We will begin the quarter by studying the operation of tensor product. This operation is of fundamental importance in a variety of fields, including differential geometry and general relativity, commutative algebra and algebraic geometry, number theory, algebraic topology and representation theory. It is this last application which we will explore this quarter. Although tensor product is in general an extremely subtle operation, for many applications it suffices to work with tensor products over fields, which are more easily tractable.

1. Tensor product of modules

Let $R$ be a commutative ring, and $M_1, \ldots, M_n$ modules over $R$. We define the tensor product as follows:

**Definition 1.1.** The tensor product $M_1 \otimes \cdots \otimes M_n$ of the $M_i$ over $R$ is the quotient

$$F_{M_1 \times \cdots \times M_n}/R_{M_1 \times \cdots \times M_n},$$

where $F_{M_1 \times \cdots \times M_n}$ is the free $R$-modules generated by $M_1 \times \cdots \times M_n$ (considered as a set), and $R_{M_1 \times \cdots \times M_n}$ is the submodule generated by elements of the form

$$(m_1, \ldots, m_i + m'_i, \ldots, m_n) - (m_1, \ldots, m_i, \ldots, m_n) - (m_1, \ldots, m'_i, \ldots, m_n)$$

and

$$(m_1, \ldots, cm_i, \ldots, m_n) - c(m_1, \ldots, m_i, \ldots, m_n),$$

for $c \in R$. (*)

Note that the tensor product is by construction an $R$-module. In situations where there is the possibility of confusion, we can make the commutative ring $R$ explicit in the tensor product notation by writing $M_1 \otimes_R \cdots \otimes_R M_n$.

**Notation 1.2.** Given $(m_1, \ldots, m_n) \in M_1 \times \cdots \times M_n$, the corresponding element in $M_1 \otimes \cdots \otimes M_n$ is denoted by $m_1 \otimes \cdots \otimes m_n$.

**Example 1.3.** We have $m_1 \otimes \cdots \otimes m_n = 0$ in $M_1 \otimes \cdots \otimes M_n$ if any $m_i$ is equal to 0, since although $(m_1, \ldots, m_n)$ still gives a nonzero element of the free module used to define $M_1 \otimes \cdots \otimes M_n$, according to the scalar multiplication relation we can factor out the 0, so

$$m_1 \otimes \cdots \otimes 0 \otimes \cdots \otimes m_n = 0(m_1 \otimes \cdots \otimes 0 \otimes \cdots \otimes m_n) = 0.$$

An important special case is the tensor product of vector spaces over a field.

**Example 1.4.** If $k$ is a field, and $V_1, \ldots, V_n$ are vector spaces over $k$, and $(e_{i,j})_{j}$ are bases of $V_i$ for $i = 1, \ldots, n$, then elements of the form

$$e_{1,j_1} \otimes \cdots \otimes e_{n,j_n}$$

form a basis of $V_1 \otimes \cdots \otimes V_n$. In particular, if the $V_i$ are finite-dimensional, with $\dim V_i = d_i$, then $V_1 \otimes \cdots \otimes V_n$ has dimension $\prod_{i=1}^{n} d_i$.

We will give a proof of a more general statement shortly.
In applications to fields such as representation theory and differential geometry, the most common occurrence of tensor products is in the above context of vector spaces, frequently over a specific classical field such as $\mathbb{R}$ or $\mathbb{C}$. In general, tensor products can behave in surprising ways.

**Example 1.5.** Considering $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ as $\mathbb{Z}$-modules, we have that $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z}$ is the zero module. Indeed, for any $a \in \mathbb{Z}/2\mathbb{Z}$, $b \in \mathbb{Z}/2\mathbb{Z}$, we have

$$a \otimes b = 3a \otimes b = 3(a \otimes b) = a \otimes 3b = a \otimes 0 = 0.$$  

A similar argument shows that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = 0$ whenever $m$ and $n$ are relatively prime.

The importance of the tensor products arises from its universal property. Recall:

**Definition 1.6.** Given $R$-modules $M_1, \ldots, M_n$ and $N$, a **multilinear map** $\varphi : M_1 \times \cdots \times M_n \to N$ is a function such that for any $i$, and any $m_j \in M_j$ for $j \neq i$, the composed map

$$M_i \to M_1 \times \cdots \times M_n \to N$$

induced by $m_i \mapsto (m_1, \ldots, m_i, \ldots, m_n)$ and $\varphi$ is $R$-linear. That is, for all $m_i, m'_i \in M_i$, we have

$$\varphi(m_1, \ldots, m_i + m'_i, \ldots, m_n) = \varphi(m_1, \ldots, m_i, \ldots, m_n) + \varphi(m_1, \ldots, m'_i, \ldots, m_n)$$

and for all $c \in R$, $m_i \in M_i$ we have

$$\varphi(cm_1, \ldots, cm_i, \ldots, m_n) = c\varphi(m_1, \ldots, m_i, \ldots, m_n).$$

That is, a multilinear map is one which is additive and respects scalar multiplication when working in any one coordinate at a time. One might wonder if, given $M_1, \ldots, M_n$, there is some module $M$ such that for all $N$, multilinear maps $M_1 \times \cdots \times M_n \to N$ are in natural bijection with standard $R$-module homomorphisms $M \to N$. This turns out to be precisely the universal property of the tensor product.

**Proposition 1.7.** Given $R$-modules $M_1, \ldots, M_n$ and $N$, and a multilinear map $\varphi : M_1 \times \cdots \times M_n \to N$, the map $M_1 \otimes \cdots \otimes M_n \to N$ induced by $m_1 \otimes \cdots \otimes m_n \mapsto \varphi(m_1, \ldots, m_n)$ is a well-defined $R$-module homomorphism.

Furthermore, this construction defines a bijection between multilinear maps $M_1 \times \cdots M_n \to N$ and $R$-module homomorphisms $M_1 \otimes \cdots \otimes M_n \to N$.

**Remark 1.8.** Before proving the proposition, we observe a consequence of it: the identity map $M_1 \otimes \cdots \otimes M_n \to M_1 \otimes \cdots \otimes M_n$ corresponds to a multilinear map

$$\varphi : M_1 \times \cdots \times M_n \to M_1 \otimes \cdots \otimes M_n.$$  

This is Lang’s “universal multilinear map.” The proposition can then be stated equivalently in this context as asserting that any multilinear map $M_1 \times \cdots \times M_n \to N$ is obtained by composing $\varphi$ with an $R$-module homomorphism $M_1 \otimes \cdots \otimes M_n \to N$.

As is standard with universal properties, the tensor product is in fact uniquely determined by this universal property. More specifically, suppose $M$ is an $R$-module, with a multilinear map

$$\varphi_M : M_1 \times \cdots \times M_n \to M,$$

such that for any $N$, any multilinear map $M_1 \times \cdots \times M_n \to N$ is obtained by composing $\varphi_M$ with an $R$-module homomorphism $M \to N$. We then claim that $M \cong M_1 \otimes \cdots \otimes M_n$, and in fact there is a unique isomorphism which commutes with $\varphi$ and $\varphi'$.

Indeed, according to the universal property, the multilinear map $\varphi'$ determines a unique homomorphism $f : M_1 \otimes \cdots \otimes M_n \to M$ such that $\varphi' = f \circ \varphi$. By hypothesis, $\varphi$ also determines a unique homomorphism $g : M \to M_1 \otimes \cdots \otimes M_n$ such that $\varphi = g \circ \varphi'$. Then $\varphi = g \circ \varphi' = g \circ f \circ \varphi$. Thus, the
homomorphism $g \circ f$ induces $\varphi$ when composed with $\varphi$. But the identity map on $M_1 \otimes \cdots \otimes M_n$ also induces $\varphi$, so by the uniqueness of the universal property, we conclude that $g \circ f = \text{id}$. A similar argument shows that $f \circ g = \text{id}$, so we conclude that $f$ is an isomorphism, as desired.

**Proof.** By the definition of a free module, we get a module homomorphism $F_{M_1 \times \cdots \times M_n} \to N$ induced by applying $\varphi$ to the generators. This gives a homomorphism on the tensor product because the multilinearity relations precisely imply that the generators of $R_{M_1 \times \cdots \times M_n}$ are sent to 0.

To see that we obtain a bijection in this manner, it suffices to observe that we can reverse the above procedure – a homomorphism $M_1 \otimes \cdots \otimes M_n \to N$ trivially induces a function $M_1 \times \cdots \times M_n \to R_{M_1 \times \cdots \times M_n}$, and the fact that the resulting function is a multilinear map follows from the vanishing of the lifted homomorphism on $R_{M_1 \times \cdots \times M_n}$. □

We next observe that the tensor product is associative and commutative in a natural sense.

**Proposition 1.9.** Let $M_1, M_2, M_3$ be $R$-modules. Then the map

$$(M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes M_2 \otimes M_3$$

determined by

$$(m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes m_2 \otimes m_3$$

is well-defined and an isomorphism of $R$-modules, and similarly for

$$M_1 \otimes (M_2 \otimes M_3) \to M_1 \otimes M_2 \otimes M_3.$$ 

In particular, $(M_1 \otimes M_2) \otimes M_3 \cong M_1 \otimes (M_2 \otimes M_3)$.

In addition, the map

$$M_1 \otimes M_2 \to M_2 \otimes M_1$$

determined by

$$m_1 \otimes m_2 \mapsto m_2 \otimes m_1$$

is well-defined and an isomorphism of $R$-modules.

The proof is routine, and we omit it. In general, we will tend to omit proofs which are routine and explained adequately in Lang’s book, while we will go through proofs which either involve interesting ideas or are not covered adequately in the book.

**Warning 1.10.** Proposition 1.9 says that the tensor product is commutative on the level of modules, but this is not the case on the level of elements. That is, if we have $m_1, m_2 \in M$, then typically $m_1 \otimes m_2 \neq m_2 \otimes m_1$ inside $M \otimes M$.

Lastly, we mention that maps of modules induce maps of the corresponding tensor products.

**Proposition 1.11.** Let $M_1, \ldots, M_n$ and $N_1, \ldots, N_n$ be $R$-modules, and suppose $f_i : M_i \to N_i$ is an $R$-module homomorphism for $i = 1, \ldots, n$. Then we obtain an $R$-module homomorphism

$$M_1 \otimes \cdots \otimes M_n \to N_1 \otimes \cdots \otimes N_n$$

induced by

$$m_1 \otimes \cdots \otimes m_n \mapsto f_1(m_1) \otimes \cdots \otimes f_n(m_n).$$

Once again, we omit the proof.
2. Basic properties

We now discuss some basic properties of the tensor product, starting with their interaction with direct sums. Recall that if $I$ is a set, and we have an $R$-module $M_i$ for each $i \in I$, then the direct sum $\bigoplus_{i \in I} M_i$ admits homomorphisms $M_i \to \bigoplus_{i \in I} M_i$ for each $i \in I$, and has the universal property that composing with these homomorphisms induces a bijection for any $R$-module $N$ between homomorphisms $\bigoplus_{i \in I} M_i \to N$ and tuples of homomorphisms $M_i \to N$. Formally, we have

$$\text{Hom}(\bigoplus_{i \in I} M_i, N) \cong \prod_{i \in I} \text{Hom}(M_i, N).$$

**Proposition 2.1.** Given a set $I$ and $R$-modules $M_i$ for each $i \in I$, and an $R$-module $N$, there is a natural isomorphism

$$\bigoplus_{i \in I} (M_i \otimes N) \cong \left( \bigoplus_{i \in I} M_i \right) \otimes N$$

induced by the homomorphisms

$$M_i \otimes N \to \left( \bigoplus_{i \in I} M_i \right) \otimes N$$

sending $m_i \otimes n$ to $m_i \otimes n$.

**Proof.** Both $\bigoplus_{i \in I} (M_i \otimes N)$ and $\left( \bigoplus_{i \in I} M_i \right) \otimes N$ are generated by elements of the form $m_i \otimes n$ for $m_i \in M_i$, $n \in N$. The natural map sends $m_i \otimes n$ to $m_i \otimes n$, so it is necessarily surjective. Moreover, to check that it is an isomorphism, it is enough to check that we have a well-defined map of modules in the opposite direction determined by sending $(\sum_i m_i) \otimes n$ to $\sum_i (m_i \otimes n)$. This just involves checking that the relations in the definition of the tensor product are all sent to sums of relations $\bigoplus_{i \in I} (M_i \otimes N)$, which is routine.

For a more abstract proof, see Lang. □

**Warning 2.2.** It is not the case that tensor products commute with (infinite) direct products. The reason is very concrete: although we have a natural map

$$\left( \prod_{i \in I} M_i \right) \otimes N \to \prod_{i \in I} (M_i \otimes N),$$

there is no reason this should be surjective, as on the righthand side, the elements of $N$ are allowed to vary over the infinitely many terms in the product, while on the lefthand side, only finitely many elements of $N$ can occur.

We next study the behavior of tensor products with free modules.

**Proposition 2.3.** Suppose that $M$ is an $R$-module, and $N$ is a free $R$-module, with basis $\{v_i\}_{i \in I}$. Then every element of $M \otimes N$ may be written uniquely as

$$\sum_{i \in I} m_i \otimes v_i,$$

with $m_i \in M$, and all but finitely many $m_i$ equal to 0.

**Proof.** We first consider the case that $N$ has rank 1, and $v$ is a generator. We have the map $M \to M \otimes N$ defined by $m \mapsto m \otimes v$; the statement of the proposition in this case is equivalent to showing that this map is an isomorphism. But we can define the inverse explicitly: since every element of $N$ is $cv$ for a unique $c \in R$, we can define a map $M \otimes N \to M$ by $m \otimes cv \mapsto cm$. 6
We check directly that this vanishes on the tensor relations and is thus well-defined, and because \( m \otimes cv = cm \otimes v \), we see that it is inverse to \( m \mapsto m \otimes v \), so we conclude that both maps are isomorphisms, as desired.

Now, for the general case, it follows from the case of rank 1 that every element of \( \bigoplus_{i \in I} M \otimes R \) is uniquely of the form \( \sum_{i \in I} m_i \otimes 1 \), with all but finitely many \( m_i \) equal to 0. According to Proposition 2.1 we conclude that every element of \( M \otimes (\bigoplus_{i \in I} R) \) is uniquely of the form \( \sum_{i \in I} m_i \otimes 1_i \), where \( 1_i \in \bigoplus_{i \in I} R \) denotes the element with a 1 in the \( i \)th entry, and 0's elsewhere. Finally, the basis \( \{v_i\} \) determines an isomorphism \( N \cong \bigoplus_{i \in I} R \), so by the functoriality of Proposition 1.11 we conclude that every element of \( M \otimes N \) is uniquely of the form \( \sum_{i \in I} m_i \otimes v_i \), as desired. \( \square \)

As an immediate consequence of Proposition 2.3, we finally conclude a generalized version of the statement promised for vector spaces in Example 1.4.

**Corollary 2.4.** If \( M, N \) are free \( R \)-modules, with bases \( \{v_i\}_{i \in I}, \{w_j\}_{j \in J} \) respectively, then \( M \otimes N \) is free, with basis \( \{v_i \otimes w_j\}_{i \in I, j \in J} \).

In particular, if \( M \) and \( N \) have finite rank, then \( \text{rk } M \otimes N = \text{rk } M \cdot \text{rk } N \).

Notice that the corresponding statement for \( n \)-fold tensor products also follows, by induction.

For free modules of finite rank, the Hom module may also be expressed in terms of tensor products, as is frequently done in differential geometry.

**Proposition 2.5.** Let \( M, N \) be \( R \)-modules. Let \( M^\vee = \text{Hom}_R(M, R) \) be the dual of \( M \). Then there is a unique module homomorphism

\[
M^\vee \otimes N \rightarrow \text{Hom}_R(M, N)
\]

defined by \( \nu \otimes n \mapsto (m \mapsto \nu(m)n) \). Moreover, if \( M \) is free of finite rank, then this homomorphism is an isomorphism.

**Proof.** It is routine to check that the map is a well-defined homomorphism. To check that it is an isomorphism in the case that \( M \) is free of finite rank, let \( \{v_i\} \) be a basis for \( M \), and \( \{v_i^\vee\} \) the resulting dual basis for \( M^\vee \). Then an element \( f \in \text{Hom}_R(M, N) \) is freely and uniquely determined by specifying \( f(v_i) = w_i \) for each \( i \). We thus obtain a map \( \text{Hom}_R(M, N) \rightarrow M^\vee \otimes N \) by sending \( f \) to \( \sum_i v_i^\vee \otimes w_i \), which is easily seen to be inverse to the given map. \( \square \)

\( (*) \)

**Exercise 2.6.** If \( M, N \) are \( R \)-modules, show that there is a homomorphism

\[
M^\vee \otimes N^\vee \rightarrow (M \otimes N)^\vee
\]

sending \( (\mu, \nu) \) to the map determined by

\[
m \otimes n \mapsto \mu(m) \otimes \nu(n).
\]

Show that if \( M \) is free of finite rank, then this map is an isomorphism.

Next, we observe that functoriality of the tensor product induces a natural map (which one checks easily is \( R \)-linear)

\[
(2.6.1) \quad \text{End}_R(M) \otimes \text{End}_R(N) \rightarrow \text{End}_R(M \otimes N)
\]

for any \( R \)-modules \( M, N \).

**Exercise 2.7.** If \( M, N \) are free \( R \)-modules of finite rank, then (2.6.1) is an isomorphism.

Next up is a fundamental exactness property of tensor products.


**Proposition 2.8.** Suppose

\[ 0 \to M' \to M \to M'' \to 0 \]

is an exact sequence of \( R \)-modules. Then for any \( R \)-module \( N \), the induced sequence

\[ M' \otimes N \to M \otimes N \to M'' \otimes N \to 0 \]

is exact.

**Proof.** Since \( M'' \otimes N \) is generated by the elements of the form \( m'' \otimes n \), and each \( m'' \) can be lifted to some \( m \in M \) by hypothesis, we conclude the surjectivity of \( M \otimes N \to M'' \otimes N \) immediately. Similarly, an element of \( M \otimes N \) is in the image of \( M' \otimes N \) if and only if it can be written in the form \( \sum_i m_i \otimes n_i \) where each \( m_i \) is in the image of \( M' \). Any such element maps to \( \sum_i 0 \otimes n_i = 0 \) in \( M'' \otimes N \), so we have that the image is contained in the kernel. To complete the proof, we need only show the opposite containment.

Let \( I \subseteq M \otimes N \) be the image of \( M' \otimes N \). We have shown that we have a (surjective) map \( f : (M \otimes N)/I \to M'' \otimes N \), and we will conclude the desired exactness if we show that \( f \) is in fact injective. According, we construct a map \( g \) in the other direction, with \( g \circ f = \text{id} \). For each \( m'' \in M'' \), choose an element \( m \in M \) mapping to \( m'' \). We claim that we have a well-defined homomorphism \( M'' \otimes N \) mapping each \( m'' \otimes n \) to the class of \( m \otimes n \), which moreover is independent of the choice of each \( m \) mapping to \( m'' \). Indeed, if \( m_1, m_2 \) both map to \( m'' \), then \( m_1 \otimes n - m_2 \otimes n = (m_1 - m_2) \otimes n \in I \), since \( m_1 - m_2 \) maps to 0 in \( M'' \), and must therefore be in the image of \( M' \). To check that \( g \) is a well-defined homomorphism from \( M'' \otimes N \), we need to check that it vanishes on the tensor relations, and this follows from the independence of the choice of the elements \( m \) defining \( g \).

Moreover, \( M \otimes N \) is generated by elements of the form \( m \otimes n \), and therefore so is \( (M \otimes N)/I \). Because we have checked that the definition of \( g \) is independent of the choice of elements \( m \) lifting each element \( m'' \), we see that on an element of the form \( m \otimes n \), the map \( g \circ f \) is the identity. Thus, \( g \circ f \) is the identity on all of \( (M \otimes N)/I \), and we conclude that \( f \) is injective, as desired. \( \square \)

**Warning 2.9.** Note that tensor product is very much not left exact, in general. As an example, consider the exact sequence of \( \mathbb{Z} \)-modules

\[ 0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0. \]

If we tensor by \( \mathbb{Z}/m\mathbb{Z} \), we obtain maps

\[ 0 \to \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0. \]

The first map \( \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is induced from multiplication by \( m \), so it is the zero map. The second map is induced from the quotient map, so is the identity. We thus see that (as guaranteed by Proposition 2.8) the sequence is exact in the middle and on the right, but it is not exact on the left.

We conclude our survey of the basic properties by looking at what happens when we mod out by an ideal.

**Proposition 2.10.** Let \( M \) be an \( R \)-module, and \( I \subseteq R \) an ideal. Then the map

\[ M \otimes (R/I) \to M/IM \]

induced by \( m \otimes [r] \mapsto [rm] \) is well-defined, and an isomorphism of \( R \)-modules.

**Proof.** As usual, to check the map is well-defined just involves checking that the tensor relations are sent to 0, which is straightforward. We construct an inverse explicitly: given \([m] \in M/IM \), we send it to \( m \otimes 1 \in M \otimes (R/I) \). We have to check that this is well-defined: that is, if
When at least one of the two modules in a tensor product is itself a ring, we obtain additional structure for the tensor product. We begin by considering the case of a commutative ring:

**Proposition 3.1.** If $M$ is an $R$-module, and $R'$ is a commutative $R$-algebra, then $M \otimes_R R'$ has the natural structure of an $R'$-module, induced by multiplication on the righthand side.

**Proof.** By definition, $M \otimes_R R'$ is an $R$-module, so it is enough to see that multiplication on the right gives a well-defined module structure. That is, for $r'_1 \in R'$ and $m \otimes r'_2 \in M \otimes_R R'$, define the product of $r'_1$ with $m \otimes r'_2$ to be $m \otimes r'_1 r'_2$. One needs to check that this gives a well-defined multiplication map, and that it satisfies the axioms of a module, but both of these are straightforward. □

The tensor product thus gives us a way of taking an $R$-module and making it into an $R'$-module, whenever we have an $R$-algebra $R'$.

**Applying Proposition 2.3, we have:**

**Proposition 3.2.** Suppose $M$ is a free $R$-module, with basis $\{v_i\}$. Then $M \otimes_R R'$ is a free $R'$-module, with basis $\{v_i \otimes 1\}$.

**Example 3.3.** Suppose we have a vector space $V$ over $\mathbb{R}$, and we want to “complexify” it; that is, we want to allow ourselves to use complex coordinates instead of real ones. If we choose a basis $\{e_i\}$ for $V$, we can do this simply by declaring $\{e_i\}$ to be a basis over $\mathbb{C}$ for a new complex vector space. However, this is somewhat ad hoc, especially if $V$ doesn’t come naturally with a basis. The tensor product gives us an intrinsic way of doing this: according to Proposition 3.1, the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$ has the natural structure of a $\mathbb{C}$-vector space, and by Proposition 3.2 a basis of $V$ gives a natural basis of $W$, so is the space we want.

**Example 3.4.** If $R' = R/I$ for some ideal $I$, then we have seen that $M \otimes_R R' = M/IM$. This is naturally a module over $R/I$, so we can view it as an example of Proposition 3.1.

A basic property of tensoring with rings is transitivity:

**Proposition 3.5.** If $R'$ is an $R$-algebra, and $R''$ is an $R'$-algebra, the map

$$M \otimes_R R'' \to (M \otimes_R R') \otimes_{R'} R''$$

induced by

$$m \otimes r'' \mapsto (m \otimes 1) \otimes r''$$

is an isomorphism.
Proposition 3.7 that $R$ and we can conclude the desired isomorphism from Proposition 2.10. □

Now, suppose that we tensor together two (not necessarily commutative) $R$-algebras. We see that the tensor product is still an $R$-algebra. The following proposition is routine, and we omit the proof.

Proposition 3.6. Suppose $A$ and $B$ are $R$-algebras (not necessarily commutative). Then $A \otimes_R B$ has the natural structure of an $R$-algebra induced by

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'.$$

In the case that $A$ and $B$ are commutative, then $A \otimes_R B$ defines a fibered coproduct in the category of rings, which is extremely important in algebraic geometry. It is used to define products of varieties, and fibers of maps, among other constructions. The tensor product of rings is a subtle object, but from one point of view it is also rather concrete:

Proposition 3.7. Suppose that $R[[x_i]_{i \in I}]$ is a polynomial ring (possibly in infinitely many variables), and $\{f_j\}_{j \in J}$ is a collection of elements. Then for any commutative $R$-algebra $R'$, the map

$$R[[x_i]_{i \in I}]/(\{f_j\}_{j \in J}) \otimes_R R' \to R'[\{x_i\}_{i \in I}]/(\{f_j\}_{j \in J})$$

induced by sending $g \otimes r'$ to $r'g$ is an isomorphism.

In the definition of the map, recall that if $R'$ is an $R$-algebra, there is a natural ring homomorphism $R \to R'$ induced by $r \mapsto r \cdot 1$, so we can think of a polynomial with coefficients in $R$ as a polynomial with coefficients in $r'$.

Proof. It is routine to check that the given map is a well-defined homomorphism. To check that it is an isomorphism, we first do so in the case that there are no $f_j$, so that we have only polynomial rings. In this case, we have from Proposition 2.3 that $R[[x_i]_{i \in I}] \otimes_R R'$ is freely generated by the monomials in the $x_i$, and since the same is true of $R'[\{x_i\}_{i \in I}]$, we immediately conclude the desired isomorphism. Finally, for the general case, by transitivity of tensor production (Proposition 3.5) and by the case of polynomial rings, we have

$$R[[x_i]_{i \in I}]/(\{f_j\}_{j \in J}) \otimes_R R' = R[[x_i]_{i \in I}]/(\{f_j\}_{j \in J}) \otimes_{R[[x_i]_{i \in I}]} (R[x\{x_i\}_{i \in I}] \otimes_R R') = R[[x_i]_{i \in I}]/(\{f_j\}_{j \in J}) \otimes_{R[[x_i]_{i \in I}]} R'[\{x_i\}_{i \in I}],$$

and we can conclude the desired isomorphism from Proposition 2.10. □

Example 3.8. One of the simplest examples is polynomial rings over a field: it follows from Proposition 3.7 that

$$k[x_1, \ldots, x_m] \otimes k[y_1, \ldots, y_n] = k[x_1, \ldots, x_m, y_1, \ldots, y_n].$$

Exercise 3.9. Show that $C \otimes_R C \cong C \oplus C$.

Exercise 3.10. Let $\mathbb{R}(t)$ be the field of rational functions over $\mathbb{R}$. Describe $\mathbb{R}(t) \otimes_{\mathbb{R}} C$.

Example 3.11. For a noncommutative example, recall that according to Exercise 2.7 we have

$$\text{End}_R(M) \otimes \text{End}_R(N) \cong \text{End}_R(M \otimes N),$$

when both sides are considered as modules. But $\text{End}_R(M)$ is naturally an $R$-algebra with multiplication given by composition of endomorphisms. Thus, we now have multiplication rules on both sides, and one can check that the above isomorphism is compatible with the multiplication, and gives an isomorphism of $R$-algebras.
4. The tensor algebra

Given an \( R \)-module \( M \), we can construct a ring out of it as follows:

**Definition 4.1.** If \( M \) is an \( R \)-module, the **tensor algebra** \( T(M) \) is the \( R \)-algebra described as follows: as an \( R \)-module,

\[
T(M) = \bigoplus_{i=0}^{\infty} T^i(M),
\]

where \( T^0(M) = R \), and \( T_i(M) = \text{\( i \) times } M \otimes \cdots \otimes M \). Multiplication is induced from the maps \( T^i(M) \times T^j(M) \to T^{i+j}(M) \) sending \( (m_1 \otimes \cdots \otimes m_i, m'_1 \otimes \cdots \otimes m'_j) \) to \( (m_1 \otimes \cdots \otimes m_i \otimes m'_1 \otimes \cdots \otimes m'_j) \).

It is easy to check that \( T(M) \) is an associative (but in most cases noncommutative) \( R \)-algebra. It is also by definition a graded ring, meaning roughly that we can talk about elements of different degrees (with a degree-\( d \) element being one in \( T^d(M) \)), and the algebra is direct sum over the different degrees, with multiplication acting additively on degrees.

If \( M \) is free of finite rank of \( R \), then \( T(M) \) may be thought of as a “noncommutative polynomial algebra” over \( R \).

Much of the importance of the tensor algebra comes from two quotient algebras one can construct out of it: the symmetric algebra, and the exterior (or alternating) algebra. The exterior algebra is an important topic unto itself, but for now we will only discuss the symmetric algebra.

**Definition 4.2.** Given an \( R \)-module \( M \), the **symmetric algebra** \( S(M) \) is defined to be the quotient of \( T(M) \) by the ideal generated by elements of the form

\[
m_1 \otimes \cdots \otimes m_i - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(i)}
\]

for any \( i \) and any permutation \( \sigma \in S_i \).

In fact, the ideal in question is equal to the \( R \)-submodule generated by the same generators, and thus the symmetric algebra can be defined one degree at a time.

Like the tensor algebra, the symmetric algebra is naturally graded. Unlike the tensor algebra, the symmetric algebra is commutative, since we have modded out precisely by switching order of multiplication. In the case that \( M \) is free of finite rank, the symmetric algebra is particularly easy to understand:

**Proposition 4.3.** Suppose that \( M \) is a free module of rank \( n \). Then a choice of basis \( \{v_i\} \) of \( M \) induces an isomorphism

\[
R[x_1, \ldots, x_n] \sim S(M)
\]

by \( x_i \mapsto v_i \).

**Proof.** That the map is uniquely determined is an immediate consequence of the fact that \( S(M) \) is a commutative \( R \)-algebra, and \( R[x_1, \ldots, x_n] \) is a free commutative \( R \)-algebra. Moreover, it is clear from the definitions that the elements \( v_i \) generate \( S(M) \) as a ring, so the map is necessarily surjective. On the other hand, we can construct a map

\[
S(M) \to R[x_1, \ldots, x_n]
\]

in the obvious way, by sending \( v_{j_1} \otimes \cdots \otimes v_{j_i} \) to \( x_{j_1} \cdots x_{j_i} \). It is straightforward to check that this is compatible with the relations we used to define \( S(M) \), and it is moreover clear that the composed map \( R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n] \) is the identity on the \( x_i \), and hence the identity. We conclude that the map \( R[x_1, \ldots, x_n] \to S(M) \) is also injective, and hence an isomorphism.
5. Flatness

Recall that if we have an injective map of modules, then the tensor product with an arbitrary does not necessarily preserve injectivity. However, tensoring with some modules does preserve injectivity. This leads to the subject of flatness.

**Proposition 5.1.** Let $N$ be an $R$-module. Then the following are equivalent:

(i) For any injection $M' \to M$, the induced map $M' \otimes N \to M \otimes N$ is injective.

(ii) For any short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

the induced sequence

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is exact.

(iii) For any very short exact sequence

$$M' \to M \to M''$$

the induced sequence

$$M' \otimes N \to M \otimes N \to M'' \otimes N$$

is exact.

(iv) For any exact sequence of $R$-modules, the induced sequence after tensoring with $N$ remains exact.

**Proof.** Since an injection can always be extended to a short exact sequence by adding in the cokernel, and we already know that tensor is right exact, we have that (i) is equivalent to (ii). In addition, since checking exactness can always be done in terms of three consecutive modules at a time, (iii) is equivalent to (iv). We clearly have that (iv) implies (ii), so it is enough to check that (ii) implies (iii). Given

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

exact, we obtain the short exact sequence

$$0 \to \text{im } f \to M \to \text{im } g \to 0,$$

and by (ii) we conclude that

$$0 \to \text{im } f \otimes N \to M \otimes N \to \text{im } g \otimes N \to 0$$

is exact. Moreover, since tensor products preserve surjectivity, we have that $M' \otimes N \to \text{im } f \otimes N$ is a surjection, so

$$M' \otimes N \to M \otimes N \to \text{im } g \otimes N \to 0$$

is exact. Finally, again applying (ii) we conclude that $\text{im } g \otimes N \to M'' \otimes N$ remains injective, so

$$M' \otimes N \to M \otimes N \to M'' \otimes N$$

is exact, as desired. 

**Remark 5.2.** Note that the right exactness of tensor product does not imply condition (iii) for arbitrary $R$-modules, since we really used the surjectivity of the second map in checking that exactness in the middle is preserved.

The following properties are easy to check::

**Proposition 5.3.** Flatness satisfies the following:

(i) $R$ is flat as an $R$-module;
(ii) $\bigoplus M_i$ is flat if and only if every $M_i$ is flat;
(iii) A free $R$-module is flat.

The last part of the proposition has a converse for local rings:

**Theorem 5.4.** Suppose that $M$ is a finitely generated flat $R$-module, and $R$ is a local ring. Then $M$ is free.

The theorem is relatively easy for Noetherian rings, but not so easy in general; see Lang for the proof.

Since every vector space over a field is free, we conclude:

**Corollary 5.5.** When $R$ is a field, tensoring always preserves exactness.

**Exercise 5.6.** Show that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module which is not free.

**Exercise 5.7.** Show that if $R$ is an integral domain, and $M$ has torsion (that is, there are non-zero elements $m \in M$ and $r \in R$ such that $rm = 0$), then $M$ is not flat. What about if $R$ is not an integral domain?

We mention without proof a couple more properties of flat modules.

**Proposition 5.8.** Suppose that

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence, and $M''$ is flat. Then for any $R$-module $N$, the sequence

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is still exact.

**Proposition 5.9.** A module $M$ is flat if and only if for every ideal $I$ of $R$, the map $I \otimes M \to M$ sending $i \otimes m$ to $im$ is injective.
CHAPTER 2

Semisimplicity

1. Remarks on non-commutative rings

We begin with some reminders and remarks on non-commutative rings. First, recall that if $R$ is a not necessarily commutative ring, an element $x \in R$ is invertible if it has both a left (multiplicative) inverse and a right (multiplicative) inverse; it then follows that the two are necessarily equal. Note however, that unlike the example of $n \times n$ matrices over a field, in a general noncommutative ring having a left inverse does not imply the existence of a right inverse, and vice versa. When $R$ is not commutative, we will use the term “module” over $R$ to mean ”left module” over $R$.

Furthermore, the standard definitions of matrices and matrix multiplication all work perfectly over non-commutative rings, and in particular given a non-commutative ring $R$, we have the ring $\text{Mat}_n(R)$ of $n \times n$ matrices with coefficients in $R$. We say that $R$ is a division ring if every nonzero elements is invertible; that is, a division ring is a field which is not necessarily commutative. Then, much of the theory of vector spaces over a field extends to modules over a division ring: in particular, they are all free, and any two bases of a given module have the same cardinality. We will still refer to modules over division rings as “vector spaces”, and the cardinality of a basis as the “dimension.”

Example 1.1. The quaternions form a division ring over the real numbers.

Working over a division ring, homomorphisms between finite-dimensional vector spaces still correspond to matrices, as in the field case. However, we will want to work with a somewhat more general version of this correspondence. Suppose $R$ is any ring, and let $M = M_1 \oplus \cdots \oplus M_n$, $N = N_1 \oplus \cdots \oplus N_m$ for some $R$-modules $M_i$, $N_i$. Then the group $\text{Hom}_R(M,N)$ is in natural bijection with “matrices” of the form
\[
\begin{pmatrix}
\varphi_{11} & \cdots & \varphi_{1n} \\
\vdots & \ddots & \vdots \\
\varphi_{m1} & \cdots & \varphi_{mn}
\end{pmatrix},
\]
where $\varphi_{i,j} \in \text{Hom}_R(M_j, N_i)$. As a special case of this:

Proposition 1.2. Let $M$ be an $R$-module. Then for any $n > 0$ we have a ring isomorphism
\[
\text{End}_R(E^\oplus n) \cong \text{Mat}_n(\text{End}_R(E)).
\]

Warning 1.3. There is a subtlety which arises in the noncommutative case, which is that if $M$ is a one-dimensional vector space over a division ring $D$, with basis $\{v\}$, then we have a natural map of groups $D \to \text{End}_D(M)$ sending $c \in D$ to the map sending $v$ to $c \cdot v$. However, although this map is a bijection, it is not a ring isomorphism, as one can see that it reverses the order of multiplication.

Warning 1.4. Recall also that the definition, given $R$-modules $M, N$, of the $R$-module structure on $\text{Hom}_R(M, N)$ required $R$ to be commutative in order to get a module structure. (*) huh?
2. Semisimple modules

**Definition 2.1.** A module $M$ over a ring $R$ is **simple** if it is nonzero and has no submodules other than 0 and itself. $M$ is **semisimple** if it is a direct sum of simple modules.

**Example 2.2.** $\mathbb{Z}/p\mathbb{Z}$ is a simple $\mathbb{Z}$-module for any $p$, and accordingly any direct sum of such modules is semisimple. However, $\mathbb{Z}/p^2\mathbb{Z}$ is not semisimple: it is clearly not a direct sum of two or more submodules, but on the other hand, it is not simple, since it contains the submodule $p\mathbb{Z}/p^2\mathbb{Z}$.

One of the most basic facts about simple and semisimple modules is that it is relatively easy to describe their endomorphism rings. We begin with the case of simple modules.

**Proposition 2.3** (Schur’s lemma). **Suppose** $M, N$ **are simple** $R$-modules. **Then** every nonzero homomorphism $M \to N$ **is an isomorphism. In particular,** $\text{End}_R(M)$ **is a division ring.**

**Proof.** Let $\varphi : M \to N$ be a nonzero homomorphism. Its kernel and image are submodules of $M$ and $N$ respectively, and since $\varphi \neq 0$, simplicity implies that we must have $\ker \varphi = 0$ and $\im \varphi = N$. Thus, $\varphi$ is bijective, and hence an isomorphism. $\square$

We can now apply this to semisimple modules.

**Proposition 2.4.** **Suppose** $M = M_1^\oplus n_1 \oplus \cdots \oplus M_r^\oplus n_r$, **with** the $M_i$ **simple** $R$-modules, and pairwise non-isomorphic. **Then** the $M_i$ and $n_i$ **are uniquely determined up to permutation, and the ring** $\text{End}_R(M)$ **can be realized as a ring of matrices of the form**

\[
\begin{pmatrix}
N_1 & \cdots & 0 \\
\vdots & N_2 & \vdots \\
0 & \cdots & N_r
\end{pmatrix},
\]

\((*)\) **where** $N_i = \text{Mat}_{n_i}(D_i)$, **with** $D_i$ **the division ring** $\text{End}_R(M_i)$.

**Proof.** The matrix description follows immediately from our previous discussion, together with Proposition 2.3. It thus remains to see the asserted uniqueness. Suppose we have also $M \cong N_1^\oplus m_1 \oplus \cdots \oplus N_s^\oplus m_s$, so that we have an isomorphism

\[M_1^\oplus n_1 \oplus \cdots \oplus M_r^\oplus n_r \cong N_1^\oplus m_1 \oplus \cdots \oplus N_s^\oplus m_s.\]

Since this map is injective, we see that for each $i$, the induced map $M_i \to N_j^\oplus m_j$ is nonzero, and thus that for some $j$, the induced map $M_i \to N_j^\oplus m_j$ is nonzero. This last map is determined by $m_j$ maps $M_i \to N_j$, at least one of which must be nonzero. We conclude by Proposition 2.3 that $M_i \cong N_j$. Thus, each $M_i$ is isomorphic to some $N_j$, and considering the inverse map we conclude that $r = s$ and (up to isomorphism) the $N_j$ are just a permutation of the $M_i$.

It is thus enough to see that if we have a simple module $M'$, with $(M')^\oplus n \cong (M')^\oplus n'$, then $n = n'$. But we have $\text{End}_R(M')^\oplus n \cong \text{Mat}_n(\text{End}_R(M'))$, and $\text{End}_R(M')^\oplus n' \cong \text{Mat}_{n'}(\text{End}_R(M'))$, and we can check that these isomorphisms are actually isomorphisms of vector spaces over $\text{End}_R(M')$ (which is a division ring, by Proposition 2.3). Since isomorphic vector spaces have the same dimension, we conclude that $n^2 = (n')^2$, and hence that $n = n'$. $\square$

We now move on to give some equivalent characterizations of semisimple modules:

**Proposition 2.5.** **Suppose** that $M$ **is an** $R$-module. **Then** the following are equivalent:

(i) $M$ **is semisimple;**

(ii) $\text{End}_R(M)$ **is a division ring;**

(iii) $M$ **is a direct sum of simple modules;**

(iv) $M$ **is a direct product of simple modules.**
modules.

\( M \) and \( M_\phi \) if it is 0, we can add \( i \) of \( v \) submodule of \( M \) by letting the surjective, as claimed.

By hypothesis, we have some submodule \( N \) sending \( r \) for some \( M \). Then every \( Rv \) then have that \( Rv \) is semisimple.

2.5 (ii), we conclude that \( N \) would yield a proper submodule of \( Rv \), which contradicts the definition of \( M \), as desired.

By Proposition 2.5 (iii), we can write \( \varphi_j : M' \oplus \bigoplus_{j \in J} M_j \to M \) is injective, then in fact \( \varphi_j \) is an isomorphism. It suffices to show that \( M_i \) is contained in the image of \( \varphi_j \) for all \( i \in I \). By the simplicity of the \( M_i \), we have that \( M_i \cap \text{im} \varphi_j \) is either 0 or \( M_i \). But if it is 0, we can add \( i \) to \( J \), contradicting the maximality. Thus we conclude that \( \varphi_j \) is indeed surjective, as claimed.

Our claim proves both that (ii) implies (i), by setting \( N = 0 \), and also that (i) implies (iii), by letting the \( M_i \) be the simple modules in a direct sum decomposition of \( M \). It thus remains to see that (iii) implies (ii), for which the crucial point is that under hypothesis (iii), every nonzero submodule of \( M \) must contain a simple module. This is equivalent to showing that for every nonzero \( v \in M \), the module \( Rv \) contains a simple submodule. Let \( I \subseteq R \) be the kernel of the map \( R \to M \) sending \( r \) to \( rv \). Then because \( v \neq 0 \), we have \( I \neq R \), and by Zorn’s lemma \( I \) is contained in some maximal left ideal \( m \) of \( R \). We then have that \( mv \) is a maximal submodule of \( Rv \), not equal to \( Rv \). By hypothesis, we have some submodule \( N \subseteq M \) such that \( mv \oplus N = M \). Because \( mv \subseteq Rv \), we then have that \( Rv = mv \oplus (N \cap Rv) \). Then we have that \( N \cap Rv \), since any proper submodule would yield a proper submodule of \( Rv \) strictly containing \( mv \), contradicting maximality. We thus conclude that \( N \cap Rv \) is the desired simple submodule of \( Rv \).

We can now easily conclude (ii): let \( M' \subseteq M \) be the sum of all simple submodules of \( M \). If \( M' \neq M \), we have \( M = M' \oplus N \) for some \( N \subseteq M \) nonzero. But then \( N \) contains a simple submodule, contradicting the definition of \( M' \). We therefore conclude that the three conditions are equivalent, and our argument has also shown that every nonzero submodule of \( M \) contains a simple submodule, as desired. \( \square \)

**Corollary 2.6.** If \( M \) is a semisimple \( R \)-module, then every submodule or quotient module of \( M \) is semisimple.

**Proof.** Let \( N \subseteq M \) be a submodule, and let \( N' \subseteq N \) be the sum of all simple submodules. By Proposition 2.5 (iii), we can write \( M = N' \oplus M' \) for some \( M' \subseteq M \). Since \( N' \subseteq N \), we have \( N = N' \oplus (M' \cap N) \). If \( M' \cap N \neq 0 \), then by Proposition 2.5 it must contain a simple submodule, but this violates the definition of \( N' \), so we conclude that \( M' \cap N = 0 \) and \( N = N' \). By Proposition 2.5 (ii), we conclude that \( N \) is semisimple.

Now suppose that \( N = M/N' \) for some \( N' \). Then by Proposition 2.5 (ii), we have \( M = N' \oplus M' \) for some \( M' \subseteq M \). Then the quotient map \( M \to M/N' \) yields an isomorphism \( M' \to N \), and we have \( M' \) semisimple by the case of submodules above, so we conclude that \( N \) is likewise semisimple. \( \square \)

**Corollary 2.7.** Suppose that a ring \( R \) is semisimple when considered as a module over itself. Then every \( R \)-module is semisimple.

**Proof.** It is clear from the definition that a direct sum of semisimple modules is semisimple, so if \( R \) is semisimple over itself, then so is every free \( R \)-module. Since every module is a quotient of a free module, we conclude from Corollary 2.6 that every \( R \)-module is semisimple. \( \square \)
**Example 2.8.** A commutative ring \( R \) is simple as a module over itself if and only if its only ideals are 0 and \( R \), if and only if \( R \) is a field.

It is almost as rare for \( R \) to be semisimple as a module over itself. Specifically, suppose that \( R = \bigoplus I_i \) for some simple ideals \( I_i \). Then for any \( i \neq j \), we have \( I_i I_j \subseteq I_i \cap I_j = (0) \), so we see that (assuming \( R \) is not a field, so there is more than one of the \( I_i \)) we cannot have \( R \) an integral domain. (*)

Expanding on same argument shows we get direct sum decomposition as rings, at least in finite case. Wtf in infinite case?