A QUICK OVERVIEW OF INTERSECTION THEORY

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1. Three problems

Let's consider three problems:

Question 1.1. In how many points do two plane curves intersect?

Question 1.2. How many lines in three-space meet four fixed general lines?

Question 1.3. How many plane conics are tangent to five fixed general conics?

The answer to the first question is of course given by Bezout's theorem: the number of points, if counted with the appropriate multiplicity, is the product of the degrees of the curves in question. Moreover, one can use a Bertini's theorem argument (using the same trick as Example 8.20.2 of [2]) to show that if we fix the first curve, but let the second curve be general of a given degree, the intersection will be transverse – that is, will consist only of reduced points.

The second question is the first non-trivial example of Schubert calculus, a classical collection of enumerative problems involving counting linear spaces which meet fixed linear spaces in prescribed dimension. Despite its disarming simplicity, the question is non-trivial on its face. One can see rather easily in certain special configurations that the answer is 2, and this turns out to be the correct answer, but the argument is not easy.

The third question at first looks easier: plane conics are parametrized by nonzero homogeneous quadratic polynomials in three variables, up to scaling; this is a $\mathbb{P}^5$. Being tangent to a fixed conic is a hypersurface in this $\mathbb{P}^5$, and it is not difficult to compute explicitly that the hypersurface is of degree 6. We are thus intersecting 5 hypersurfaces of degree 6, and it seems to follow from our generalized Bezout's theorem that the answer is $6^5 = 7776$. However, this is not correct! The subtlety is that the degree-6 hypersurface we obtain from a general conic is not a general hypersurface of degree 6. In fact, we see that inside our $\mathbb{P}^5$ of conics, there is a surface of non-reduced conics, the doubled lines. Each of these will be tangent to every curve in the plane. Hence, although we are intersecting five hypersurfaces in $\mathbb{P}^5$, the intersection is not finite, but rather contains a surface! This is a first example of what is called an excess intersection problem, and the answer we want is the number of points of intersection lying outside this surface. This turns out to be 3264, which means that the surface lying inside the intersection should somehow be considered to count for 4512 points.

2. Intersection theory and its merry band of power tools

All three of the problems discussed above are prime examples of **intersection theory**. The first problem is of course stated explicitly in terms of intersections, while the second and third are stated as problems in **enumerative geometry**, i.e., in counting geometric objects. However, we quickly rephrased the third problem into a question of intersections in $\mathbb{P}^5$, considered as a moduli space of plane conics, and similarly we can rephrase the second question as an intersection problem in the Grassmannian. This highlights a general theme: intersection theory is intimately related to, and often motivated by, enumerative geometry.

Enumerative geometry is of course a classical subject that played a central role in the development of the classical Italian school of algebraic geometry. It comes up in a surprising number of contexts, and was even used by Harris, Mumford, and Eisenbud in the 1980's to prove a series of fundamental qualitative statements on the geometry of moduli spaces of curves. However, enumerative geometry arguments classically relied on heuristic arguments that could not be backed up with any rigor: most notable was the *principle of continuity*, which asserts that the answer to an enumerative problem would remain invariant as one varied the problem: for instance, moving the four general lines in Question 1.2 above, or the five conics in Question 1.3. However, without rephrasing the enumerative question as a question in intersection theory, there was typically no way to justify such a principle.

However, the reach of intersection theory extends far beyond enumerative geometry. Although the order of definitions and constructions varies from approach to approach, intersection theory has gone hand in hand with the theories of Chern (and Segre) classes, degeneracy loci, and generalizations of the Riemann-Roch theorem. The Chern class of a vector bundle can be thought of as describing the loci where collections of sections become linearly dependent. Similarly, given a map of vector bundles, the degeneracy locus of the map is the locus where the rank drops. The Porteous formula is a classical and powerful formula expressing the degeneracy locus of a map of vector bundles in terms of the Chern classes of the bundles. Finally, the Riemann-Roch theorem and its generalizations to vector bundles on higher-dimensional varieties can be expressed as an intersection-theoretic formula relating the Euler characteristic of a vector bundle to its Chern classes, and the Chern classes of the base variety.

Of course, we have already seen the variety of applications of the Riemann-Roch theorem in even the basic theory of curves, so one can imagine that its generalizations are equally important. Taken together, these topics constitute a striking number of the most important tools in classical algebraic geometry. For a coherent development of all of these topics (written – and indeed in its present form developed – after Hartshorne [2]), see Fulton [1]. An earlier summary, without proofs, is of course provided in Appendix A of [2].

3. Intersection theory on surfaces

To get some idea of what intersection theory should look like, we begin with the case of surfaces, and more specifically Bezout's theorem. The key feature here is that if we are only interested in the number of intersection points, the specific curves we are intersecting are (almost) irrelevant; as long as they intersect in the correct dimension, all that matters is their degrees. We can rephrase this by considering curves of degree d to be the divisors associated to sections of the line bundle $\mathscr{O}(d)$ for each d, and we are then saying that linearly equivalent divisors yield equivalent intersections.

This then generalizes handily to surfaces:

Theorem 3.1. *If X is a smooth projective surface, there exists a unique intersection pairing*

$$\cdot : \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$$

which is symmetric and additive, depends only on linear equivalence classes of divisors, and has the property that if D, D' intersect in a finite set of points, $D \cdot D'$ is given by the length of the intersection.

See Theorem V.1.1 and Proposition V.1.4 of [2] for a proof. We remark that this attaches a number even to divisors which intersect in a one-dimensional set, and in particular assigns a self-intersection number to every divisor D. If D can move freely, so that there is a D' linearly equivalent to D such that $D \cap D'$ is a finite set, then $D \cdot D = D \cdot D'$ is the length of that set, and in particular non-negative. However, it is possible that D does not move, and in that case we can get negative intersection numbers.

Example 3.2. Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$, and let D be a "vertical line". Then D is linearly equivalent to any other vertical line, and any two such lines are disjoint, so we see that $D \cdot D = 0$.

Next consider $X = \mathrm{Bl}_x \mathbb{P}^2$ for some $x \in \mathbb{P}^2$, and let E be the exceptional divisor of the blowup. Then one can check that $E \cdot E = -1$.

The reason that a negative intersection number can happen is that E doesn't move, and that in fact if one adds divisors to E in order to get it to move, and then uses additivity and subtracts off the contributions from the added divisor, one is left with a negative number.

There is a general formula for the self-intersection of a divisor, at least when D is a smooth curve (one can make sense of the formula more generally): $D \cdot D = \deg \mathcal{N}_{D/X}$, where $\mathcal{N}_{D/X}$ is the normal bundle of D inside X, a line bundle on D; see Example 1.4.1 of [2]. The conceptual reason for this is that if $\mathcal{N}_{D/X}$ has a global section s, we obtain a family of normal vectors at every point of D. If we "integrate" s to deform D, we will obtain some D' which continues to intersect D at precisely the points where s vanishes. Thus, $D \cdot D = D \cdot D' = \deg \mathcal{N}_{D/X}$. Of course, in the category of algebraic varieties this is nothing more than heuristic reasoning, but it turns out to produce the correct answer.

4. The nature of a general intersection theory

Intersection theory in general attempts to generalize these ideas to higher-dimensional varieties, and higher-codimensional subvarieties. As before, we want to impose an equivalence relation on subvarieties in the hopes of obtaining more tractable computations. Ultimately intersection theory constructs a ring, the Chow ring, which has properties quite similar to the homology ring of algebraic topology. When we are lucky, this ring will be finitely presented, and we will be able to give a presentation and therefore work quite effectively with explicit computations on subvarieties and their intersection.

The preliminary formal definitions are as follows:

Definition 4.1. A **cycle** of codimension r on a variety X is a formal sum of subvarieties of X of codimension r.

Remark 4.2. For any subscheme $Z \subseteq X$ of codimension r, we can associate a cycle as the sum of the components of Z of codimension r, which coefficients given by the multiplicity of X along each component.

A key distinction between surfaces and the general case is that in general, we will work with all (or at least a large class of) varieties simultaneously, and in particular varieties of different dimensions. This allows us to leverage our relatively good understanding of divisors into a theory that applies in higher codimension as well. We begin by defining a pushforward operation on cycles of given dimension, which we use to define our equivalence relation.

Definition 4.3. Given $f : X \to X'$, and $Y \subseteq X$ a cycle, we define $f_*(Y)$ to be 0 if $\dim \overline{f(Y)} < \dim Y$, and otherwise set $f_*(Y) = [K(Y) : K(\overline{f(Y)})]\overline{f(Y)}$.

This is a fancy version of taking the image; however, our definition ensures that for every m we get a well-defined map from cycles of dimension m to cycles of dimension m, whether or not the image of a given cycle has full dimension.

Definition 4.4. Given a variety X, we define **rational equivalence** to be, for each r, the equivalence relation on cycles of codimension r generated by the following equivalences: for any $V \subseteq X$ a subvariety, and $\tilde{V}$ its normalization, let $\pi : \tilde{V} \to X$ be the induced map. Then for any D, D' linearly equivalent, we declare $\pi_* D \sim \pi_* D'$ as cycles in X.

We give a rather trivial example:

Example 4.5. Any two points on $\mathbb{P}^2$, and more generally on $\mathbb{P}^n$, are rationally equivalent, since we can set V in the above definition to be the line connecting them.

However, by the same token we note that rational equivalence is a rather strong condition: two points can be rationally equivalent only if there is some curve on which they are rationally equivalent, which requires that they can be connected by a chain of rational curves. However, there are many varieties with no rational curves on them at all, and for these we see that no two points are rationally equivalent. This is addressed somewhat by the weaker equivalence notion of **algebraic equivalence**, with which one can also make a viable intersection theory. See Exercise V.1.7 of [2] for the notion of algebraic equivalence of divisors on a surface.

Definition 4.6. Let $A^r(X)$ be the group of cycles of codimension r, up to rational equivalence. Set $A(X) = \oplus_r A^r(X)$.

We will obtain a ring structure on $A(X)$ from maps $A^r(X) \times A^s(X) \to A^{r+s}(X)$ provided by intersection theory. $A(X)$ is then called the **Chow ring** of X.

Theorem 4.7. *There is a unique intersection theory on the class of smooth, quasiprojective varieties, satisfying certain natural axioms which generalize those of Theorem 3.1, and which describe the relationship between intersection and pushforward of cycles.*

The axioms can be simplified considerably with the introduction of a notion of **pullback** of cycles, using the intersection maps and the product $X \times_k X'$.

Definition 4.8. Given $f : X \to X'$, and Y' a cycle on X', define $f^*(Y') = p_{1*}(\Gamma_f \cdot p_2^{-1}(Y'))$.

Note that set-theoretically this is essentially just the preimage of Y' inside X. For a list of suitable axioms, see §A.1 of [2].

Historically, the proof of Theorem 4.7 involved two major components: first, one needed a good notion of intersection multiplicity; and second, one needs a "moving lemma" so that a normalization axiom which specifies the intersection maps for cycles meeting transversely is enough to characterize the intersections of arbitrary cycles.

Both of these turn out to be rather subtle. For intersection multiplicities, it is not enough in general to consider the intersection scheme alone; the answer can depend also on the subvarieties being intersected. Serre gave a compact but technical definition of multiplicities in terms of Tors. However, we mention that whenever we are intersecting Cohen-Macaulay subvarieties which meet in the expected codimension, the more technical notions of intersection multiplicity will coincide with the naive notion obtained by considering lengths, and in particular in this case it does suffice to look only at the intersection scheme, independent of the subvarieties being intersected.

The idea of the moving lemma is that given two cycles, one ought to be able to find linearly equivalent cycles which meet transversely, and whose intersection is therefore determined. It is of course not necessarily the case, as we already saw in the case of surfaces, that any given cycle will move, so in general if we are given effective cycles Y and Z, one must show that we can find effective cycles Z' and $Z'' \sim Z' + Z$ such that Y meets Z' and Z'' transversely; one can then deduce the value of $Y \cdot Z$.

We conclude by mentioning that Fulton and MacPherson turned the classical arguments on their heads, constructing an intersection theory (which worked in a considerably more general setting) directly without either defining intersection multiplicities or proving a moving lemma, and then comparing their results to the classical case. This is the approach exposed by Fulton in [1].

References

1. William Fulton, *Intersection theory*, second ed., Springer-Verlag, 1998.
2. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.