$ be a morphism of schemes, and $\mathcal{L}$ a line bundle on Y. Then given $s \in H^0(Y, \mathcal{L})$, we have that $Z(f^*s) = f^{-1}(Z(s))$ (as closed subschemes of X).*

Proposition 2.3. *Let $f : X \to \mathbb{P}^n_k$ be a morphism given by $(\mathcal{L}, (s_0, \dots, s_n))$, with X a scheme over $\mathbb{P}^n_k$. Then:*

- (i) *f is* **degenerate***; i.e., factors through some hyperplane $H \subseteq \mathbb{P}^n_k$, if and only if the s_i are linearly dependent.*
- (ii) *Composition of f with a linear change of coordinates $X_i \mapsto \sum_j a_{i,j} X_j$ corresponds to applying the same transformation to the s_i.*

The first proposition is checked easily from the fact that $\mathcal{L}$ is locally isomorphic to $\mathcal{O}_Y$, and left as an exercise.

Part (i) of the next proposition then follows, since f factors through a hyperplane H if and only if $f^{-1}(H) = X$, and writing H as the zero locus of a linear combination $\sum a_i X_i \in H^0(\mathbb{P}^n_k, \mathcal{O}(1))$, by the first proposition this is true if and only if the pullback section $\sum_i a_i s_i$ is indentically 0.

(ii) follows immediately from the above theorem.

Thus, if f is non-degenerate, and $V := < s_0, \dots, s_n >$, then composing with a change of coordinates is equivalent to a change of basis of V. Thus, if we only care about f up to automorphism of $\mathbb{P}^n_k$, it is enough to remember the space V rather than the choice of basis s_i. This motivates the following classical definitions:

Definition 2.4. A **linear series** (of dimension n) on X is a pair $(\mathcal{L}, V)$, with $V \subseteq H^0(X, \mathcal{L})$, and $\dim_k V = n + 1$. We say $x \in X$ is a **base point** of $(\mathcal{L}, V)$ if for every $s \in V$, we have $s \in \mathfrak{m}_x \mathcal{L}_x$.

We see immediately from the above proposition that:

Corollary 2.5. *Given a scheme X over* $\operatorname{Spec} k$*, pullback of $\mathcal{O}(1)$ and coordinate forms induces a bijection from the set of non-degenerate morphisms $X \to \mathbb{P}^n_k$ over* $\operatorname{Spec} k$*, up to linear change of coordinate on $\mathbb{P}^n_k$, to the set of basepoint-free linear series of dimension n on X.*

We will want to know:

Lemma 2.6. *If $(\mathcal{L}, V)$ is a linear series on a scheme X, the locus of base points of $(\mathcal{L}, V)$ is naturally a closed subset of X.*

Proof. We have already associated a closed subscheme of X to the vanishing locus of any section $s \in V$. The base locus is thus an intersection of closed subsets, and hence closed. □

We can translate our criterion for closed immersions into the context of linear series as follows:

Proposition 2.7. *Let X be a proper scheme of finite type over $\operatorname{Spec} k$, with $k = \bar{k}$, and suppose we are given a linear series $(\mathcal{L}, V)$ of dimension n. This induces a closed immersion $X \hookrightarrow \mathbb{P}^n_k$ if and only if:*

(i) *for each $x, x' \in X$ distinct closed points, there exists a section $s \in V$ with $s \in \mathfrak{m}_x \mathcal{L}_x$, but $s \notin \mathfrak{m}_{x'} \mathcal{L}_{x'}$;*
(ii) *for each $x \in X$ closed, the set of $s \in V$ with $s \in \mathfrak{m}_x \mathcal{L}_x$ spans the k-vector space $\mathfrak{m}_x \mathcal{L}_x / \mathfrak{m}_x^2 \mathcal{L}_x$.*

Proof. First note that by the above lemma, it is enough to check that $(\mathcal{L}, V)$ is basepoint free on closed points, and this follows from (i). Next, since X is proper, if we write $f : X \to \mathbb{P}^n_k$ for the morphism induced by $(\mathcal{L}, V)$, then f is automatically proper (see Corollary II.4.2(e) of [1]). By Proposition 1.6, it suffices to see that (i) and (ii) imply that f is injective on closed points and their tangent spaces, respectively.

(i) is satisfied if and only if for any $x, x' \in X$ distinct closed points, we have $s \in V$ with s vanishing at x but not x'; this is equivalent to the existence of a hyperplane $H \subseteq \mathbb{P}^n_k$ such that $f^{-1}(H)$ contains x but not x', or equivalently, with $f(x) \in H$ but $f(x') \notin H$, and this in turn is equivalent to having $f(x) \neq f(x')$.

On the other hand, (ii) is clearly equivalent to surjectivity of

$$f^* : \mathfrak{m}_{f(x)} \mathcal{O}(1)_{f(x)} / \mathfrak{m}^2_{f(x)} \mathcal{O}(1)_{f(x)} \to \mathfrak{m}_x \mathcal{L}_x / \mathfrak{m}_x^2 \mathcal{L}_x,$$

which one can check is equivalent to surjectivity of the map induced by f on cotangent spaces, which in turn is equivalent to injectivity on tangent spaces. We therefore conclude the desired statement from Proposition 1.6. □

3. The case of curves

From this point on, we let C be a smooth, proper, and geometrically connected curve over $\operatorname{Spec} k$, of genus g. Given a field $k' \supseteq k$, we denote by $C_{k'}$ the base change $C \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ over $\operatorname{Spec} k'$.

We begin with a brief discussion of the degree of a linear series:

Definition 3.1. We define the **degree** of a linear series $(\mathcal{L}, V)$ on C to be the degree of $\mathcal{L}$.

Since the degree of a line bundle $\mathcal{L}$ is by definition equal to the degree of the divisor associated to a section s of $\mathcal{L}$, we see in particular that if $(\mathcal{L}, V)$ is basepoint free, hence induces a morphism $f : C \to \mathbb{P}^n_k$, the degree is also given as the degree of the effective divisor $f^*(H)$ for H any hyperplane in $\mathbb{P}^n_k$.

Example 3.2. Suppose that $(\mathcal{L}, V)$ is a basepoint-free 1-dimensional linear series. Then its degree is the length of any fiber of the corresponding map $f : C \to \mathbb{P}^1_k$, which we saw on last semester's final exam is the same as the degree as a morphism of curves (i.e., the degree of the induced extension of function fields).

Example 3.3. Suppose that $(\mathcal{L}, V)$ induces a closed imbedding $C \hookrightarrow \mathbb{P}^n_k$. Then the degree is the length of the intersection of C with a hyperplane H.

Example 3.4. If further $n = 2$, we have that C is $V(F)$ for some homogeneous polynomial of some degree d, and hence C intersects H with length d; thus we see that the degree of $(\mathcal{L}, V)$ is none other than the degree of the polynomial defining C.

We also want the following observation:

Lemma 3.5. *Let $(\mathcal{L}, V)$ be a linear series on C, and $P \in C(k)$. Then $\dim_k(V \cap H^0(C, \mathcal{L}(-P))) = \dim V - \delta$, where $\delta = 0$ if P is a base point of $(\mathcal{L}, V)$, and $\delta = 1$ otherwise.*

Proof. Since $\mathcal{L}(-P)$ is the subsheaf of $\mathcal{L}$ consisting of sections vanishing at P, it is clear that $\delta = 0$ if P is a base point, and $\delta > 0$ otherwise. It thus suffices to see that $\delta \leq 1$. Suppose $s \in V$ is non-vanishing at P; we claim it generates $V/(V \cap H^0(C, \mathcal{L}(-P)))$. But since $k = k(P)$, for any $s' \in V$ we have a k-linear combination of s and s' vanishing at P, hence in $V \cap H^0(C, \mathcal{L}(-P))$, which is what we needed to show. □

We can finally rephrase our criterion for closed immersions as follows in the case of linear series on curves:

Corollary 3.6. *Let $(\mathcal{L}, V)$ be a linear series on C, and suppose k is algebraically closed. Then $(\mathcal{L}, V)$ is basepoint free and induces a closed immersion to $\mathbb{P}^{\dim V - 1}_k$ if and only if for all $P, Q \in C$ not necessarily distinct closed points, we have $\dim(V \cap H^0(C, \mathcal{L}(-P - Q))) = \dim V - 2$.*

Proof. We check the conditions of 2.7. By the above lemma, we must have for any P, Q that Q is not a base point of $(\mathcal{L}(-P), V \cap H^0(C, \mathcal{L}(-P)))$. For $P \neq Q$, this is precisely (i) of the proposition, while for $P = Q$, this is condition (ii), since we note that $\mathcal{L}(-P)_P = \mathfrak{m}_P \mathcal{L}_P$, and $\mathfrak{m}_P \mathcal{L}_P / \mathfrak{m}_P^2 \mathcal{L}_P$ is one-dimensional. □

The main consequence of Riemann-Roch which we will use is the following:

Corollary 3.7. *(Corollary 3.5 of [2]) Let $\mathcal{L}$ be a line bundle with $\deg \mathcal{L} > 2g - 2$. Then $h^0(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$.*

Corollary 3.8. *Let $\mathcal{L}$ be a line bundle of degree $d > 2g$ on C. Then the linear series $(\mathcal{L}, H^0(C, \mathcal{L}))$ defines a closed immersion $C \hookrightarrow \mathbb{P}^{d-g}_k$.*

Proof. This is almost immediate from the preceding two corollaries. The only point that needs to be verified is that it suffices to check that we have a closed immersion after base change to $\bar{k}$, which follows from the following two lemmas (see also Proposition 1.9 and Lemma 3.2 of [2]), whose proofs are left as exercises. □

Lemma 3.9. *X be a scheme over $\operatorname{Spec} k$, with linear series $(\mathcal{L}, V)$, and k' a field containing k. Let X' and $(\mathcal{L}', V')$ be the base change to k'. Then $(\mathcal{L}, V)$ is basepoint free on X if and only if $(\mathcal{L}', V')$ is basepoint free on X'.*

Lemma 3.10. *Let $f : X \to Y$ be a morphism of schemes over* $\operatorname{Spec} k$*, and k' a field containing k. Let $f' : X' \to Y'$ be the base change to k'. Then f is a closed immersion if and only if f' is a closed immersion.*

We conclude with several elementary corollaries on genus 0 and 1 curves that make no direct reference to linear series or the cohomological machinery.

Corollary 3.11. *Suppose that C has genus 0. Then $C \cong \mathbb{P}^1_k$ if and only if $C(k) \neq \emptyset$.*

Proof. Surely, if $C \cong \mathbb{P}^1_k$, then $C(k) \neq \emptyset$. Conversely, suppose $P \in C$ is a k-valued point. Then P defines a divisor of degree 1, so $\mathcal{O}_C(P)$ is a line bundle of degree 1, and by Corollary 3.8 defines a closed immersion $C \hookrightarrow \mathbb{P}^1_k$. But since C and $\mathbb{P}^1_k$ are both curves, this must be an isomorphism. □

Example 3.12. Note that it is not automatic that $C(k) \neq \emptyset$: consider the plane conic defined by (in homogeneous coordinates) $X_0^2 + X_1^2 + X_2^2 = 0$ in $\mathbb{P}^2_k$. If $k = \mathbb{Q}$, or for that matter, $\mathbb{R}$, then C has no k-valued points. However, one checks that C does have genus 0 (for instance, by checking that over the algebraic closure, there is an isomorphism with $\mathbb{P}^1_{\bar{k}}$).

However, the above example is in essence the only way a genus 0 curve can fail to be $\mathbb{P}^1_k$:

Corollary 3.13. *Suppose C has genus 0. Then C can be realized in $\mathbb{P}^2_k$ as a plane conic.*

Proof. By Corollary 3.8, it suffices to produce a line bundle on C of degree 2. But we observe that $\Omega^1_{C/k}$ has degree -2, so $(\Omega^1_{C/k})^\vee$ has degree 2. □

Combining the previous two corollaries, we find:

Corollary 3.14. *Suppose C has genus 0. Then there exists $k' \supseteq k$ of degree ≤ 2 over k and such that $C_{k'} \cong \mathbb{P}^1_{k'}$.*

Proof. If we realize C as a plane conic, we can choose any non-zero values in k for X_1, X_2, and then finding a point on C is equivalent to solving a quadratic equation for X_0 over k. □

Corollary 3.15. *Suppose C has genus 1, and $C(k) \neq \emptyset$. Then C can be realized in $\mathbb{P}^2_k$ as a plane cubic.*

Proof. Let $P \in C$ be a k-valued point. Then by Corollary 3.8, the line bundle $\mathcal{O}_C(3P)$ defines a closed immersion of C in $\mathbb{P}^2_k$ as a cubic curve. □

References

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