A CONCISE ACCOUNT OF THE WEIL CONJECTURES AND ETALE COHOMOLOGY

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1. The early history

The story of the Weil conjectures and the development of etale cohomology is the story of one of the great triumphs of 20th century algebraic geometry. The problem treated is exceedingly elementary – counting solutions of polynomials over finite fields – but the proofs require and indeed motivated the creation of an astonishing array of sophisticated technical machinery.

The earliest echoes of the Weil conjectures may be found in two very different sources: Gauss' work on counting solutions to polynomial equations modulo p, which arose in relation to Gauss sums; and Riemann's study of the zeroes of the zeta function, leading to the Riemann hypothesis. These themes were brought together by E. Artin in his thesis in the 1920's, when he developed an analogue of the zeta function associated to curves over finite fields, verified the Riemann hypothesis in some examples, and conjectured that it holds for all curves.

Following further work of F. K. Schmidt on the form of the zeta function of curves over finite fields, and Hasse on the Riemann hypothesis for elliptic curves, Weil was able to conclude the story for curves by proving the Riemann hypothesis. It is interesting to note that it was precisely in order to make his arguments rigorous that Weil wrote *Foundations of Algebraic Geometry*, in which he introduced the notion of an abstract algebraic variety, obtained by gluing together affine varieties. In 1949, Weil went further, in what must surely be the most monumental paper to have appeared in the *Bulletin of AMS*. In it, Weil generalized what was known for curves, giving a precise conjecture as to the form of the zeta function for any smooth, projective variety over a finite field.

2. The statement

We now turn from history to mathematical content, and allow ourselves the following anachronistic definition:

Definition 2.1. Let X be a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. Then the **zeta function** of X is defined by:

$$\zeta(X,s)=\prod_{x\in X}\frac{1}{1-|k(x)|^{-s}},$$

where the product is over closed points $x \in X$, and $|k(x)|$ denotes the order of the residue field of x, which will always be a finite field.

Example 2.2. If $X = \operatorname{Spec}\mathbb{Z}$, we obtain the Riemann zeta function, while for $\mathscr{O}_K$ the ring of integers of a number field K, we obtain the Dedekind zeta function associated to K.

Example 2.3. If X is a variety of finite type over some finite field $\mathbb{F}_q$, it is an easy exercise to check that

$$\zeta(X,s) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{(q^{-s})^r}{r}\right),$$

where $N_r = |X(\mathbb{F}_{q^r})|$ is the number of points of X over $\mathbb{F}_{q^r}$.

Thus, in this case the zeta function is an encoding of the number of points of X over all (finite) extension fields of $\mathbb{F}_q$, and we see that Artin's zeta function indeed combined the counting of points over finite fields with Riemann's study of the zeta function.

Weil's conjecture then asserted that if X is a smooth projective variety over $\mathbb{F}_q$, we should have the following description of its zeta function:

(i) $\zeta(X,s)$ can be written as a rational function of q^{-s};

(ii) more explicitly, if $\dim X = n$, and we set $t = q^{-s}$, we can write

$$\zeta(X,s) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)},$$

where each root of each $P_i(t)$ is a complex number of norm $q^{-i/2}$;

(iii) the roots of $P_i(t)$ are the same as the roots of $t^{\deg P_{2n-i}} P_{2n-i}(1/q^n t)$;

(iv) if X is the reduction modulo p of a variety $\tilde{X}$ defined over a subfield of $\mathbb{C}$, then $b_i := \deg P_i(t)$ is the ith topological Betti number of $\tilde{X}$ in the analytic topology.

Recall that the ith Betti number of a topological space X is described via singular cohomology as $\dim_{\mathbb{Q}} H^i(X,\mathbb{Q})$.

Note that the last part of (ii) asserts in terms of the variable s, the zeros and poles of $\zeta(X,s)$ lie on the lines $\mathrm{Re}(s) = i/2$ with $i = 0,\ldots,2n$. This is thus the **Riemann hypothesis** for zeta functions of varieties. Also in terms of s, (iii) is a functional equation for the substitution $s \mapsto n - s$.

Example 2.4. We examine the case of curves: suppose $n = 1$, and X is a curve of genus g. The first observation is that we have $b_0 = b_2 = 1$, and $b_1 = 2g$. This is well-known for complex curves of genus g, which topologically are simply oriented compact surfaces with g holes. However, it is a fact (not true in higher dimensions) that every curve in positive characteristic may be obtained as the reduction mod p of a curve in characteristic 0, so we can apply (iv).

Thus $P_0(t)$ and $P_2(t)$ are linear, and (ii) says their roots have absolute value 1 and q^{-1} respectively, so we see we can write

$$\zeta(X,s) = \frac{P(t)}{(1-t)(1-qt)},$$

where $P(t)$ is a polynomial of degree $2g$, whose roots each have norm $q^{-1/2}$.

We observe in particular that as in the classical Riemann hypothesis, the roots of $\zeta(X,s)$ are all on the line $\mathrm{Re}(s) = 1/2$. In the context of the broader Weil conjectures, this suggests that the (conjectural) fact that all the zeros of the Riemann zeta function lie on a single line is something of a fluke, perhaps due arising because $\dim \mathrm{Spec}\,\mathbb{Z} = 1$.

We conclude by mentioning that the Riemann hypothesis for varieties over finite fields implies extremely strong estimates on the number of points on those varieties.

For instance, if X is a hypersurface of degree d and dimension n over $\mathbb{F}^q$, Weil's conjectures imply that

$$||X(\mathbb{F}_q)| - (1 + q + \cdots + q^n)| \leq bq^{n/2},$$

where we can even describe b explicitly – as the nth Betti number of a smooth complex n-dimensional hypersurface of degree d. In the case of curves, the resulting estimates are in fact equivalent to the Riemann hypothesis, which is one reason why that case was so much easier than the higher-dimensional case.

3. Towards a new cohomology theory

Weil's conjectures did not simply come out of the blue, but rather, as he himself made clear in his 1954 ICM address, arose from a daring consideration of what would happen if the varieties in question had been defined over the complex numbers, so that we could apply classical algebraic topology.

To see how this relates, recall that for a variety X defined over $\mathbb{F}_q$, we have the **Frobenius map** $\Phi_q : X \to X$ obtained on coordinates by the map $(x_1, \ldots, x_m) \mapsto (x_1^q, \ldots, x_m^q)$. Moreover, for any $r \geq 1$, we can recover $X(\mathbb{F}_{q^r}) \subseteq X(\bar{\mathbb{F}}_q)$ as the fixed points of $(\Phi_q)^r$.

Then begins the game of "what if?" Suppose for the moment that X and Φ_q (!) had both been defined over $\mathbb{C}$. Then we have the singular cohomology groups $H^i(X, \mathbb{Q})$, each of some finite dimension b_i over $\mathbb{Q}$, and Φ_q acts linearly on each cohomology group, say with eigenvalues $\alpha_{i,j}$. Recall that we have the Lefschetz fixed-point theorem:

Theorem 3.1. *(Lefschetz) The number of fixed points of $f : X \to X$ (counted with appropriate multiplicity) is given by $\sum_i (-1)^i \operatorname{Tr}(f|_{H^i(X,\mathbb{Q})})$.*

In the complex case, as long as there are finitely many fixed points, the multiplicities are always positive, and are equal to 1 exactly when the graph of f intersects $\Delta \subset X \times X$ transversely. One can argue that because Φ_q is inseparable (!), this transversality always occurs, so we don't need to worry about multiplicities. Thus we find

$$N_r = \sum_i (-1)^i \sum_j \alpha_{i,j}^r.$$

Elementary manipulations (see §A.4 of [2]) then show that we obtain a rational form for the zeta function as asserted, with $P_i(t) = \det(I - t(\Phi_q|_{H^i(X,\mathbb{Q})}))$ a polynomial of degree b_i. Furthermore, Poincare duality gives the functional equation. However, the analogue of the Riemann hypothesis is not so clear; it's possible that Weil based this part of his conjectures simply on generalization of what was already known, as well as what he computed in simple examples like Grassmannians.

At this point, the story gets slightly murky. What is clear is that by 1958, Serre and Grothendieck had decided that a possible approach to proving Weil's conjectures would be to develop a cohomology theory of algebraic varieties over arbitrary fields which would have enough of the same properties as singular cohomology to make the above arguments rigorous. In his 1958 ICM address, Grothendieck explicitly discussed this possibility, terming such a cohomology theory a "Weil cohomology," while Serre had already attempted to construct such a theory, his Witt vector cohomology, in a paper in the proceedings of a topology symposium which was held in 1956. Despite Grothendieck's implicit attribution, it seems unlikely that

Weil himself ever formulated such an approach: at least, he never made any indication in his writings that he was thinking along these lines, and on the contrary, when discussed the role of singular cohomology as a motivation for his conjectures, stated that such thinking could never be made into a proof. His later distaste for the more abstract cohomology theories also appears to be widely known.

Nonetheless, after Serre introduced the cohomology of coherent sheaves in 1954, it was clear that cohomology could become a powerful tool for varieties over arbitrary fields, and soon thereafter the search was on for a cohomology theory which more closely mirrored the properties of singular cohomology on manifolds, with the explicit aim of proving the Weil conjectures.

4. Etale cohomology

In 1960, Dwork managed to prove the rationality and functional equation using p-adic analysis rather than cohomology, but the Riemann hypothesis seemed as far out of reach as before. Meanwhile, Grothendieck was in the process of developing his theory of schemes (with generalized cohomology theory and the Weil conjectures already in mind, as can be seen from the introduction to Chapter I of *EGA*), and not long thereafter he introduced his theory of **etale cohomology**, whose properties he developed jointly with M. Artin. Furthermore, building on ideas of Serre, Grothendieck noted that the full Weil conjectures would follow from a series of conjectures on etale cohomology inspired by some very difficult theorems in the classical setting; in what was either a remarkable burst of optimism or Grothendieck's idea of a joke, he termed these the "standard conjectures".

The idea behind etale cohomology is incredibly natural in hindsight, but nonetheless requires a profound shift in perspective. Grothendieck's basic insight was the following: in order to define sheaf cohomology, we are more interested in the category of sheaves on a topological space than the topological space itself, and the definition of a sheaf lends itself naturally to generalization. This is clear for presheaves: recall that a presheaf (of, for instance, sets) on X is nothing other than a functor from the category of open sets of X (with morphisms given by inclusion) to the category of sets. By replacing the category of open sets of X with a more general category, it is clear how to generalize the notion of presheaf.

For sheaves, the situation is only slightly more complicated. The definition of sheaf in the classical setting posits that the sections over any U are uniquely determined by tuples of sections on any open cover $\{U_i\}$ on U, which agree on the intersections $\{U_i \cap U_j\}$. We note that in the category of open sets of X, $U_i \cap U_j = U_i \times_X U_j$, so in the more general setting, we will replace intersection with fiber product. Grothendieck introduced the notion of a "topology" on a category, consisting of the data of a collection of morphisms which are supposed to correspond to coverings, and therefore satisfy a few natural axioms (such data is now called a **Grothendieck topology** on a category). Given a Grothendieck topology on a category C, one can define a sheaf formally as a functor on C which satisfies the usual gluing axiom relative to the covers. Note that this not only generalizes the usual notion of sheaves, but also encompasses the idea of "Zariski sheaves" we introduced when studying representable functors. In the latter case, our category is the category of all schemes, with covers corresponding to Zariski covers.

We then define etale cohomology in terms of the cohomology of certain constant sheaves on categories of schemes, where we define the covers of our Grothendieck topology – the **etale topology** – to consist of (collections of) morphisms which are etale and (set-theoretically) cover their target. We can thus define sheaves on the etale topology, and using the usual derived functor machinery, we obtain a notion of **etale cohomology** of sheaves for the etale topology. Note replacing open covers by etale maps is not unreasonable: if X is a complex variety, they consist of the maps which are local isomorphisms for the associated analytic spaces, and are therefore not so far from being open coverings in the analytic topology.

A worthwhile exercise for the arithmetically inclined is the following: for $X = \operatorname{Spec} k$, etale sheaves of groups on X are equivalent to Galois modules over k, and the etale cohomology agrees with the Galois cohomology. This fact has a discouraging consequence: one can use it to show that if X is normal and $\mathscr{F}$ any constant sheaf, the etale cohomology $H^i_{\text{et}}(X, \mathscr{F})$ is torsion for $i > 0$. In particular, if we were hoping to obtain our cohomology theory by setting $\mathscr{F} = \mathbb{Q}_X$, the constant sheaf associated to $\mathbb{Q}$, and obtain etale cohomology groups which are finite-dimensional vector spaces over $\mathbb{Q}$, our hopes appear to be dashed. However, the situation is salvageable. It turns out that etale cohomology with torsion coefficients is well-behaved, so we choose an $\ell \neq p$ (if the characteristic of our base field is p), and define

$$H^i_{\text{et}}(X, \mathbb{Q}_\ell) := H^i_{\text{et}}(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell := \varprojlim_{m>0} H^i_{\text{et}}(X, \mathbb{Z}/\ell^m\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Artin's comparison theorem then states that if X is a complex variety, we have $H^i_{\text{et}}(X, \mathbb{Q}_\ell) \cong H^i(X, \mathbb{Q}_\ell)$, where the cohomology on the right is the usual singular cohomology. The argument is inductive, using fibrations by curves to reduce to looking at the first cohomology groups, which are then compared using the comparison theorem for the algebraic fundamental group, itself a highly non-trivial result.

5. Deligne finishes the job

In the early 1970's, Deligne ended the story rather decisively by proving the Weil conjectures. A few years earlier, Deligne had already shown that the Weil conjectures imply Ramanujan's conjecture on the coefficients of the modular form $\Delta(q)$. Specifically, if we write

$$\Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n,$$

Ramanujan's conjecture states that for any prime p, we have the inequality $|\tau(p)| \leq 2p^{11/2}$. Earlier work of Eichler, Shimura, Kuga, and Ihara interpreted this in terms of counting points of varieties, and nearly reduced it to the Weil conjectures, but it was Deligne who completed the argument, showing in fact that the more general Ramanujan-Petersson conjecture follows from the Weil conjectures, on coefficients of a certain class of cusp forms.

Deligne's proof caught many by surprise because it was not by way of the standard conjectures. Indeed, most of the standard conjectures remain open, although Deligne was able to conclude one of them – the "hard Lefschetz theorem" – by extending the techniques he used to prove the Weil conjectures. Instead, his argument is a rather elaborate computation which ultimately boils down to an induction of dimension, using the notion of Lefschetz pencils, introduced by Lefschetz in the classical setting to study the cohomology of complex inductively in terms of the cohomology of their hyperplane sections. Roughly, the idea is to take a one-dimensional family of hyperplane sections of X, which then gives a one-dimensional family of subvarieties of X which together cover X. The general varieties in the family will be smooth, but a finite number have mild singularities, and these play a key role in the theory. Deligne also realized that ideas of Rankin and Langlands on using L-series over $\mathbb{Z}$ to prove the Ramanujan conjecture could be transferred to L-series over curves over finite fields. In this context, the L-series were more tractable due to the existence of cohomological methods, and this combined with the Lefschetz theory allowed Deligne to complete a proof of the Weil conjectures, obtaining for himself a Fields medal in the process. For more details, we refer the reader to the survey article [3] of Katz, as well as the book [1] of Freitag and Kiehl.

We conclude by mentioning that beyond the Ramanujan-Petersson conjecture, and the bounds for the number of points on smooth hypersurfaces (and more generally smooth projective varieties) over finite fields mentioned earlier, the Weil conjectures have found a wide array of applications both to explicit and to theoretical problems. For instance, one can use the Weil conjectures to derive estimates for various exponential sums, including Kloosterman sums. Few mathematical problems can lay claim to such a powerful combination of elementary nature, breadth of applications, and depth of theory inspired in the search for a proof.

References

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