A SHORT DESCRIPTION OF THE THEORY OF DEFORMATIONS OF IMBEDDED SCHEMES AND MORPHISMS

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1. Generalities on deformation problems and their relationship to global moduli problems

Deformation theory is often an important tool in dealing with a variety of questions related to moduli spaces. We have discussed the basic idea before: we study families of objects over Artinian rings restricting to a given object over $\operatorname{Spec} k$, and this tells us, for instance, the dimension of the tangent space of a moduli space, or whether it is smooth at a point. We begin by developing this general picture a bit further.

We will restrict to the situation of being over a field. If one is interested in mixed-characteristic situations, all these ideas to categories of Artin Λ-algebras, where Λ is a fixed complete Noetherian (such as a Witt vector ring), but the setup gets a bit wordier in this case.

Suppose we are given a functor $F : \mathrm{Sch}_k \to \mathrm{Set}$, and an element $\eta_0 \in F(\operatorname{Spec} k)$. We can define a deformation problem (functor) associated to η_0 and F as the functor $F_{\eta_0} : \mathrm{Art}(k) \to \mathrm{Set}$ associating to each $A \in \mathrm{Art}(k)$ the set of objects $\eta \in F(\operatorname{Spec} A)$ restricting to $\eta_0 \in F(\operatorname{Spec} k)$. Here $\mathrm{Art}(k)$ denotes the category of Artin local k-algebras with residue field k, so we are given a canonical map $\operatorname{Spec} k \to \operatorname{Spec} A$.

If F is not representable (for instance, if it is a moduli functor parametrizing objects with automorphisms), this will not necessarily produce a well-behaved deformation functor. As in our earlier discussion of deformations of smooth varieties, we would want to functor to include a choice of morphism $\eta_0 \to \eta$, which doesn't make sense in the general setting of functors, and points towards the usefulness of stacks.

However, if F is represented by a locally Noetherian scheme X, and η_0 corresponds to $x \in X$ with $k = k(x)$, then F_{η_0} has several nice properties:

- The tangent space $F_{\eta_0}(k[\epsilon]/(\epsilon^2))$ is the tangent space of X at x.
- If X is locally of finite type over $\operatorname{Spec} k$, it is smooth over $\operatorname{Spec} k$ at x if and only if F_{η_0} is formally smooth: that is, for each $A \twoheadrightarrow A'$ in $\mathrm{Art}(k)$, we have $F_{\eta_0}(A) \twoheadrightarrow F_{\eta_0}(A')$.
- It is pro-representable (meaning it is representable, but we have to enlarge from the category of Artinian rings to the category of complete Noetherian rings in order to find a representing object), and in fact is represented by $\operatorname{Spec} \hat{\mathscr{O}}_{X,x}$.

Sketch of proof. The first two statements are direct rephrasings of the usual description of the tangent space, and the formal criterion for smoothness. For the third statement, we want to see that for $A \in \mathrm{Art}(k)$, maps $A \to X$ restricting to the given inclusion $i_x : \operatorname{Spec} k(x) \to X$ always factor uniquely through $\operatorname{Spec} \hat{\mathscr{O}}_{X,x} \to X$. It is clear that they factor uniquely through $\operatorname{Spec} \mathscr{O}_{X,x} \to X$, so we want to see that every homomorphism $\mathscr{O}_{X,x} \to A$ factors uniquely through $\mathscr{O}_{X,x} \to \hat{\mathscr{O}}_{X,x}$. Uniqueness follows from the injectivity of $\mathscr{O}_{X,x} \to \hat{\mathscr{O}}_{X,x}$ (since we assumed X is locally Noetherian). But because A is Artinian, we have that some power $\mathfrak{m}^n$ of the maximal ideal of $\mathscr{O}_{X,x}$ must map to 0, and it follows formally that the map factors through $\hat{\mathscr{O}}_{X,x}$. □

We therefore see that we can get a substantial amount of finer information about the geometry of X at x if we understand the associated deformation problem well enough to recover properties of $\hat{\mathscr{O}}_{X,x}$. For instance, $\hat{\mathscr{O}}_{X,x}$ tells us the dimension and singularity type (if any) of X at x.

We have already discussed the idea of unobstructedness for measuring smoothness. The more general notion of an obstruction theory will measure how far a deformation functor is from being smooth, and will also help us produce information on the dimension. Roughly, the idea is the following: an obstruction theory taking values in a vector space V assigns an obstruction element in V to each pair of a deformation over A, and a ring extension $A' \twoheadrightarrow A$, such that the obstruction is zero if and only if the deformation can be lifted to A'. More formally, we have:

Definition 1.1. An **obstruction theory** taking values in a k-vector space V consists of the following data: for each surjective map $A' \twoheadrightarrow A$ in $\operatorname{Art}(k)$, having kernel I with $\mathfrak{m}I = 0$, and each object $\eta_A \in F(A)$, one has an obstruction element $\mathrm{ob}_{\eta_A, A'} \in V \otimes_k I$ which is zero if and only if η_A can be lifted to an element $\eta_{A'} \in F(A')$. Moreover, $\mathrm{ob}_{\eta_A, A'}$ should be functorial in A' in the sense that if A'' is a quotient lying between A' and A, the image of $\mathrm{ob}_{\eta_A, A'}$ is equal to $\mathrm{ob}_{\eta_A, A''}$.

Often, obstruction theory is used in the case that $V = 0$, in which case we can conclude smoothness immediately (and don't need the functoriality property). However, it can also be used in more refined ways. The basic idea of the following theorem is due to Mori, although he stated the result only in a special case, and it fell to later authors to give the definition of an obstruction theory and state the result in this generality.

Theorem 1.2. *(Mori) Suppose that a deformation functor F is prorepresented by* $\operatorname{Spec} R$*, has tangent space T_F, and an obstruction theory taking values in a vector space V. Then we have:*

$$\dim T_F - \dim V \leq \dim \operatorname{Spec} R \leq \dim T_F.$$

The argument is reasonably direct, but clever and somewhat involved, so we do not give it here. The original proof is quite readable; see Proposition 3 of [2].

2. Reducing to quotients redux

We will now focus our attention on deformations of subschemes and morphisms, showing that just as in the case of Hilbert and Hom schemes, we will be able to reduce deformations of subschemes and of morphisms to deformations of quotient sheaves; in fact, the reduction for morphisms will hold more generally in the setting of deformations. We begin by defining out deformation problems:

Definition 2.1. Given X/k and $Z_0 \subseteq X$, the functor of **deformations of the subscheme** Z_0 assigns to each $A \in \operatorname{Art}(k)$ the set of subschemes $Z_A \subseteq X_A$, with Z_A flat over A and restricting to Z_0 over k. Here $X_A := X \times_k A$.

Definition 2.2. Given $X, Y/k$ and $f_0 : X \to Y$ a morphism over k, the functor of **deformations of the morphism** f_0 assigns to each $A \in \mathrm{Art}(k)$ the set of morphisms $f_A : X_A \to Y$ over k restricting to f_0 over k. Here $X_A := X \times_k A$.

Definition 2.3. Given X/k, $\mathscr{F}$ quasicoherent on X, and $\mathscr{Q}_0$ a quotient of $\mathscr{F}$, the functor of **deformations of the quotient** $\mathscr{Q}_0$ assigns to each $A \in \mathrm{Art}(k)$ the set of quotients $\mathscr{F}_A \to \mathscr{Q}_A$ with $\mathscr{Q}_A$ flat over A and restricting to $\mathscr{Q}_0$ over k. Here $\mathscr{F}_A$ is the pullback of $\mathscr{F}$ to $X \times_k A$.

Note that these deformation functors can be defined (and studied) without any reference to the corresponding global moduli functors. While one often first proves representability of a global moduli functor and then uses deformation theory to study its local behavior, since Artin's work on general algebraic stacks, deformation theory has also played an important role in proving representability (usually by an algebraic stack, rather than a scheme) of the global functor.

Our reduction to the case of quotient sheaves proceeds as follows:

Proposition 2.4. *Suppose we are given X/k.*

(i) *Given a closed subscheme $Z \subseteq X$, deformations of Z correspond to deformations of $\mathscr{O}_Z$ as a quotient of $\mathscr{O}_X$.*
(ii) *Given another scheme Y/k, with both X and Y locally of finite presentation, and given a morphism $f : X \to Y$, deformations of f correspond to deformations of Γ_f in $X \times_k Y$.*

Proof. The first assertion is immediate from the correspondence between closed subschemes and ideal sheaves.‘ ‘ If we didn't mind imposing properness hypotheses, we could use Lemma 2.3 of [3] to prove the second assertion, just as is done for the global moduli problems in *loc. cit.* However, because the reduction is true more generally for deformation problems, we instead conclude the desired statement from the following lemma, by applying it to a morphism $p_1 : Z \to X_A$, with $Z \subseteq X_A \times_k Y$ and p_1 being an isomorphism after restriction to $\operatorname{Spec} k$. □

Lemma 2.5. *Let $f : X \to Y$ be a morphism of schemes locally of finite presentation over a base scheme S, and suppose that X is flat over S. Then f is an isomorphism if and only if it is an isomorphism on the fibers over each $s \in S$.*

Proof. This is Corollary 17.9.5 of [4]. The idea is that one can check unramifiedness, non-extension of residue fields, and (set-theoretic) bijectivity on fibers in general, and under the flatness hypothesis one can also check flatness on fibers. We then have an etale map which doesn't extend residue fields, which is then locally an open immersion, and bijectivity implies it is an isomorphism. □

3. Deformations and obstructions for quotients, subschemes, and morphisms

We now come to our main purpose: describing the tangent and obstruction spaces for deformations of quotient sheaves, and applying this description as well to subschemes and morphisms, obtaining in particular bounds on the dimensions of Quot, Hilbert, and Hom schemes. We do not attempt to give the most general possible statements, contenting ourselves instead with certain hypotheses that significantly simplify the results. For more detailed accounts, see Hartshorne's lecture notes [1].

Proposition 3.1. *Suppose that we are given a quotient $\mathcal{Q}_0$ of $\mathscr{F}$ on X, with $\mathscr{G} := \ker(\mathscr{F} \twoheadrightarrow \mathcal{Q}_0)$. Then the first-order deformations of $\mathcal{Q}_0$ are parametrized by $H^0(X, \mathcal{H}om(\mathscr{G}, \mathcal{Q}_0))$.*

Furthermore, if for any $A, A' \in \text{Art}(k)$, and any deformation $\mathcal{Q}_A$ of $\mathcal{Q}_0$, locally on X there exists a deformation of $\mathcal{Q}_0$ over A' extending $\mathcal{Q}_A$, then we have an obstruction theory taking values in $H^1(X, \mathcal{H}om(\mathscr{G}, \mathcal{Q}_0))$.

Idea behind proof. The proof can be a bit technical, so instead the sketch the basic idea of why such a statement should hold.

We begin with the case of a quotient W of a vector space V, with kernel K. Deforming the quotient is the same as deforming the kernel, and the idea is that if we want to deform over $k[\epsilon]/(\epsilon^2)$, we should give a deformation $v + \epsilon v'$ for each vector $v \in K$. Thus any element of $\text{Hom}(K, V)$ gives a deformation. However, if two vectors differ by an element of K, the deformation will not give a different subspace, so the actual space of deformations is $\text{Hom}(K, V/K) = \text{Hom}(K, W)$. Similarly, if we start with vector bundles $\mathscr{F} \twoheadrightarrow \mathcal{Q}$ on X, at each point we are deforming $\mathscr{F}_x$ as a subspace of $\mathcal{Q}_x$, so it makes sense that the deformations are given by $\mathcal{H}om(\mathscr{G}, \mathcal{Q}_0)$.

To see why the statement on obstructions holds, we take an open cover $\{U_i\}$ and extensions of the given deformation over each open set in the cover; the question is whether they can be adjusted to agree on each U_{ij}. But the choices of extensions on each U_i are, by the same argument as above, parametrized by $\mathcal{H}om(\mathscr{G}, \mathcal{Q}_0)$, so taking the difference of the chosen extensions on each U_{ij} gives a Cech 1-cocyle, which gives a trivial cohomology class if and only if the extensions on each U_i can indeed be adjusted to glue together and give an extension on all of X. □

Specializing to the case of the Hilbert scheme, we have:

Proposition 3.2. *Suppose that $Z_0 \subseteq X$ is a local complete intersection. Then a deformation of Z_0 inside X can always be extended locally on X, so we have that the first-order deformations are given by $H^0(Z_0, \mathcal{N}_{Z_0/X})$, and there is an obstruction theory with values in $H^1(Z_0, \mathcal{N}_{Z_0/X})$.*

Proof. We first need to see that the deformations of Z_0 inside X are locally unobstructed. This takes a little work, but boils down to the following: any deformation of Z_0 inside X will still be a local complete intersection, and if we want to lift any such deformation locally, we simply lift a minimal set of generators of the defining ideal.

Now, in our case we have $\mathscr{G} = \mathcal{I}_{Z_0}$ and $\mathcal{Q}_0 = \mathscr{O}_{Z_0} = \mathscr{O}_X/\mathcal{I}_{Z_0}$, so

$$\mathcal{H}om(\mathscr{G}, \mathcal{Q}_0) = \mathcal{H}om(\mathcal{I}_{Z_0}, \mathscr{O}_X/\mathcal{I}_{Z_0}) = \mathcal{H}om(\mathcal{I}_{Z_0}/\mathcal{I}_{Z_0}^2, \mathscr{O}_{Z_0}) = \mathcal{N}_{Z_0/X},$$

so we obtain the desired statement. □

Applying this result to morphisms we find:

Corollary 3.3. *Suppose that Y is smooth over* Spec k*, and we are given a morphism $f_0 : X \to Y$ over* Spec k*. Then first-order deformations of f_0 are parametrized by $H^0(X, f_0^*\mathcal{T}_Y)$, and there is an obstruction theory taking values in $H^1(X, f_0^*\mathcal{T}_Y)$.*

Proof. We need to see that under the stated hypotheses, the graph Γ_{f_0} is a local complete intersection, and its normal bundle is $f_0^*\mathcal{T}_Y$.

We note that both are clear, at least conceptually, when also X is smooth: then Γ_{f_0} is smooth inside a smooth variety, so automatically a local complete intersection, and we have $\mathcal{N}_{\Gamma_{f_0}/X\times_k Y} = \mathcal{T}_{X\times_k Y}/\mathcal{T}_{\Gamma_{f_0}}$, so at each point $(x, f_0(x))$ of Γ_{f_0} we have the tangent space of $X \times_k Y$, which is $T_x(X) \times T_{f_0(x)}(Y)$, and are modding out by $T_x(X)$ (using the isomorphism $p_1 : \Gamma_{f_0} \xrightarrow{\sim} X$), which leaves $T_{f_0(x)}(Y)$.

The proof in general is more formal: Γ_{f_0} is the pullback under $f_0 \times \mathrm{id} : X \times_k Y \to Y \times_k Y$ of the diagonal $\Delta_Y \subseteq Y \times_k Y$. By the smoothness of Y, we have that Δ_Y is a local complete intersection, so it follows that Γ_{f_0} is, also. Moreover, $\mathcal{I}_Y/\mathcal{I}_Y^2 = \Omega^1_{Y/k}$, so $f_0^*\Omega^1_{Y/k} = \mathcal{I}_{\Gamma_{f_0}}/\mathcal{I}^2_{\Gamma_{f_0}}$, and taking duals we find $f_0^*\mathcal{T}_Y = \mathcal{N}_{\Gamma_{f_0}/X\times_k Y}$, as desired. □

Applying Theorem 1.2, we therefore obtain in particular the following explicit description of the local geometry of Hilbert and Hom schemes:

Corollary 3.4. *Suppose we are given X projective over k, and a closed subscheme Z_0 of X. Then we have*

$$h^0(X, \mathcal{N}_{X/Z_0}) - h^1(X, \mathcal{N}_{X/Z_0}) \leq \dim_{Z_0} \mathrm{Hilb}(X/S) \leq h^0(X, \mathcal{N}_{X/Z_0}),$$

and if $H^1(X, \mathcal{N}_{X/Z_0}) = 0$, we have that $\mathrm{Hilb}(X/S)$ is smooth at Z_0.

Corollary 3.5. *Suppose we are given X and Y projective over k, with X flat and Y smooth. Let $f_0 : X \to Y$ be a morphism. Then we have*

$$h^0(X, f_0^*\mathcal{T}_Y) - h^1(X, f_0^*\mathcal{T}_Y) \leq \dim_{f_0} \mathrm{Hom}(X, Y/S) \leq h^0(X, f_0^*\mathcal{T}_Y),$$

and if $H^1(X, f_0^*\mathcal{T}_Y) = 0$, we have that $\mathrm{Hom}(X, Y/S)$ is smooth at f_0.

References

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