A BRIEF SKETCH OF THE MINIMAL MODEL PROGRAM

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1. Classification of varieties

The pursuit of general structure theorems on algebraic varieties often leads naturally to questions of classification. One approach to such theorems is via the notion of birational equivalence, and more specifically, can be broken into the following two steps:

(i) Understand the relationship between any two birationally equivalent varieties.
(ii) Find natural "simplest" representatives of every birational equivalence class, called a **minimal model**.

One can easily imagine that if such a program were carried out, it would constitute a powerful tool in proving structure theorems. Indeed, one could first prove a theorem for the minimal models provided by (ii), which are intended to be easier to work with, and then use (i) to deduce the statement for arbitrary varieties.

A hint of the complexity of (ii) is that the following is already a very deep and powerful theorem:

Theorem 1.1. *(Hironaka, 1964) Every birational equivalence class of varieties over a field of characteristic* 0 *contains at least one smooth projective variety.*

This is **resolution of singularities**, and Hironaka earned himself a Fields medal by proving this statement (in a much sharper form) in [2]. In characteristic p, the question (even in its weak form) remains open, at least for the present. This result is important not only for the study of singular varieties, but also for the study of smooth, non-proper varieties, and hence arises in many other contexts as well, as we see shortly.

Part (i) has recently been given a rather complete answer:

Theorem 1.2. *(Wlodarczyk, 2003* [4]*) Let* $f : X \dashrightarrow Y$ *be a birational map of smooth proper varieties over a field of characteristic* 0*. Then* f *may be factored into a sequence of blowups and blowdowns along smooth subvarieties.*

Moreover, every intermediate subvariety can be chosen to be smooth, and if f *is an isomorphism on an open subset* U*, then the blowups and blowdowns can be chosen to occur on the complement of* U*.*

This theorem is not our focus, but we mention that its proof relies heavily both on resolution of singularities (hence the requirement for characteristic 0), and on the theory of toroidal varieties and various refinements thereof. We also mention that the above result is known as the "weak factorization theorem" because a refined version is conjectured: namely, that f may always be factored as a sequence of blowups, followed by a sequence of blowdowns. However, it is unclear whether there are applications requiring the stronger statement.

For the remainder of our discussion, we focus on question (ii) above, the existence of minimal models. We will make no serious attempt to give proper historical attribution for the various ideas mentioned, leaving that instead to more systematic surveys such as [3].

We assume throughout that we are working with varieties over an algebraically closed field of characteristic 0. For curves and surfaces, this restriction is unnecessary, but for higher-dimensional varieties the need for extensive use of resolution of singularities has been a stumbling block to progress in positive characteristic, although we do mention that some current work of Cascini, Hacon and McKernan is attempting to study the minimal program in positive characteristic under the hypothesis of resolution of singularities.

2. Curves and surfaces

We now consider the question of producing, in each birational equivalence class of varieties, a minimal model which should be in some sense the simplest in its class. This is more or less vacuous in the case of curves, and comparatively straightforward in the case of surfaces, although even there we will see that the geometry of the surfaces in a given class will affect what we can say about a minimal model.

The case of curves. We begin by recalling the following answer for the case of curves:

Theorem 2.1. *There is a unique smooth, proper (equivalently, projective) curve in each birational equivalence class.*

In modern terms, the uniqueness follows easily from the valuative criterion for properness, from which one concludes that any rational map from a smooth curve to a proper one extends to a morphism, and therefore that any birational map between smooth proper curves is an isomorphism. Existence is even easier, as one starts with any curve, takes the closure of an affine piece in projective space, and then normalizes to obtain a smooth projective model.

Thus, minimal model theory has nothing to tell us about smooth curves (although we nonetheless frequently study singular curves via their normalizations, which has a similar spirit).

Producing minimal surfaces. The theory for surfaces, although still classical, has considerably more substance. For the classical argument (as given in Chapter V of [1]), understanding minimal models is closely tied to understanding birational maps, and the strong factorization conjecture for the case of surfaces is in fact an old theorem of Zariski.

A preliminary theorem is the following:

Theorem 2.2. *Every surface is birationally equivalent to a smooth projective surface.*

However, unlike the case of curves, the smooth projective model is far from unique. Indeed, we know that given any smooth projective surface, we can blow up any point to obtain another smooth projective surface, birationally equivalent to the first. This motivates the need for a good theory of minimal smooth projective surfaces, i.e., minimal models. We use slightly nonstandard terminology here, because the standard terminology is conflicting:

Definition 2.3. A smooth projective surface X is **classically minimal** if every birational morphism $f : X \to X'$ to some smooth projective surface X' is an isomorphism.

It is helpful to have an explicit description of which surfaces are classically minimal, and here we first have (in what will become a theme) rational curves and intersection theory playing an important role.

Definition 2.4. Let X be a smooth projective surface, and E a curve of X. We say that E is a (-1)**-curve** if $E \cong \mathbb{P}^1$, and $E \cdot E = -1$.

It is not difficult to check that if X is the blowup of a smooth surface at a point, and E is the exceptional divisor, then E is a (-1)-curve. In fact, this is an equivalence:

Theorem 2.5. *A curve E on a smooth projective surface X is a (-1)-curve if and only if it is the exceptional divisor of the blowup of a smooth surface at a point.*

Thus, every (-1)-curve can be **contracted**, yielding a new smooth surface in which the original curve maps to a point. See Theorem V.5.7 of [1].

It is thus clear that if a surface has a (-1)-curve, it cannot be classically minimal. It turns out that this too is an equivalent condition.

Theorem 2.6. *Every smooth projective surface X admits a birational morphism $f : X \to X'$, with X' classically minimal. Moreover, X' is classically minimal if and only if there is no (-1)-curve on X'.*

See Theorem V.5.8 of [1]. As mentioned earlier, the proof of this theorem relies heavily on the strong factorization theorem for surfaces, which is Theorem V.5.5 of *loc. cit.*

Classifying minimal surface. This of course leaves open the question of uniqueness and classification of minimal models. To complete our picture of surfaces, we describe in more detail the classically minimal models. We need some preliminary definitions:

Definition 2.7. A variety X is **rational** if it is birationally equivalent to projective space.

A smooth projective surface X is **ruled** if it admits a morphism $f : X \to C$ to some smooth curve C such that each fiber is isomorphic to $\mathbb{P}^1$. X is **birationally ruled** if it is birationally equivalent to a ruled surface.

A special case of the first definition is that a proper integral curve is rational if its normalization is isomorphic to $\mathbb{P}^1$. Rational curves play a vital role in minimal model theory, not only for surfaces but in higher dimensions as well.

We can construct ruled surfaces explicitly by taking the projective bundle associated to any locally free sheaf of rank 2 on any C. In fact this in an equivalence, and in particular, there is the following classification of surfaces which are both rational and ruled:

Theorem 2.8. *Every ruled surface over a curve C is the projective bundle associated to $\mathcal{E}$ for some locally free sheaf $\mathcal{E}$ of rank 2 on C. Furthermore, any X which is rational and ruled is a Hirzebruch surface X_n, where for any $n \geq 0$, we define X_n to be the projective bundle associated to $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n)$.*

These are cases where X is covered by rational curves; we consider the geometry of X to be relatively simple, although there will be a cost in non-uniqueness of minimal models:

Theorem 2.9. *Let X be a smooth projective surface. Then exactly one of the following holds.*

(i) *X is not birationally ruled, and there is a unique classically minimal surface birationally equivalent to X.*
(ii) *X is birationally ruled and rational, and the classically minimal surfaces birationally equivalent to X are $\mathbb{P}^2$, and the Hirzebruch surfaces X_n, for $n = 0$ and $n > 1$.*
(iii) *X is birationally ruled over a curve C which has positive genus, and the classically minimal surfaces birationally equivalent to X are the ruled surfaces over C.*

See V.5.8.2-V.5.8.4 of [1] for partial explanations and further references.

Themes for the case of surfaces. To summarize, the classical situation for surfaces presents the following themes:

(i) Understanding minimal models is a consequence of understanding birational maps.
(ii) To move from more complicated to simpler surfaces, we contract rational curves with negative intersection numbers.
(iii) There is a posthoc dichotomy between surfaces with well-behaved (i.e., unique) minimal models, and surfaces which can be described well in terms of curves.

3. Mori and the case of threefolds

Although at least the statements for surfaces were well-understood by Italian school in the early 20th century, there was not even a conjecture for the behavior of threefolds until Mori's groundbreaking work in the 1980's, which earned him a Fields medal. To grossly oversimplify, the reason for this delay was that not only did fundamentally new ideas have to be introduced, but the approach had to revamped in such a way that Mori's program, when applied back to surfaces, not only completely reformulated the main results, but even changed the idea of what a minimal model should be.

From surfaces to threefolds. The ruthless contraction of rational curves with negative intersection numbers still plays a central role in higher dimensions. However, rather than attempting to find a minimal model in each birational equivalence class, Mori's approach to understanding higher-dimensional varieties calls for a combination of minimal models and induction on dimension, with the latter carried out by producing especially nice fibrations. The key differences can be summarized as follows:

(i) The posthoc dichotomy from the case of surfaces is built fundamentally into the techniques.
(ii) Both the construction a minimal model via birational contractions and the construction of nice fibrations are carried out within a single framework.

(iii) The canonical bundle K_X, which only lurked in the background of the surface case, is elevated to a central role, used to evaluate when curves should be considered "negative".
(iv) It turns out we have to allow our minimal models to be singular.

Definitions and statements. In order to state the final theorem, we need some preliminary definitions.

Definition 3.1. A smooth, projective variety is **Fano** if the anticanonical bundle (usually denoted $-K_X$, following divisorial notation) is ample.

A morphism $f : X \to Y$ is a **Fano contraction** if the fibers of f are positive-dimensional, and the general fiber is Fano.

Example 3.2. The prototypical Fano varieties are projective spaces, and similarly the map from a projective space bundle to its base gives a Fano fibration.

Example 3.3. Among curves, the only Fano variety is $\mathbb{P}^1$. Going further, we have the basic trichotomy, that elliptic curves are those curves with K_X trivial, and curves of genus ≥ 2 (often called hyperbolic) are those with K_X ample.

The notion of a Fano variety is very classical, and the idea is that Fano varieties should be very special, and relatively easy to work with, while "most" varieties have K_X closer to being ample than $-K_X$, and are correspondingly more complicated.

We now need some definitions related to singularity theory.

Definition 3.4. A variety X is **$\mathbb{Q}$-factorial** if for every effective Weil divisor D, some multiple mD is an effective Cartier divisor.

This is a condition on the singularities of X; in particular, if X is smooth it is automatically $\mathbb{Q}$-factorial. Note that if X is $\mathbb{Q}$-factorial, we can always define the intersection number of a Weil divisor with a curve: indeed, it turns out that one can always define the intersection number of a curve with an effective Cartier divisor, and hence we can set $D \cdot C = \frac{1}{m}(mD \cdot C)$, where mD is an effective Cartier divisor. This is in general a rational number, but we will not allow this to concern us.

We also mention that K_X will still be an effective Weil divisor if X is $\mathbb{Q}$-factorial, so we have a notion of $K_X \cdot C \in \mathbb{Q}$, for any curve C in X.

Another condition on the singularities is the notion of having only **terminal** singularities. The definition of this is more technical, but can nonetheless be expressed in an elementary manner; see Definition 11.9 of [3].

Finally, we have a notion of positivity which is related to, but weaker than ampleness:

Definition 3.5. A divisor D on a $\mathbb{Q}$-factorial variety X is called **nef** if for every curve C on X, we have $D \cdot C \geq 0$.

With these preliminary definitions out of the way, we can define minimal models and state a version of Mori's theorem:

Definition 3.6. A projective variety X is a **minimal model** if it is $\mathbb{Q}$-factorial, has only terminal singularities, and has K_X nef.

Mori initiated and completed the proof of the following theorem, although many other people were involved in between.

Theorem 3.7. *Let X be a projective variety, with $\dim X \leq 3$. Then X is birationally equivalent to some X' which is $\mathbb{Q}$-factorial, projective, has only terminal singularities, and satisfies exactly one of the following two conditions:*

(I) *There is a Fano contraction $f : X' \to Y$.*
(II) *X' is a minimal model.*

Remark 3.8. This gives a new interpretation of the case of surfaces. We first comment that for surfaces, there is no such thing as a terminal singularity, so Theorem 3.7 remains in the smooth setting. (I) then corresponds to the case in which X is birationally ruled. If X is not rational, then we have Y a curve of positive genus, and $f : X' \to Y$ having fibers isomorphic to $\mathbb{P}^1$, while if X is rational, we could have $X' \cong \mathbb{P}^2$ and Y simply a point. In this case, classically minimal is not the same as minimal. On the other hand, (II) corresponds to the remaining case, where we had a unique classically minimal model, which will also be a minimal model.

It is natural to ask how minimal models are related. Further work of many people has introduced the notion of a certain class of birational maps called **flops**, and shown:

Theorem 3.9. *Any birational map $f : X \dashrightarrow X'$ of minimal models of dimension 3 is obtained via a sequence of flops.*

The idea of the proof. We now sketch the basic idea of the proof of Theorem 3.7.

In 1982, Mori initiated the minimal model program for threefolds, with an analysis of smooth threefolds X for which K_X is not nef. In this case, there is some curve C' with $C' \cdot K_X < 0$, and Mori first showed that one can use such a negative curve to produce an "extremal" curve C, which will always be rational. He then showed that one can produce a contraction map $f : X \to Y$, contracting C (and possibly many other curves), and he carried out a careful analysis of the possibilities. If $\dim Y < \dim X$, we have a Fano contraction, and are done. If f contracts a divisor in X, one can repeat the process, and a simple numerical argument shows that it must terminate eventually. Finally, it could be a **small contraction**, contracting a curve, in which case Mori shows we are forced to consider singular threefolds in order to proceed.

In the argument for the existence of C, a key step is to produce rational curves on X in a controlled manner. Mori accomplished this by applying his remarkable "bend-and-break" technique, first developed in his 1979 paper proving Hartshorne's conjecture. We will discuss this theory in more detail in a future lecture, but for now we cannot resist mentioning that although the results are for complex varieties, the only known proof involves reduction to characteristic p. In any case, his techniques can also be applied to prove the following, showing that Fano varieties are indeed quite special:

Theorem 3.10. *Suppose that X is a Fano variety, or more generally that $f : X \to Y$ is a Fano fibration. Then rational curves cover a dense open subset of X.*

Following Mori's initial paper, many people then contributed to developing the minimal model program for threefolds, introducing the correct notion of singularities, showing that comparable contractibility theorems hold even with such singularities, and describing what ought to be done in the case of small contractions. It turns out that for a small contraction, one ought to perform an operation called a **flip**; this is like a flop, but assymmetric, and therefore Shokurov was able to show that for threefolds, any sequence of flips must terminate. The final step of the proof of Theorem 3.7 was again carried out by Mori, in 1988, when he proved the existence of flips.

4. The higher-dimensional case

The general concept of the minimal model program, as well as the results constituting several of its steps, generalize to varieties of any dimension. In particular, one expects that Theorem 3.7 should hold verbatim, without the restriction on dimension. The main obstruction to completing the proof in this generality is to prove existence and termination of flips. In 2005, Hacon and McKernan proved that existence of flips in dimension n follows from termination in dimension $n-1$, in the hope of setting up an induction.

Termination of flips remains open, but in 2006 Birkar, Cascini, Hacon and McKernan were able to prove a major result which, while not fully proving the minimal model program, is nonetheless strong enough to deduce many of its consequences. Their paper is built around an idea which goes back many years, but had not played a key role in the proof of the minimal model program in dimension 3.

Definition 4.1. For a variety X, define the **canonical ring** of X by

$$R(X) = \oplus_{m \geq 0} H^0(X, K_X^{\otimes m}).$$

This ring is a birational invariant for smooth varieties, and therefore, if it is finitely generated, one can take its Proj, and this would be an alternate approach to producing either a minimal model (more precisely, a variant known as a **canonical model**, with slightly worse singularities) or a fibration. The former case would occur when X has **general type**, i.e. for some $m > 0$, the image of the rational map from X to projective space determined by $K_X^{\otimes m}$ has the same dimension as X. Traditionally, this had been seen as more or less in competition with the minimal model progam, although it was known that results on either side would imply results on the other. For instance, Mori concluded that the canonical ring of a threefold was finitely generated as a consequence of Theorem 3.7.

The main insight of Birkar, Cascini, Hacon and McKernan was that by blending the approach suggested by the minimal model program with a study of the canonical ring, and using the Hacon-McKernan theorem on existence of flips in a key way, it is possible to construct an induction argument which proves fundamental results in both arenas: in particular, they are able to prove that if X is of general type, the canonical ring is finitely generated, and X has a minimal model.

References

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