AN ABRIDGED EXPOSITION OF MORI'S BEND AND BREAK ARGUMENT

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1. The statement and overview of proof

Mori's "bend and break" argument was originally introduced in 1979 in order to prove Hartshorne's conjecture, showing that (at least in characteristic 0) one can recognize projective spaces as precisely the smooth projective varieties with ample tangent bundles (here for a vector bundle $\mathcal{E}$ to be ample means that if $\mathbb{P}(\mathcal{E})$ is the associated projective bundle, $\mathscr{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample). He then applied the same arguments shortly thereafer in his groundbreaking work on the structure of threefolds. Mori's argument stands out as one of the most spectacular instances of proving a theorem for complex varieties using reduction to positive characteristic, and we begin by stating and outlining the proof of his result.

Recall the following terminology: a **rational curve** is an integral curve with normalization isomorphic to $\mathbb{P}^1$; a **Fano** variety is a smooth projective variety with $-K_X$ ample; and a divisor D is **nef** if and only if $D \cdot C \geq 0$ for every curve $C \subseteq X$.

Theorem 1.1. *(Mori, 1979) Let X be a Fano variety. Then X contains a rational curve.*

In fact, Kollar has observed that Mori's argument can be strengthened to show that X is covered by rational curves.

The proof may be summarized as follows. First, using deformation theory and the existence of the Frobenius map, show the following:

Theorem 1.2. *If X is a smooth projective variety over an algebraically closed field k of characteristic $p > 0$, and K_X is not nef, then X contains a rational curve.*

Next, one wants to conclude that if X has $-K_X$ ample, then it has rational curves, because it has rational curves modulo p for all primes p. However, in order to make this argument work, one needs a bound on the degrees of the rational curves which is independent of p. Such a bound is obtained by using deformation theory to show that if the degree is too high, one can always deform the curve until it breaks into a reducible union of curves of smaller degree. Specifically, Mori showed:

Theorem 1.3. *Let X be a smooth projective variety, and C a rational curve in X. If $-K_X \cdot C \geq \dim X + 2$, then C may be deformed to a union of at least two rational curves.*

Note that here the union may be in a scheme-theoretic sense: we allow C to be deformed to a single curve with multiplicity.

It is then not difficult to put together these theorems and prove Theorem 1.1.

2. The deformation theory

Mori considers a slight variant of the Hom scheme we have already discussed. Specifically, given X, Y over S, Mori wants to study morphisms from X to Y which induce a given morphism on some closed subscheme $Z \subseteq X$. For instance, when X is a curve we will specify the image of certain points on X.

We thus start with schemes X and Y of finite type over S Noetherian, and suppose we have a closed subscheme $Z \subseteq X$, and a fixed morphism $p : Z \to Y$. We can define our functor as follows.

Definition 2.1. The functor $\mathcal{H}om_p(X, Y/S)$ associates to each scheme T/S the morphisms $f : X_T \to Y$ over S such that $f|_{Z_T} = p|_{Z_T}$.

We have the following easy observation:

Proposition 2.2. *If X and Z are flat and projective over S, and Y is quasiprojective, then $\mathcal{H}om_p(X, Y/S)$ is represented by a scheme $\operatorname{Hom}_p(X, Y/S)$, which is a closed subscheme of $\operatorname{Hom}(X, Y/S)$.*

Proof. We know in this situation that $\operatorname{Hom}(X, Y/S)$ is representable, as is $\operatorname{Hom}(Z, Y/S)$. The inclusion $Z \hookrightarrow X$ induces a morphism $\operatorname{Hom}(X, Y/S) \to \operatorname{Hom}(Z, Y/S)$, and we also have that p corresponds to a section $S \to \operatorname{Hom}(Z, Y/S)$, which is a closed immersion since $\operatorname{Hom}(Z, Y/S)$ is separated. Then $\mathcal{H}om_p(X, Y/S)$ is represented by the fiber product $\operatorname{Hom}(X, Y/S) \times_{\operatorname{Hom}(Z,Y/S)} S$, and this is a closed subscheme of $\operatorname{Hom}(X, Y/S)$ since p gives a closed immersion $S \to \operatorname{Hom}(Z, Y/S)$. □

Similarly, setting $S = \operatorname{Spec} k$ we can define the corresponding deformation problem:

Definition 2.3. Given $f : X \to Y$ over $S = \operatorname{Spec} k$, the deformations of f (fixing Z) over $A \in \operatorname{Art}(k)$ are the morphisms $f_A : X_A \to Y$ which restrict to f on $\operatorname{Spec} k$, and restrict to p on Z_A.

Proposition 2.4. *Suppose that Y is smooth over k. Then first-order deformations of f (fixing Z) are parametrized by $H^0(X, f^*\mathcal{T}_Y \otimes \mathcal{I}_Z)$, and there is an obstruction theory with values in $H^1(X, f^*\mathcal{T}_Y \otimes \mathcal{I}_Z)$.*

This is proved the same way as the standard statement on deforming morphisms from X to Y; it is not hard to see that deformations fixing Z are exactly those which come from sections on $f^*\mathcal{T}_Y$ vanishing along Z.

Corollary 2.5. *If X and Z are flat and projective, and Y is smooth and quasiprojective over k, and we have a morphism $f : X \to Y$ over k, we have the following dimension bound: $h^0(X, f^*\mathcal{T}_Y \otimes \mathcal{I}_Z) - h^1(X, f^*\mathcal{T}_Y \otimes \mathcal{I}_Z) \leq \dim_f \operatorname{Hom}_p(X, Y/k) \leq h^0(X, f^*\mathcal{T}_Y \otimes \mathcal{I}_Z)$.*

When X is a curve, and Z a divisor, the lower bound is the Euler characteristic of a vector bundle on X, which is computed by the Riemann-Roch theorem for vector bundles. In this case, one can compute:

Corollary 2.6. *If further X is a nonsingular curve of genus g, and Z is a divisor on X, we have*

$$\dim_f \operatorname{Hom}_p(X, Y/k) \geq \chi f^*\mathcal{T}_Y \otimes \mathscr{O}_X(-Z) = -\deg f(K_Y \cdot f(X)) - \dim Y \deg Z + \dim Y(1-g).$$

This is the ultimate result of our deformation theory calculations, and will allow us to produce certain families of curves in X necessary for Mori's arguments.

3. Producing rational curves in positive characteristic

We now apply Corollary 2.6 to prove Theorems 1.2 and 1.3. The idea is to produce rational curves using rational maps:

Lemma 3.1. *Suppose that $S \dashrightarrow X$ is a rational map from a smooth surface to a variety, which cannot be extended to a morphism. Then X contains a rational curve.*

Proof. It is not difficult to see that the rational map to X can be resolved to a morphism $\widetilde{S} \to X$ by a sequence of blowing up points, and then it follows from Zariski's main theorem (see the version given as Theorem V.5.2 of [1]) that the exceptional divisor of the final blowup is a rational curve in $\widetilde{S}$ which has a nonconstant map to X, giving the desired rational curve in X. □

Proof of Theorem 1.2. We have by hypothesis that K_X is not nef, so there exists some curve $Z \subseteq X$ such that $K_X \cdot Z < 0$. The strategy is as follows: we will produce a morphism $f : C \to X$ with image Z, where C is a nonsingular projective curve, such that f fits into a family of morphisms $C \to X$ with base D, where D is a nonsingular *non*projective curve. We then get a morphism $C \times D \to X$ which does not extend to a morphism $C \times D' \to X$, where D' is the nonsingular compactification of D. This completes the proof of the theorem by the above lemma.

It is therefore sufficient to produce C and the asserted family of morphisms over D. We see from Corollary 2.6 that if $\tilde{Z}$ is the normalization of Z, to make the dimension of the Hom scheme positive, we'd like to increase $\deg f$ without increasing the genus of $\tilde{Z}$. But this is possible in positive characteristic, using the Frobenius morphism. We can take D to be any curve in the resulting Hom scheme, and it follows that D is not proper by the rigidity lemma below. □

Lemma 3.2. *(Rigidity) Let X, Y, Z be varieties, with X proper, and suppose we have a morphism $f : X \times Y \to Z$, such that $f(X \times \{y\})$ is a point for some $y \in Y$ (i.e., f induced the constant map $X \times \{y\} \to Z$. Then f induces the constant map for all $y \in Y$; that is, f factors through $p_2 : X \times Y \to Y$.*

Proof of Theorem 1.3. The proof of this is more involved, so we settle for a summary. Roughly, the role of deformation theory is similar to the previous proof, except that instead of using positive characteristic and the Frobenius morphism to obtain a positive-dimensional family of maps, we use our hypothesis on the intersection number. Here, Mori uses maps with two points fixed instead of one, and via a careful study of a one-dimensional family of such maps, shows that if it is extended to a proper family, some fibers will have to be reducible. That is, the deformation constitutes "bending" the rational curve, and in the limit it "breaks" into a reducible curve. □

4. Reduction argument

The basic general lemmas we need to conclude the argument are the following:

Lemma 4.1. *Suppose that S is of finite type over $\operatorname{Spec} \mathbb{Z}$, and $f : X \to S$ is a morphism of finite type. Suppose that $f(X)$ contains every $s \in S$ such that $\operatorname{char} \kappa(s) \neq 0$. Then f is surjective.*

Proof. One proves this by multiple applications of Chevalley's theorem on constructibility of the image, first to show that every closed point of S must have residue field with positive characteristic, and then to conclude that if $f(X)$ contains every closed point of S, it must be all of S. □

Note that this statement is blatantly false without the finite type hypotheses.

Lemma 4.2. *Suppose X is smooth and projective over a field k of characteristic 0. Then there is a model X' of X, smooth and projective over some* $\operatorname{Spec} R$*, with $R \subseteq k$ finitely generated over $\mathbb{Z}$, and such that $X' \times \operatorname{Spec} k \cong X$. Furthermore, if $\mathscr{L}$ is ample on X, we can choose R and X' so that $\mathscr{L}$ extends uniquely to a line bundle $\mathscr{L}'$, and such that $\mathscr{L}'$ is ample in all fibers of X'.*

Proof. Since X is projective, it can be defined as the zero set of a finite collection of polynomials with coefficients in k. Let R' be the ring generated by the consequent finite collection of coefficients. We then have R' finitely generated over $\mathbb{Z}$, and X can be extended to a projective variety over $\operatorname{Spec} R'$ using the same defining polynomials. Finally, by openness of smoothness, we can take an affine open set $\operatorname{Spec} R \subseteq \operatorname{Spec} R'$ over which we obtain the desired smooth projective X'. Now, ampleness in fibers is an open condition, and it is not hard to see that at least locally on $\operatorname{Spec} R$ we can extend $\mathscr{L}$ uniquely to a line bundle $\mathscr{L}'$, so we conclude that after a possible further localization of R, we have all the desired conditions. □

We can now conclude the proof of the main theorem.

Proof of Theorem 1.1. If $-K_X$ is ample, we first produce a model X' over $\operatorname{Spec} R$ as in the lemma above, and the extension of $-K_X$ to X' will be $-K_{X'/R}$, the inverse of the relative canonical bundle. Choose any $\mathfrak{p} \in \operatorname{Spec} R$ with positive residue characteristic. Then $X'_{\mathfrak{p}}$ is Fano and has characteristic p, so by Theorem 1.2 there is a rational curve C' on $X'_{\mathfrak{p}}$. Moreover, we claim that Theorem 1.3 implies there is a rational curve C with $-K_X \cdot C \leq \dim X + 1$. Indeed, if C' does not satisfy this bound, we have that C' is algebraically equivalent to some union of rational curves $\sum_{i=1}^m C_i$ for $m \geq 2$, and since $-K_X$ is ample, we have $-K_X \cdot C_i > 0$, so

$$-K_X \cdot C_1 = -K_X \cdot C - \sum_{i=2}^{m} (-K_X \cdot C_i) < -K_X \cdot C,$$

so by induction we obtain the desired statement.

Let Y be the set of components of $\operatorname{Hom}_p(\mathbb{P}^1, X')$ corresponding to nonconstant f with $-K_{X'/R} \cdot f(\mathbb{P}^1) \leq \dim X + 1$; we have produced a point on Y for infinitely many p, so if we show that Y has finite type, Lemma 4.1 implies that Y has a point over every point of $\operatorname{Spec} R$, and we are done. But one can check that the bound on $-K_{X'/R} \cdot f(\mathbb{P}^1)$, because $-K_{X'/R}$ is ample, implies that only finitely many Hilbert polynomials of graphs occur, so we conclude that Y is of finite type, as desired. □

References

1. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.