A PITHY LOOK AT THE QUOT, HILBERT, AND HOM SCHEMES

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1. The moduli functors

Moduli spaces have played a central role in algebraic geometry, and one of the most basic moduli spaces has been the space of subvarieties of a given variety, called the Hilbert scheme. Another useful moduli space is the Hom scheme, parametrizing morphisms between a pair of fixed varieties. It turns out that both of these are representable quite generally, so we have a fine moduli scheme. The construction uses special cases of Grothendieck's Quot scheme, which is a generalization of the Grassmannian. We begin by giving precise definitions of the functors in question.

Here we work always in the category of schemes over a fixed scheme S, and if we have X/S, and some other T/S, we denote by X_T the base change $X \times_S T$.

Definition 1.1. Given schemes X over S, and $\mathscr{F}$ a quasicoherent $\mathscr{O}_X$-module, the **Quot functor** $\mathcal{Q}uot(\mathscr{F}/X/S)$ is the functor assigning to an S-scheme T the set of quotients $\mathcal{Q}$ of $\mathscr{F}_T$ which are quasi-coherent and flat over T, where $\mathscr{F}_T$ is the pullback of $\mathscr{F}$ to X_T.

Here we consider two quotients $\mathscr{F}_T \twoheadrightarrow \mathcal{Q}$ and $\mathscr{F}_T \twoheadrightarrow \mathcal{Q}'$ to be equivalent if there is an isomorphism $\mathcal{Q} \overset{\sim}{\to} \mathcal{Q}'$ commuting with the quotient maps.

Note that this generalizes the Grassmannian in two ways: first, we are allowed to work on arbitrary schemes over S, instead of on S itself, and second, we impose very few conditions on $\mathscr{F}$.

Remark 1.2. Here we use quotients rather than subsheaves of $\mathscr{F}$ largely because surjectivity of sheaf maps is preserved under base change, while injectivity (in general) is not. Also, the key flatness condition turns out to be for the quotient (recall that it is not enough, even for the Grassmannian, to assume that the subsheaf is flat). The price is that we have to work with an equivalence relation.

An alternative definition to avoid the equivalence relation would be in terms of subsheaves whose quotients are flat.

The definitions of the functors represented by the Hilbert and Hom schemes are the following:

Definition 1.3. Given schemes X over S, the **Hilbert functor** $\mathcal{H}ilb(X/S)$ is the functor assigning to an S-scheme T the set of all closed subschemes of $X_T := X \times_S T$ which are flat over T.

Definition 1.4. Given schemes X and Y over S, the **Hom functor** $\mathcal{H}om(X, Y/S)$ is the functor assigning to an S-scheme T the set of morphisms $f : X_T \to Y$ over S.

The Quot and Hilbert schemes play an even more important role in moduli theory than their definitions might suggest: they play a key role also in constructions of moduli spaces of abstract objects, such as schemes (not imbedded in a larger scheme), or vector bundles (not arising as quotients of a fixed vector bundle). For instance, to construct a moduli space of curves (either as a coarse moduli space or a stack), one might first show that all curves arise in Hilbert schemes, and then realize the desired moduli space as a quotient of an appropriate Hilbert scheme obtained by equated two curves which are isomorphic but imbedded differently.

Remark 1.5. One might ask: what happens if we are interested in the classical picture, i.e., smooth varieties rather than schemes? This can be interpreted in two senses, and has two different answers:

First, because smoothness is an open condition in families, we will have an open subfunctor of $\mathcal{H}ilb(X/S)$ parametrizing smooth subvarieties of X, at least as long as X is proper over S.

Second, even if we are only interested in smooth subvarieties (and even if we set $X = \mathbb{P}^n$), it turns out that the Hilbert scheme (which is of course uniquely determined by the Hilbert functor) can be very singular – Mumford gave an example of an everywhere nonreduced component for a Hilbert scheme of smooth curves, and more recently Vakil has shown in [4] that arbitrarily bad singularities occur in the same Hilbert schemes. Of course, one could always take the reduced induced structure on the Hilbert scheme, but then one would lose information about the deformation theory of the subvarieties, which is often an important part of the picture.

Remark 1.6. We have followed Grothendieck in not applying additional conditions in the initial definitions of our functors. However, if one wants to deduce representability in the quasiprojective case from representability in the projective case, one ought to impose the further condition that the support of a quotient sheaf or subscheme must be proper over T.

2. Reducing to the Quot scheme

Much like proving NP-completeness, for the most part we only prove actually prove that a single functor is representable, and prove that everything else is representable by reducing to known cases. That single functor is the Quot functor (arguably, the Grassmannian, since the proof for Quot ultimately reduces to the Grassmannian). We now explain why the construction of the Quot scheme will also give us Hilbert and Hom schemes.

Proposition 2.1. *Given any X/S, we have*

$$\mathcal{H}ilb(X/S) = \mathcal{Q}uot(\mathscr{O}_X/X/S),$$

with the equality denoting canonical isomorphism.

Proof. Indeed, closed subschemes Z of X_T are in one-to-one correspondence with quasicoherent ideal sheaves on X_T, and hence quasicoherent quotient sheaves of $\mathscr{O}_{X_T}$. Moreover, flatness of Z over T is by definition equivalent to flatness of the corresponding quotient sheaf, so we get the desired equality. □

The reduction for Hom schemes is slightly subtler, but not too difficult:

Proposition 2.2. *Given X, Y proper schemes over S, with X also flat over S and S locally Noetherian, we have that $\mathcal{H}om(X, Y/S)$ is an open subfunctor of $\mathcal{H}ilb(X \times_S Y/S)$.*

In particular, under these hypotheses if $\mathcal{H}ilb(X \times_S Y/S)$ is representable, then $\mathcal{H}om(X, Y/S)$ is represented by an open subscheme.

Proof. The basic idea is of course that in complete generality, morphisms $f : X_T \to Y$ over T are in correspondence with their graphs, which are (locally closed) subschemes $\Gamma_f \subseteq X_T \times_S Y$ such that $p_1 : \Gamma_f \to X_T$ is an isomorphism. Under the hypothesis that Y is separated, we get that graphs Γ_f are necessarily closed subschemes, and if X is flat then Γ_f is too, so $\mathcal{H}om(X, Y/S)$ is naturally a subfunctor of $\mathcal{H}ilb(X \times_S Y/S)$, without further hypotheses.

It remains to analyze the condition on p_1 to show that under our additional hypotheses we obtain an open subfunctor. This follows immediately from the following background result, applied to $p_1 : Z \to X$, where Z is a closed subscheme of $X \times_S Y$, flat over S by definition of the Hilbert functor. Finally, it is easy to see from the definition that an open subfunctor of a representable functor is representable, by an open subscheme. □

Lemma 2.3. *Suppose we are given X, Y proper schemes over a locally Noetherian base scheme S, with X flat over S, and a morphism $f : X \to Y$ over S. Then the locus of points $s \in S$ such that $f_s : X_s \to Y_s$ is an isomorphism is an open subset U of S, and f is an isomorphism on the preimage of U.*

Proof. This is Proposition 4.6.7 (ii) of [5]. The proof first shows the desired assertion for closed immersions without the flatness hypothesis by direct analysis of the sheaf maps and an application of Nakayama's lemma. The statement for isomorphisms is deduced from a lemma on injectivity of sheaf maps in fibers under the additional flatness hypothesis. □

Remark 2.4. Note that a flatness hypothesis is necessary: otherwise a counterexample is provided by the inclusion of the x- and y-axes into the plane, considered as a morphism over the x-axis; this is an isomorphism only at $x = 0$. This shows that flatness of fibers of a morphism is not an open condition without some flatness hypothesis on the source.

Similarly, without a properness hypothesis we would have examples such as the inclusion of the complement of $xy = 1$ into the plane, which is again an isomorphism only above $x = 0$. This could even happen with f proper: let Y be the plane, and X the disjoint union of the plane with the same hyperbola.

3. Constructing the Quot scheme: statement and consequences

We will prove the existence of the Quot scheme under projective hypotheses. This is not a result of lazyness, but rather the same set of hypotheses used by Grothendieck, who expressed frustration that after having gone to the trouble to remove projective hypotheses throughout his foundational work, he was ultimately unable to prove existence of such basic objects as Quot and Hilbert schemes in the same generality. Moreover, the Hilbert functors for non-projective schemes need not exist; indeed, Hironaka's example of a non-projective threefold (see Example B.3.4.1 of [2]) also gives an example for which the Hilbert functor is not representable.

However, we do remark that the quasiprojective case can be deduced from the projective case (assuming, as remarked earlier, a proper support condition).

We assume we have the following situation:

Situation 3.1. Suppose that X is projective over S, with S Noetherian and $\mathscr{F}$ coherent on X. Fix a very ample line bundle $\mathscr{O}_X(1)$ on X. We work in the restricted category of locally Noetherian schemes over S.

Note that if $\mathscr{O}_X(1)$ is a very ample sheaf on X, it makes sense to talk about the Hilbert polynomial of a quotient of $\mathscr{F}$. Because Hilbert polynomials are invariant in flat families, this will give a decomposition of the Quot scheme into an infinite disjoint union of subschemes, each of which will be projective.

The main existence result is then:

Theorem 3.2. *In Situation 3.1, the functor $\mathcal{Q}uot(\mathscr{F}/X/S)$ is representable by a locally Noetherian scheme* $\mathrm{Quot}(\mathscr{F}/X/S)$*. Furthermore,* $\mathrm{Quot}(\mathscr{F}/X/S)$ *is a countable disjoint union of projective schemes* $\mathrm{Quot}^P(\mathscr{F}/X/S)$ *parametrizing quotients with fixed Hilbert polynomial P.*

Remark 3.3. Note that here we use Grothendieck's definition of projectivity, which is weaker than Hartshorne's. This means that both the hypotheses for the theorem and its conclusion are slightly weaker, although there are other variants using stronger definitions of projective.

Remark 3.4. It is not difficult to show that one obtains the same result in the full category of schemes over S, via standard limit arguments used to remove Noetherian hypotheses.

As a consequence of the theorem and Propositions 2.1 and 2.2, we can conclude:

Corollary 3.5. *Suppose S is Noetherian, and X is projective over S. Then $\mathcal{H}ilb(X/S)$ is represented by a scheme* $\mathrm{Hilb}(X/S)$*, which moreover is a countable disjoint union of projective schemes* $\mathrm{Hilb}^P(X/S)$*, parametrizing subschemes of X with fixed Hilbert polynomial P.*

Suppose that Y is another scheme projective over S, and also X is flat. Then $\mathcal{H}om(X,Y/S)$ is represented by a scheme $\mathrm{Hom}(X,Y/S)$*, and we may write it as a disjoint union of quasiprojective schemes* $\mathrm{Hom}^P(X,Y/S)$ *parametrizing morphisms whose graphs have fixed Hilbert polynomial P.*

Remark 3.6. We make a brief comment on the last statement, since it is not so clear *a priori* what kind of conditions the Hilbert polynomial of a graph of a morphism impose on the morphism. However, these conditions can often be understood quite geometrically: for instance, if X and Y are curves over S, then it is not hard to see that the possible Hilbert polynomials will correspond precisely to the possible degrees of the morphisms from X to Y, so we find that $\mathrm{Hom}(X,Y/S)$ is a disjoint union of quasiprojective schemes parametrizing morphisms of each given degree. Of course, the precise Hilbert polynomial corresponding to each degree will depend on the choice of a very ample line bundle on $X \times_S Y$.

We conclude this section with the statement of the only general positive result on the structure of Hilbert schemes, which was (appropriately enough) proved by Hartshorne in his thesis:

Theorem 3.7. *(Hartshorne) For any polynomial P, the Hilbert scheme* $\mathrm{Hilb}^P(\mathbb{P}^n_S/S)$ *is always connected for S connected.*

Although much work has been done on studying irreducible components of Hilbert schemes, it remains the case that very little is known except in special cases.

4. Constructing the Quot scheme: overview of proof

The construction of the Quot scheme, as is frequently the case for moduli schemes, breaks into three steps: the first step is to identify natural discrete invariants of the situation, and write the problem as a disjoint union over these; the second step is to write each piece as an increasing union of open subfunctors, each of which is shown to be representable; and the final step is to show "boundedness," that for each piece the increasing open subfunctors stabilize, so that one is enough. In fact, if one just wants representability, one could stop after the second step, but in order to conclude that our functor is represented by a disjoint union of schemes of finite type, or something sharper such as projective schemes, the third step is vital, and is often the hardest part.

Here we give an overview of the proof, referring the reader to the original article of Grothendieck [1] or the updated survey by Nitsure [3] for a more detailed account.

As asserted earlier, the first step – stratifying the Quot scheme by Hilbert polynomials – follows easily from the invariance of the Hilbert polynomial in flat families. It will therefore suffice to prove that each $\mathcal{Q}uot^P$ is represented by a projective scheme.

The next step is conceptually fairly simple, but does involve some non-trivial technical points. We first note that we can reduce to the case that $X = \mathbb{P}^m_S$, and $\mathscr{F} = \mathscr{O}^r_X$. Indeed, the first reduction is trivial since X is assumed projective, and quotients of $\mathscr{F}$ are the same on X and on $\mathbb{P}^m_S$. The second reduction is non-trivial, but the point is that if $\mathscr{G} \twoheadrightarrow \mathscr{F}$ is any surjection of coherent sheaves, we have that $\mathcal{Q}uot(\mathscr{F}/X/S)$ is naturally a closed subfunctor of $\mathcal{Q}uot(\mathscr{G}/X/S)$. This follows easily from a general result that for any map between coherent sheaves with the second sheaf flat, there is a closed subscheme representing the vanishing locus of the map. We can then write $\mathscr{F}$ as a quotient of $\mathscr{O}_X(n)^r$ for some n, r, reducing to $\mathscr{F} = \mathscr{O}_X(n)^r$, and we obtain an isomorphism of Quot functors for $\mathscr{O}_X(n)^r$ and $\mathscr{O}^r_X$ by tensoring with $\mathscr{O}_X(-n)$.

Having made this reduction, the main definition is:

Definition 4.1. For each $N \geq 0$, denote by $\mathcal{Q}uot^P_N(\mathscr{F}/X/S)$ the subfunctor of $\mathcal{Q}uot^P(\mathscr{F}/X/S)$ consisting of quotients $\mathcal{Q}$ of $\mathscr{F}$ such that for all $n \geq N$:

(I) (cohomological vanishing) $R^i f_*(\mathscr{F}(n)) = R^i f_*(\mathcal{Q}(n)) = 0$ for all $i > 0$;

(II) (generation by global sections) the natural map $f^* f_* \mathcal{Q}(n) \to \mathcal{Q}(n)$ is surjective.

Of course, one can make this definition more generally: the point however is that with our reduction, we have $\mathscr{F}$ also flat over S, from which it follows from the theory of cohomology and base change that the properties imposed above are preserved under base change. This is necessary in order to obtain a subfunctor.

Moreover, it follows from the theory of cohomology and base change that the cohomological vanishing conditions are open conditions, and Nakayama's lemma implies that the surjectivity condition is likewise open. Finally, Serre's theorem says that for any coherent quotient, and N sufficiently large, both conditions are satisfied. We conclude:

Corollary 4.2. *The subfunctors $\mathcal{Q}uot^P_N(\mathscr{F}/X/S)$ form a nested cover of $\mathcal{Q}uot^P(\mathscr{F}/X/S)$ by open subfunctors.*

The next claim is that each $\mathcal{Q}uot^P_N(\mathscr{F}/X/S)$ can be realized as a subscheme of a Grassmannian. Roughly, the idea is that under our hypotheses, for $n \geq N$ every quotient of $\mathscr{F}$ corresponding via tensoring to a locally free quotient of $\mathscr{F}(n)$, and this gives a quotient of $f_*\mathscr{F}(n)$, also locally free and of rank $P(n)$. Moreover, this association is unique, so we obtain that $\mathcal{Q}uot^P_N(\mathscr{F}/X/S)$ is a subfunctor of the Grassmannian functor $\mathcal{G}r(P(n), f_*\mathscr{F}(n))$. The tricky bit is analyzing which quotients of $f_*\mathscr{F}(n)$ arise as flat quotients of $\mathscr{F}(n)$; one uses the notion of flattening stratifications to show that in fact $\mathcal{Q}uot^P_N(\mathscr{F}/X/S)$ is a locally closed subfunctor of $\mathcal{G}r(P(n), f_*\mathscr{F}(n))$, and is therefore represented by a subscheme of the Grassmannian, and is in particular quasiprojective. This completes the second step of the construction, and as noted earlier is enough to conclude the existence of the Quot scheme.

The remaining step is to prove boundedness: that for any given Hilbert polynomial P, a single choice of sufficiently large N will satisfy the above conditions for all quotients of $\mathscr{F}$ having Hilbert polynomial P. This is a difficult and rather technical step, and we will content ourselves with saying that there have been two main approaches: Grothendieck's original approach in [1] using "Chow coordinates", and a later approach of Mumford (explained in [3]) using his theory of Castelnuovo-Mumford regularity. Given boundedness, we can conclude that each $\mathcal{Q}uot^P(\mathscr{F}/X/S)$ is equal to $\mathcal{Q}uot^P_N(\mathscr{F}/X/S)$ for some N, and hence is represented by a quasiprojective scheme.

To complete the proof of Theorem 3.2, we would like to show that each $\mathrm{Quot}^P(\mathscr{F}/X/S)$ is in fact proper, hence projective. This is easy to check directly from the definition of the functor, using the valuative criterion for properness.

References

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