THE HILBERT POLYNOMIAL AND DEGREE

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1. The Hilbert polynomial

Definition 1.1. Let X be a projective variety over k, with very ample invertible sheaf $\mathscr{O}_X(1)$, and suppose we are given a coherent sheaf $\mathscr{F}$ on X. Then the **Hilbert polynomial** of $\mathscr{F}$ with respect to $\mathscr{O}_X(1)$ is defined to be the function $\chi(\mathscr{F}(n))$, where $\chi(\mathscr{F}(n))$ is the Euler characteristic of $\mathscr{F}(n)$. When $\mathscr{F} = \mathscr{O}_X$, this is also called the Hilbert polynomial of X.

The terminology is justified by:

Proposition 1.2. *The Hilbert polynomial is a polynomial in n, of degree equal to $\dim \operatorname{Supp} \mathscr{F}$, and with positive leading coefficient. It can be expressed as an integer combination of binomial polynomials $\binom{n}{r}$. Finally, for $n >> 0$, we have $\chi(\mathscr{F}(n)) = h^0(\mathscr{F}(n))$.*

Proof. The first assertion is Exercise III.5.2 of [1]. The degree assertion can be deduced by modifying the proof of the same exercise to remove the first term of the exact sequence, using Corollary 4.1 of [2] (and noting that invariance of cohomology under field extension means we can reduce immediately to the case of an infinite field). The last equality follows from Serre's theorem on vanishing of higher cohomology. This then implies that $\chi(\mathscr{F}(n)) \geq 0$ for $n >> 0$, hence that the leading coefficient is positive. Finally, the expression in terms of binomials is a general fact on polynomials taking integer values, see Proposition I.7.3 of [1]. $\square$

It is best to think of the Hilbert polynomial as having less to do with $\mathscr{F}$ than with $\mathscr{O}_X(1)$: for instance, if we replace $\mathscr{O}_X(1)$ by some power, the Hilbert polynomial of $\mathscr{F}$ goes up by the appropriate variable scaling. Another way to say this is that the Hilbert polynomial should be thought of in terms of the given imbedding of X into $\mathbb{P}^n_k$. In particular, the Hilbert polynomial of X has as much to do with the imbedding of X as with the intrinsic geometry, but of course it involves both.

Example 1.3. Suppose that X is a smooth projective curve, and $\mathscr{F} = \mathscr{L}$ is an invertible sheaf of degree d, and finally that $\deg \mathscr{O}_X(1) = m$. Then Riemann-Roch tells us that the Hilbert polynomial of $\mathscr{L}$ with respect to $\mathscr{O}_X(1)$ is $mt + d + 1 - g$. In particular, the degree of $\mathscr{O}_X(1)$ is what determines the leading term, while the degree of $\mathscr{L}$ only affects the constant term.

However, an indication of the extent to which intrinsic geometry is captured by the Hilbert polynomial is the following:

Theorem 1.4. *Let T be an integral Noetherian scheme, and $X \subseteq \mathbb{P}^n_T$ a closed subscheme. For each $t \in T$, let P_t be the Hilbert polynomial of the fiber X_t with respect to its imbedding in $\mathbb{P}^n_t$. Then X is flat over T if and only if P_t is constant over all $t \in T$.*

See Theorem III.9.9 of [1].

One way in which X could fail to be flat would be if its dimension jumped in one fiber, and this would clearly be detected by the degree of the Hilbert polynomial. However, X would also fail to be flat at t if X_t had some component (either topological or imbedded) not in the closure of $X|_{t' \neq t}$. Thus, the theorem says that the existence of any such components will force the the Hilbert polynomial to be non-constant at t_0.

We will focus on the Hilbert polynomial of X itself. However, the Hilbert polynomials of more general sheaves can still be important, particularly on higher-dimensional varieties. We mention in particular that stability conditions on vector bundles, which arise in the study of moduli spaces of vector bundles, use the Hilbert polynomial in a fundamental fashion.

2. The degree of a projective variety

It follows from Proposition 1.2 that if $X \subseteq \mathbb{P}^N_k$ is a variety of dimension n, then the leading term of its Hilbert polynomial of X is of the form $\frac{d}{n!}x^n$, for some positive integer d.

Definition 2.1. The above d is the **degree** of X in $\mathbb{P}^N_k$.

We reiterate that the degree is an invariant of the imbedding of X in $\mathbb{P}^N_k$, and not of X itself. This is clarified by the classical definition, which is that the degree of a variety in $\mathbb{P}^N_k$ is the number of points of intersection with a general linear subspace of complementary dimension $N-n$. We will relate these two conditions later.

Example 2.2. Let X be a smooth projective curve. Then its degree is simply the degree of $\mathscr{O}_X(1)$.

Example 2.3. Let X be 0-dimensional over a field k. Then its degree is $h^0(X, \mathscr{O}_X)$. Indeed, every invertible sheaf is isomorphic to $\mathscr{O}_X$, which is very ample, and we see the Hilbert polynomial is constant with value $h^0(X, \mathscr{O}_X)$.

Example 2.4. The degree of $\mathbb{P}^N_k$ in itself is 1. Indeed, the Hilbert polynomial of $\mathbb{P}^N_k$ is $\binom{t+N}{N}$, as $\Gamma(\mathbb{P}^N_k, \mathscr{O}(t))$ is homogeneous polynomials in $N+1$ variables, of degree t, of which there are $\binom{t+N}{N}$. In particular, we find that the leading term is $\frac{1}{N!}$, so the degree is 1.

A basic example of degree is the following:

Proposition 2.5. *Let $X \subseteq \mathbb{P}^N_k$ be the zero-set of a homogeneous polynomial F of degree d. Then X has degree d.*

Proof. Indeed, we have the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^N_k}(-d) \overset{\times F}{\mathscr{O}}_{\mathbb{P}^N_k} \to \mathscr{O}_X \to 0,$$

so if $P(t)$ is the Hilbert polynomial of $\mathbb{P}^N_k$, and $Q(t)$ the Hilbert polynomial of X, we get $P(t) = P(t-d) + Q(t)$, so $Q(t) = P(t) - P(t-d)$. We then see that the leading term of $Q(t)$ is Nd times the leading term of $P(t)$, so by the above example we get the desired assertion. □

We also note:

Proposition 2.6. *Suppose that* $X = X_1 \cup X_2 \subseteq \mathbb{P}^N_k$*, with* X_1 *and* X_2 *of the same dimension, but* $X_1 \cap X_2$ *of smaller dimension. Then* $\deg X = \deg X_1 + \deg X_2$*.*

If $\dim X_1 > \dim X_2$*, then* $\deg X = \deg X_1$*.*

Proof. The short exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_{X_1} \oplus \mathscr{O}_{X_2} \to \mathscr{O}_{X_1 \cap X_2} \to 0$$

and the additivity of the Hilbert polynomial immediately imply the desired result, since $\dim X = \dim X_1 = \dim X_2 > \dim(X_1 \cap X_2)$. The same argument also gives that $\deg X = \deg X_1$ when $\dim X_1 > \dim X_2$. □

These are enough to prove:

Theorem 2.7. *Let* $X \subseteq \mathbb{P}^N_k$ *be a projective scheme of dimension* n*, and* $X_1, \dots, X_r$ *its reduced components of dimension* n*, with generic points* x_i*. Then*

$$\deg X = \sum_i \mu_{x_i} \deg X_i,$$

where μ_{x_i} *is the multiplicity of* X *along* X_i*.*

In particular, the degree of X *is not affected by components of smaller dimension or imbedded components.*

Proof. By the second sequence of Proposition 3.7 of [2] (see also Proposition 3.9 of *loc. cit.*), we can inductively place $\mathscr{O}_X$ into short exact sequences decreasing its multiplicity along each X_i by one, with cokernel $\mathscr{O}_{X_i}$, and at each step additivity tells us that the Hilbert polynomial decreases by $\deg X_i$. We will ultimately end up with a subscheme of X supported on a strictly smaller-dimensional set, so by the second assertion of the above proposition, we are done. □

3. The degree of a complete intersection

We begin with a generalization of Proposition 2.5:

Proposition 3.1. *Let* $X \subseteq \mathbb{P}^N_k$ *be the zero-set of a homogeneous polynomial* F *of degree* d*, and* $Y \subseteq \mathbb{P}^N_k$ *any closed subscheme such that* $\mathscr{O}_Y(-d) \stackrel{\times F}{\to} \mathscr{O}_Y$ *is injective. Then* $\deg(X \cap Y) = (\deg X) \cdot (\deg Y)$*.*

Proof. The proof is the same as Proposition 2.5, using the exact sequence

$$0 \to \mathscr{O}_Y(-d) \stackrel{\times F}{\to} \mathscr{O}_Y \to \mathscr{O}_{X \cap Y} \to 0.$$

Note that the hypothesis that $\times F$ is injective on Y is equivalent to saying that X does not contain any (possibly imbedded) component of Y, and in particular implies that $\dim(X \cap Y) = \dim Y - 1$. □

We can now relate our definition of degree to the classical one.

Corollary 3.2. *Suppose* k *is infinite, and* $X \subseteq \mathbb{P}^N_k$*. Then* $\deg X$ *is given as* $\dim_k(\Gamma(X \cap V, \mathscr{O}_{X \cap V})$*, where* V *is a general linear subspace of dimension* $N - \dim X$*.*

Proof. We cut inductively by hyperplanes and apply the previous proposition each time, since by Corollary 4.1 of [2] as long as we are over an infinite field we will always have the desired injectivity condition, and the dimension will decrease by 1 each time. Example 2.3 then tells us the degree of $X \cap V$ is the asserted dimension. □

Exercise 3.3. Show that under the hypotheses of the corollary, if we have X generically reduced, we also have that $X \cap V$ consists of reduced points, so that if $k = \bar{k}$, in fact $\deg X = \#\{X \cap V\}$ (see Problem Set 6). Thus our definition of degree agrees with the classical one.

We also obtain the following generalization of Bezout's theorem:

Corollary 3.4. *Let* $X \subseteq \mathbb{P}^N_k$ *be a complete intersection of hypersurfaces of degrees* $d_1, \ldots, d_r$. *Then* $\deg X = d_1 \cdots d_r$.

Proof. The only subtlety in inductively applying the corollary is the injectivity hypothesis. However, each time we intersect with a hypersurface the dimension must go down by exactly 1 in order for X to be a complete intersection, so each intermediate intersection is also complete, and by Corollary 2.4 of [2] we have no imbedded points. Since the dimension drops again in the next intersection, the next hyperplane then contains no associated points, so by Corollary 1.6 of *loc. cit.* we have the desired injectiviy. □

Remark 3.5. One frequently refines results such as the above by introducing a notion of intersection multiplicity, and working with the reduced and irreducible components of the intersection. This is straightforward, using Theorem 2.7, and works well for intersecting with a hypersurface; see also Theorem I.7.7 of [1].

However, one aspect of this is very misleading: if one makes such a statement in terms of (for instance) an arbitrary reduced scheme intersecting a hypersurface, the statement looks inductive, because the formula is expressed entirely in terms of the reduced components of the intersection. However, one cannot naively generalize these formulas to intersections in higher codimension, because the degree identities we use to decompose the intersection into its components will not commute with further intersection.

The simplest example of such a failure (and the simplest example of a reduced, equidimensional scheme which is not Cohen-Macaulay) is to let Y be the union of two general planes in $\mathbb{P}^4_k$, which meet at a single point P. Then let X be a third plane, meeting Y in finitely many points. There are two cases: if $P \notin X$, then X meets each of the planes of Y once, so its intersection with Y is two (reduced) points. On the other hand, it is an easy computation that if $P \in X$, then $X \cap Y$ is a non-reduced scheme of multiplicity 3 supported on P.

What goes wrong here is that if we first intersect with a hypersurface containing P, the degree is 2, and the intersection is supported on the union of two lines through P. However, this intersection also contains an imbedded point at P, which does not affect the degree until we intersect with a second hyperplane. This is why we cannot use such naive definitions of intersection multiplicity for intersecting arbitrary (even reduced) projective schemes, although it turns out to be sufficient in the Cohen-Macaulay case.

One final comment on the example: this is the standard example of failure to be Cohen-Macaulay, but it is not essential to the example that it be reducible. Just as the simplest node in the plane is $xy = 0$, but there are irreducible nodal curves, we could write down an integral example in $\mathbb{P}^4_k$ which would have a singularity looking "locally" like the union of two planes at a point. But the equations become much more complicated, so we typically satisfy ourselves with the reducible example.

4. A sample application

We give an example of the sort of computation with spaces of sections of sheaves which frequently arise in classical geometry. Although our only cohomological tool in this case will be Riemann-Roch, one can easily imagine that cohomology will arise more directly in similar computations, and we will give references to such examples.

Lemma 4.1. *Suppose that* $X \subseteq Y \subseteq \mathbb{P}^N_k$*, with* $\dim X = \dim Y$*. Then* $\deg X \leq \deg Y$*, and if* Y *is a complete intersection we have equality if and only if* $X = Y$*.*

Proof. The inequality is obvious: since $h^0(X, \mathscr{O}_X(n)) \leq h^0(Y, \mathscr{O}_Y(n))$ for $n >> 0$, the leading coefficients of the Hilbert polynomials must satisfy the desired inequality.

Now suppose the degrees are equal. Since Y is a complete intersection, it must be equidimensional, and by Theorem 2.7 we conclude that X and Y have the same multiplicity along every component of Y. By Corollary 2.4 of [2] Y has no imbedded points, so by Proposition 3.8 of *loc. cit.* we have $X = Y$. □

Proposition 4.2. *Let* X *be a nonhyperelliptic smooth projective curve of genus* 4*, canonically imbedded in* $\mathbb{P}^3_k$*. Then* X *is the intersection of an irreducible cubic hypersurface with a unique irreducible quadric hypersurface.*

Proof. We first note that if a quadric surface containing X exists, it must be unique and irreducible: we see irreducibility because a reducible surface is a union of planes, and X is non-degenerate in $\mathbb{P}^3_k$, so not contained in any plane; for uniqueness, we note that if X were contained in distinct irreducible surfaces, it would be contained in their intersection, which isn't possible because their intersection has degree 4.

We next consider the exact sequence

$$0 \to \mathcal{I}_X(2) \to \mathscr{O}_{\mathbb{P}^3_k}(2) \to \mathscr{O}_X(2) \to 0.$$

We see that $\mathcal{I}_X(2)$ consists of quadric forms which vanish on X, which is to say, whose zero loci are quadric surfaces containing X. Now, taking global sections, it suffices to see that $h^0(\mathbb{P}^3_k, \mathscr{O}_{\mathbb{P}^3_k}(2)) > h^0(X, \mathscr{O}_X(2))$. But the former is simply the number of homogeneous forms of degree 2 in 4 variables, which is $\binom{2+3}{3} = 10$. The latter we can compute by Riemann-Roch: the only subtlety is to recall that $\mathscr{O}_X(1)$ has degree 6 on X, so $\mathscr{O}_X(2)$ has degree 12, and by Riemann-Roch we see $h^0(X, \mathscr{O}_X(2)) = 9$. Thus we have our unique irreducible quadric surface containing X.

To see that there is an irreducible cubic surface containing X, we carry out the same computation with 3 in place of 2, and find that $h^0(\mathbb{P}^3_k, \mathcal{I}_X(3)) \geq 20 - 15 = 5$. On the other hand, X is contained in a 4-dimensional space of reducible cubics, consisting of unions of the unique quadric containing X with an arbitrary plane. We thus see that X is contained in some irreducible cubic.

Finally, X must be equal to the intersection of the quadric and cubic by the above lemma, since they both have degree 6. □

For similar but more involved examples, and using more complicated cohomological computations, see Exercise III.5.6 on curves on quadrics, and Example IV.6.4.3 of [1] as a further application to curves in $\mathbb{P}^3_k$.

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
2. Brian Osserman, *Associated points and applications*, Course notes: Math 256B, Spring 2007.