LIMIT LINEAR SERIES
AN INTRODUCTION

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Our central objects of study are nonsingular curves, which were classically studied via linear series. Limit linear series are a generalization of linear series on nonsingular curves to (certain classes of) reducible curves. Their purpose is to examine how linear series degenerate on a family of smooth curves degenerating to a reducible curve. This degeneration technique can then be used to prove classical theorems about nonsingular curves, for example Brill-Noether and Clifford. We begin with a short review of linear series on a smooth curve. Next we provide an in depth analysis of sections 1 and 2 of [1], then end with a sketch of its remaining 3 sections. The appendix will contain a short derivation of the Brill-Noether number, \( \rho \). We will generally work over \( \mathbb{C} \); at times the subscript will be dropped to simplify notation.

Linear Series Background and EH §0

Linear Series. Let \( Y \) be a nonsingular genus \( g \) curve, projective over \( \mathbb{C} \). We want to analyze \( Y \), and we do so by looking at morphisms \( Y \to Z \) for suitably chosen \( Z \). One’s first hope would be something simple, say Spec \( A \), or possibly \( \mathbb{A}^n_\mathbb{C} \), where we’d most likely choose \( A = \mathbb{C} \). However, morphisms \( Y \to \text{Spec} \; A \) are in bijection with ring maps \( A \to \Gamma(Y, \mathcal{O}_Y) \). And as \( Y \) is projective (or more generally, not affine), its ring of global sections, \( \mathbb{C} \), does not at all characterize its geometry. Our next simplest target is projective space, \( \mathbb{P}^r_\mathbb{C} \).

Suppose we have a \( \mathbb{C} \)-morphism \( \varphi : Y \to \mathbb{P}^r \) of degree \( d \). Then writing \( x_i \) for the global sections generating \( \mathcal{O}_{\mathbb{P}^r}(1) \), it is a fact that \( \varphi^*(\mathcal{O}(1)) \) is a degree \( d \) line bundle on \( Y \) which is generated by the global sections \( \sigma_i = \varphi^*(x_i) \). In fact, the converse holds: any degree \( d \) line bundle \( \mathcal{L} \) on \( X \) generated by global sections \( \sigma_0, \ldots, \sigma_r \) induces a unique degree \( d \) \( \mathbb{C} \)-morphism \( \varphi : X \to \mathbb{P}^r \) such that \( \varphi^*(\mathcal{O}(1)) \cong \mathcal{L} \) and \( \varphi^*(x_i) \cong \sigma_i \). For a proof of this, see [3] II.7.1. Additionally, if we instead consider the vector space (of dimension \( r + 1 \)) spanned by the global sections of \( \mathcal{L} \), then the morphism to projective space is specified only up to a linear change of coordinates, i.e., automorphism, on \( \mathbb{P}^r \).

By a linear series on \( Y \) we mean a pair \( L = (\mathcal{L}, V) \) where \( \mathcal{L} \) is a line bundle on \( Y \) and \( V \) is a vector subspace of the space of global sections \( \Gamma(Y, \mathcal{L}) \); at times \( V \) will be given by a basis \( s_0, \ldots, s_r \) of global sections of \( \mathcal{L} \). By our above discussion, this almost gives a map to projective space: almost, because we need that \( V \) has a basis which globally generates \( \mathcal{L} \). Such a linear series is called basepoint free. If this is not the case, then all global sections of \( V \) vanish (in the fiber of \( \mathcal{L} \)) at some point \( P \in X \). We call such \( P \) a base point of the linear series, and the family of base points the base locus or base divisor \( B \).

One can think of the morphism obtained from a linear series with basepoints \( (\mathcal{L}, V) \) in two ways. Away from the base locus, the linear series does give a morphism
to projective space. Hence we always obtain a birational map from a nonsingular curve to the complete variety $\mathbb{P}^r$. One implicitly obtains a morphism by recalling that every such birational map can be extended, [4] Theorem 2.1.

A more concrete construction is to instead consider the linear series $(\mathcal{L}(D), V)$, where $D$ is the base divisor. Then that ‘same’ subspace of global sections, interpreted now in the twisted sheaf $\mathcal{L}(D)$, is basepoint free, and immediately gives a morphism to $\mathbb{P}^r$. Moreover, if the degree of the morphism, as constructed above from the birational map, has degree $d$, then this construction yields a morphism of degree $d - \deg(D)$.

Hence we see that any linear series uniquely determines a degree $d$ morphism to $r$-dimensional projective space. A $g^d_r$ is a linear series $(\mathcal{L}, V)$ of degree $d$ with vector space of dimension $r + 1$. For a section $\sigma$ in $V$, and $P \in Y$ we write $\text{ord}_P \sigma$ to denote the order of vanishing of $\sigma$ at $P$.

Claim There are exactly $r + 1$ distinct integers for the orders of vanishing of sections of $V$.

Proof. Let $a$ denote the highest order of vanishing of sections of $V$. Consider the subvector space $V(-bP)$ consisting of sections of $V$ vanishing to order at least $c$ at $P$, where $c \geq 0$. We claim that $V(-bP)/V(-(b+1)P)$ has dimension zero or one. Suppose that $V(-bP) \neq V(-(b+1)P)$, and let $t$ be a local parameter of $Y$ at $P$. Then there is some $\sigma \in V(-bP) \setminus V(-(b+1)P)$, and we can write $\sigma = t^b \rho$, where $\rho$ is a local section (defined in some neighborhood of $P$) of $\mathcal{L}$ nonvanishing at $P$. We show that $\sigma$ spans the quotient $V(-bP)/V(-(b+1)P)$. Consider any section $\sigma' \in V(-bP)$. Then writing $\sigma' = t^b \rho'$ with $\rho'$ a rational section of $\mathcal{L}$ defined and nonvanishing at $P$, we must have $b' \geq b$. Then, as $b' - b \geq 0$, the global section $\sigma' = t^{b'-b}(\rho')/\rho(P) \sigma$ vanishes to order at least $b'+1 > b$ at $P$, and is therefore trivial in $V(-bP)/V(-(b+1)P)$. Hence the quotient space is generated by $\sigma$.

Thus we obtain a filtration of our vector space $V \supset V(-P) \supset V(-2P) \supset \cdots \supset V(-aP) \supset V(-(a+1)P) = 0$ where $V(-aP) \neq 0$ and $V(-bP)/V(-(b+1)P)$ has dimension zero or one. After removing duplicates from the left to the right (i.e., if $V(-bP) = V(-(b+1)P)$ remove $V(-bP)$), we are left with a filtration of a vector space such that each consecutive quotient has dimension 1. Thus each vector space in the filtration must decrease dimension by one, and as $V$ is $r + 1$ dimensional, there must be exactly $r + 1$ nontrivial vector subspaces in our filtration.

We label the $r + 1$ distinct orders of vanishing by $a^L_i(P)$ with $i = 0, \ldots, r$, in ascending order and call it the vanishing sequence of $L$ at $P$. Defining $\alpha^L_i(P)$ to be $a^L_i(P) - i$, the nondecreasing sequence $(\alpha^L_0(P), \ldots, \alpha^L_r(P))$ of (nonnegative) integers is called the ramification sequence of $L$ at $P$.

Caution: Even if $\{\sigma_i\}$ is a basis for $V$, it is not always the case that

$$\{\text{ord}_P \sigma_i \} = \{a^L_i(P)\},$$

though we can, for each $P$, pick a basis whose orders of vanishing are precisely the vanishing sequence. Moreover, it is impossible to choose a basis for $V$ that has this
property everywhere; indeed, if \( \sigma_i \) vanished to order \( a_i^L \) at every \( P \), then \( \sigma_1 \) through \( \sigma_r \) would be identically zero!

We say that \( L \) is unramified at \( P \) if \( \alpha_i^L(P) = 0 \) for all \( i \), otherwise, \( P \) is a ramification point of \( L \). The sum of the ramification sequence at \( P \) is the weight of \( L \) at \( P \):

\[
w^L(P) = \sum_{i=0}^{r} \alpha_i^L(P) = \left( \sum_{i=0}^{r} a_i^L(P) \right) - \left( r + 1 \right)
\]

There are only finitely many ramification points of \( L \). The proof of finiteness comes from the Plucker formula, see [2]. Because \( \alpha_i^L(P) \neq 0 \) only finitely many times, the weight is also nonzero only finitely many times, and hence we can sum the weight over the entire smooth curve. This quantity has a combinatorial description via the Plucker Formula,

\[
\sum_{P \in Y} w^L(P) = \sum_{P \in Y} \sum_{i=0}^{r} \alpha_i^L(P) = (r + 1)d + \left( \frac{r + 1}{2} \right)(2g - 2)
\]

Note that in the case of a basepoint free \( g_1^d \) (a branched cover of \( \mathbb{P}^1 \)), the formula reduces to

\[
\sum_{P \in Y} (a_i^L(P) - 1) = 2d + (2g - 2)
\]

i.e., the Riemann-Hurwitz formula

\[
\chi(Y) = \chi(\mathbb{P}^1)d - \sum_{P \in Y} (e_P - 1)
\]

where \( e_P \) denotes the ramification at \( P \).

**EH §0: Preliminaries.** By curve we mean a reduced and projective curve over \( \mathbb{C} \) having only ordinary nodes for singularities. Notice this means we can write \( X = Y_1 \cup \cdots \cup Y_n \) as a decomposition into irreducible components (possible as \( X \) is finite type over \( \mathbb{C} \), and hence noetherian), where each \( Y_i \) is a reduced and irreducible (though possible singular) projective curve over \( \mathbb{C} \). By genus of \( X \) we mean the arithmetic genus, \((-1)^{\dim X} \chi(\mathcal{O}_X) - 1 = H^1(X, \mathcal{O}_X)\). With the decomposition of \( X \) as above into irreducible components, on can quickly calculate this genus with the formula

\[
g(X) = \sum_{i=1}^{n} g(Y_i) - \#(\text{components}(X)) + \#(\text{nodes}(X)) + 1
\]

where \( g(Y_i) \) denotes the geometric genus of the normalization of \( Y_i \). This formula leads one to say ‘genus is additive’, as the genus of two curves intersecting in a node is simply the sum of the individual genera.

We define the dual graph of a curve \( X \) as follows: the dual graph has one vertex for each irreducible component of \( X \), and an edge per node, connecting vertices when the corresponding irreducible components intersect. Hence a cusp singularity will give a loop: an edge connecting a vertex to itself.

We say that a curve \( X \) has compact type if its dual graph is a tree. Recall that \( \text{Pic}^0(X) \) denotes the variety of line bundles on \( X \) whose restriction to each irreducible component is degree zero. Equivalently \( X \) is compact type if \( \text{Pic}^0(X) \) is compact. Conceptually, a line bundle on a compact type curve having only ordinary nodal singularities is uniquely determined by its restriction to each irreducible
component. In this case, \( \text{Pic}^0(X) \) is simply the product \( \text{Pic}^0(Y_i) \) over each irreducible component \( Y_i \). Conversely, if \( X \) is not of compact type, then \( \text{Pic}^0(X) \) has more freedom; namely the way in which line bundles are pasted together. As one glues around a loop in the dual graph, one obtains \( \mathbb{C}^* \) worth of choices for gluing the line bundles, and as \( \mathbb{C}^* \) isn’t compact, neither is \( \text{Pic}^0(X) \).

Note that a reducible curve with only ordinarily nodal singularities has compact type if and only if it can be drawn as a tree: there is no loop of intersecting irreducible components \( Y_1 \neq Y_2 \neq \cdots \neq Y_n \) where \( Y_i \cap Y_{i+1} \neq \emptyset \) and \( Y_1 = Y_n \).

EH §1: Limit Series

Let \( X \) be a curve of compact type. Though we can define a linear series on \( X \) in terms of invertible sheaves and a vector subspace of sections, it is actually more fruitful (in the context of deformations) to consider individual linear series on each irreducible component of \( X \). Subject to an elementary intersection inequality, this is exactly how we define a limit series on \( X \).

A crude limit \( g^r_d \), also called a crude limit series, denoted \( L \), on \( X \), is, for each irreducible component \( Y \) of \( X \), a \( g^r_d \) = \((L_Y, V_Y)\) on \( Y \), which we call the \( Y \)-aspect of \( L \). Moreover, we require the Compatibility Condition hold: for any irreducible components \( Y, Z \) of \( X \) which meet at a point \( P \in X \), the vanishing sequences satisfy

\[
\text{(CC)} \quad a^L_Y(i) + a^L_Z(r-i) \geq d
\]

for \( i = 0, 1, \ldots, r \). A crude limit series is refined providing (CC) is an equality at all times. For brevity, we will now refer to a refined crude limit series on \( X \) as a refined limit series, or even shorter, a limit series.

The compatibility condition seems quite mysterious, but the following proposition shows that at least in the case of refined limit series, we are extending linear series.

**Proposition (1.1).** (Plucker Formula) Let \( X \) be a genus \( g \) curve of compact type. If

\[
\{(L_Y, V_Y) \mid Y \text{ a component of } X\}
\]

is a crude limit \( g^r_d \) on \( X \) then

\[
\sum_{P \in X \text{ Nonsingular}} w^L(P) \leq (r + 1)d + \left( \frac{r + 1}{2} \right)(2g - 2)
\]

with equality if and only if \( L \) is a limit series.

**Proof.** We prove the case for \( X \) the union of two irreducible components, the general case done by induction. So suppose \( X = Y \cup Z \) with \( Y, Z \) nonsingular irreducible curves of genus \( g_Z, g_Y \), meeting at a point \( P \in X \). Note that \( g_Z + g_Y = g \), as genus
is additive. We have

\[
  w^L_Y(P) = \sum_{i=0}^{r} \alpha_i^L_Y(P) \\
  = \sum_{i=0}^{r} \alpha_i^L_Y(P) - i \\
  \geq \sum_{i=0}^{r} (d - a_{r-i}^L(Z)) - i \\
  = \sum_{i=0}^{r} d - a_{i}^L(Z) - i,
\]

where the inequality follows directly from (CC). Another equality we’ll need is

\[
  \sum_{i=0}^{r} (a_i^L(Z) - i) + \sum_{i=0}^{r} (d - a_i^L(Z) - i) = \sum_{i=0}^{r} (d - 2i) \\
  = (r + 1)d - (r + 1)r \\
  = (r + 1)d + \binom{r + 1}{2} (2g_Z - 2) \\
  + (r + 1)d + \binom{r + 1}{2} (2g_Y - 2) \\
  - \left[ (r + 1)d + \binom{r + 1}{2} (2g - 2) \right],
\]

for all \(Q \in Z\). Then we can compute

\[
  \sum_{P \in X^{\text{Nonsingular}}} w^L(P) = \sum_{Q \in Y} w^L_Y(P) + \sum_{Q \in Z} w^L_Z(Q) - w^L_Y(P) - w^L_Z(P) \\
  \leq (r + 1)d + \binom{r + 1}{2} (2g_Y - 2) \\
  + (r + 1)d + \binom{r + 1}{2} (2g_Z - 2) \\
  - \left[ \sum_{i=0}^{r} (d - a_i^L(Z) - i) + \sum_{i=0}^{r} (a_i^L(Z) - i) \right] \\
  = (r + 1)d + \binom{r + 1}{2} (2g - 2).
\]

\[\square\]

**EH §2: Limits of Linear Series**

The whole purpose of this section is to show that, subject to certain hypothesis, given a family of nonsingular curves degenerating to a singular curve, a family of linear series on the nonsingular curves degenerates to a refined limit series over the singular curve. Let \(\pi : X \rightarrow B\) be our family of curves. The hypothesis we impose are collectively called *Situation 2.0*: 
We can extend where one, fix a group of inclusions commute. So by base change over all of \( X \) a larger field \( K \) instead, it is a fact that if \( X \) is not defined over \( \mathbb{F} \), so that \( \mathbb{F} \) injects into \( O \). Notice that nonsingularity is not preserved by base change, and so this might introduce singularities on \( X \) (though the fibers of \( X \) remain unchanged). So we allow chains rational curves at nodes of \( X_0 \) in order to smooth \( X \). Hence we may take \( (\mathcal{L}_\eta, V_\eta) \) as a linear series on \( X_\eta \).

**Claim.** We can extend \( \mathcal{L}_\eta \) to a line bundle on \( X \), unique up to twisting by \( \mathcal{O}_X(D) \), where \( D \) is a divisor supported on \( X_0 \).

**Proof.** Since \( X \) is projective over \( B \), we have that \( X_\eta \) is projective over the field \( \mathbb{F} \). Moreover, \( X_\eta \) is an irreducible variety, and so \( \text{Pic}(X_\eta) \) injects into the class group of \( X_\eta \). Hence we may take \( sL_\eta = \mathcal{O}_{X_\eta}(E) \) for some Weil Divisor \( E \) on \( X_\eta \). Now we extend \( E \) to a divisor \( \overline{E} \) on \( X \) via the map \( P \mapsto \overline{P} \) for each prime divisor \( P \) of \( E \). Finally, as \( X \) is nonsingular over \( B \), we have a correspondence between Weil and Cartier divisors on \( X \), so \( \overline{E} \) is equivalently given by a Cartier divisor on \( X \). Hence \( \mathcal{O}_X(\overline{E}) = \mathcal{L} \) is a line bundle on \( X \) which is immediately seen to restrict to (a sheaf isomorphic to) \( \mathcal{L}_{X_\eta} \) on \( X_\eta \).

Moreover, \( \mathcal{L} \) is unique up to twisting by \( \mathcal{O}_X(D) \), where \( D \) is a divisor supported on \( X_0 \). For let \( \mathcal{L} \) and \( \mathcal{L}' \) be two extensions of \( \mathcal{L}_\eta \). Then fix an isomorphism \( \varphi: \mathcal{L}|_{X_\eta} \to \mathcal{L}'|_{X_\eta} \) and let \( s \) be a rational section of \( \mathcal{L} \). Consider the divisor \( D = \text{Div } s - \text{Div } \varphi(s) \). Now as \( \varphi \) is an isomorphism restricted to \( X_\eta \), we see that \( D \) is supported on \( X \setminus X_\eta = X_0 \). Finally, one can show that \( \varphi \) extends to an isomorphism \( \mathcal{L}' \cong \mathcal{L}(D) \).

It is clear that \( X_0 = \pi^*(\mathcal{O}(0)) = V(\pi^*(t)) \). Hence \( X_0 \) is a globally principal divisor, generated by the vanishing of the global section \( \pi^*(t) \). Then outside of \( X_0 \), \( \pi^*(t) \) is invertible. Hence \( \mathcal{O}_X(X_0) \cong \mathcal{O}_X \) simply by multiplication/division of \( \pi^*(t) \). Therefore twisting by all of \( X_0 \) gives an isomorphic sheaf; to get something new we need to twist by irreducible components of \( X_0 \).

Suppose \( \mathcal{L} \) has degree \( i \) on \( Y \) and \( d-i \) on \( Z \). We will analyze \( \mathcal{L}(Y) \). Notice that \( \mathcal{L}(Y)|_Z = \mathcal{L}|_Z(P) \) as \( Y \cap Z = P \). Then because \( Y + Z = X_0 \) is a principal divisor,
we have \( \mathcal{L}(Y)|_{Y} \cong \mathcal{L}(Y - Y - Z)|_{Y} \cong \mathcal{L}(-Z)|_{Y} = \mathcal{L}|_{Y}(-P) \), again, as \( Y \cap Z = P \).

So we see that \( \mathcal{L}(Y) \) has degree \( i - 1 \) on \( Y \) and \( d - i + 1 \) on \( Z \). Therefore we can have any pair \((i, d - i)\), where \( \mathcal{L} \) is an extension of \( \mathcal{L}_{\eta} \) of degree \( i \) on \( Y \) and \( d - i \) on \( Z \). Now \textit{a priori} there may be two nonisometric extensions of \( \mathcal{L}_{\eta} \) having the same pair of degrees on \( Y \) and \( Z \), but recall that above we showed the extension is unique up to twisting by a divisor supported on \( X_{0} \).

As the structure of divisors on \( X_{0} \) is quite simple, we see immediately that the pair \((i, d - i)\) uniquely specifies the extension of \( \mathcal{L}_{\eta} \). We will write \( \mathcal{L}_{Y} \) for the extension having degree \( d \) on \( Y \) and zero on all other irreducible components of \( X_{0} \).

Again, we use that fact that \( \mathcal{O}_{X}(-nX_{0}) = \pi^{*}(t)^{n} \cdot \mathcal{O}_{X} \) to deduce that \( \mathcal{L}_{Y}|_{X_{\eta}} \) is an \( \mathcal{O} \)-subsheaf of \( \mathcal{L}_{\eta} \), isomorphic to \( \mathcal{L}_{\eta} \) via multiplication by some power of \( \pi^{*}(t) \).

Now that we’ve extended our line bundle to all of \( X \) along a chosen irreducible component \( Y \) of \( X_{0} \), we must extend the vector space of sections \( V_{\eta} \) to all of \( X \).

Next we restrict both to \( Y \) in order to obtain a linear series on \( Y \).

\textbf{Claim} The global sections of \( \pi_{*}(\mathcal{L}_{Y}) \) form a free \( \mathcal{O} \)-module.

\textbf{Proof}. Because \( \pi \) is projective, the pushforward remains quasicoherent. So all that remains is to show freeness of the global sections of \( \pi_{*}(\mathcal{L}_{Y}) \). We have that \( \pi_{*}\mathcal{L}_{Y} \) is \( M \) for some finitely generated \( \mathcal{O} \)-module \( M \). Now \( \mathcal{O} \) is a principal ideal domain, so by the classification of finitely generated modules over \( \mathcal{O} \)-PIDs, we have that \( M \) is free if and only if \( M \) is torsion free. Let \( \sigma \in M \) be a global section. Then if \( t \cdot \sigma = 0 \), where, remember, \( t \) generates the maximal ideal of \( \mathcal{O} \), then \( t \cdot \sigma|_{X_{\eta}} = 0 \). But on \( X_{\eta} \), \( t \) is a unit. Hence we’d have \( \sigma|_{X_{\eta}} = 0 \), which would imply \( \sigma = 0 \) to begin with, as \( \mathcal{L}_{Y} \) is invertible on an integral scheme \( X_{\eta} \), and so torsion free. Therefore \( M \) is torsion free, and so the Spec \( \mathcal{O} \)-module is free. \( \Box \)

Define \( V_{Y} = V_{\eta} \cap \pi_{*}\mathcal{L}_{Y} \), i.e, sections in \( V_{\eta} \) extending to global sections of \( \mathcal{L}_{Y} \) under the isomorphism \( \mathcal{L}_{Y}|_{X_{\eta}} \cong \mathcal{L}_{\eta} \) chosen above. Notice this is a free \( \mathcal{O} \)-module of rank \( r + 1 \). Then we restrict these sections to lie in the special fiber, so \( V_{Y} = V_{Y} \otimes \kappa(0) = V_{Y} \otimes \mathcal{O}/t \) is a vector space of rank \( r + 1 \). Notice the rank stays constant, because we had a free module to begin with. Our construction is now complete: \( (\mathcal{L}_{Y}|_{Y}, V_{Y}) \) is now a \( t^{n} \)-module on \( Y \).

For each irreducible component \( Y \subseteq X_{0} \), the linear series \( (\mathcal{L}_{Y}|_{Y}, V_{Y}) \) is called the \textit{Y-aspect} of \((\mathcal{L}_{\eta}, V_{\eta}) \), and the collection \( L \) of \( Y \)-aspects over each irreducible component is the \textit{limit} of \((\mathcal{L}_{\eta}, V_{\eta}) \).

We now analyze the limit \( L \), ultimately showing it is always a crude limit and providing a characterization for it to be a refined limit.

For \( \sigma \in V_{Y} \), let \( \tilde{\sigma}_{Y} \) denote \( t^{n}\sigma \), where \( n \) is the least integer such that \( t^{n}\sigma \in \tilde{V}_{Y} \), and let \( \sigma^{Y} \) be the image of \( t^{n}\sigma \) in \( V_{Y} \). Let \( D_{\sigma} \) be the closure in \( X \) of the divisor \((\sigma) \) of \( X_{\eta} \).

\textbf{Proposition} (2.2). If \( Y \) and \( Z \) are components of \( X_{0} \) meeting at a point \( P \) and \( \sigma \in V_{\eta} \), then

\[ \text{ord}_{P} \sigma^{Y} + \text{ord}_{P} \sigma^{Z} = d + (D_{\sigma}Y)_{P} + (D_{\sigma}Z)_{P} \geq d. \]

Notice that \( \tilde{\sigma}^{Y} \) is a section of \( \mathcal{L}_{Y} \), and \( \sigma^{Y} \) is the restriction of a section of \( \mathcal{L}_{Y} \) to \( X_{0} \).

Recall \( \mathcal{L}_{Y} \) has degree \( d \) and \( Y \) and zero on irreducible components \( Z \) not equal to \( Y \). If \( \sigma \) was not zero to begin with, then by construction (i.e., our choice of \( n \) above), \( \tilde{\sigma}^{Y} \) cannot vanish on both \( Y \) and \( Z \). Now if \( \sigma^{Y} \) vanishes on \( Y \), then it vanishes at \( P \). Because \( \mathcal{L}_{Y} \) has degree zero on \( Z \), it would follow that \( \sigma_{Y} \) vanishes...
on \( Z \) also. Therefore we can conclude \( \sigma^Y \) doesn’t vanish on \( Y \), and similarly \( \sigma^Z \) doesn’t vanish on \( Z \).

**Proof.** For simplicity, we again take \( X = Y \cup Z \) with \( Y \cap Z = P \). Suppose \( \tilde{\sigma}^Y \) vanishes along \( Z \) to order \( a \) and \( \tilde{\sigma}^Z \) vanishes along \( Y \) to order \( b \). Because \( \text{ord}_P \sigma^Y = (Y,\{\tilde{\sigma}^Y = 0\})_P = a + (D_{\sigma} Y)_P \) and \( \text{ord}_P \sigma^Z = (Z,\{\tilde{\sigma}^Z = 0\})_P = b + (D_{\sigma} Z)_P \), it suffices to prove \( a + b = d \).

The key insight is

\[
L_Y(-dZ) = L_Z.
\]

To see this, notice that \( L_Y \) has degree \( d \) on \( Y \) and 0 and \( Z \), while \( L_Y(-dZ)|_Z = L_Y(dY)|_Z = L_Y(Z(dP)) \) has degree \( d \) on \( Z \), and so \( L_Y(-dY)|_Y \) has degree zero on \( Y \).

Now as \( \tilde{\sigma}^Y \) vanishes in \( Y \) to order \( a \), we have that \( \tilde{\sigma}^Y \in L_Y(\cdot -a) \), and interpreted in this sheaf, vanishes to order zero on \( Z \). Using our key insight above to view \( L_Y(-aZ) \) as \( L_Y((-d-a)Z) \), we have that \( t^{d-a}\tilde{\sigma}^Y \in L_Y(-dZ) = L_Z \). Then because \( t^{d-a}\tilde{\sigma}^Y \) vanishes to order zero in the sheaf \( L_Y(-dZ) = L_Z \) and because a section of \( L_Z \) is determined by its restriction to \( L_Z \), we can conclude that \( t^{d-a}\tilde{\sigma}^Y = \tilde{\sigma}^Z \) as sections of \( L_Z = L_Y(-dZ) \).

Now as \( \tilde{\sigma}^Z \) vanishes to order \( b \) on \( Y \) as a section of \( L_Z \), we can twist these sections by \( t^{-b} \). But we also want to interpret everything in \( L_Y = L_Z(-dY) \) now, so we must twist by \( t^d \). The net effect is that we can multiply our equation by \( t^{-b} \) and interpret the resulting section as an element of \( L_Y = L_Z(-dY) \). Doing so, we obtain

\[
t^{d-b} \tilde{\sigma}^Z = t^{d-b+d-a} \tilde{\sigma}^Y.
\]

Notice that this section doesn’t vanish on \( Y \), for \( \tilde{\sigma}^Z \) vanished on \( Y \) to order \( b \) as a section of \( L_Z \), and we untwisted that vanishing (the multiplication by \( t^d \) was simply to get everything in \( L_Y \)).

The section \( t^{d}\tilde{\sigma}^Y \) also doesn’t vanish on \( Y \), as a section of \( L_Z(-dY) = L_Y(-X_0) \cong L_Y \). Then because we have two sections of \( L_Y \), both coming from \( \sigma \), which don’t vanish on \( Y \), they must be equal. So we can say \( t^{d-b+d-a}\tilde{\sigma}^Y = t^d\tilde{\sigma}^Y \), and hence conclude that \( d = d - b + d - a \), proving the desired equality.

**□**

**Lemma** (2.3). (Adapted bases). If \( Y \) and \( Z \) are two irreducible components of \( X_0 \) and \( Q \in Y \), then there exists a basis \( \{\sigma_i\} \) of \( V_{\eta} \) such that

1. \( \{\sigma^Y_i\} \) is a basis of \( V_Y \);
2. \( \{\sigma^Z_i\} \) is a basis of \( V_Z \);
3. \( (\text{ord}_Q \sigma^Y_1, \ldots, \text{ord}_Q \sigma^Y_r) \) is precisely the vanishing sequence of \( (L_Y, V_Y) \) at \( Q \).

**Proof.** Omitted.  

**□**

**Remark.** A basis such as in lemma 2.3 is said to be adapted to \( Q \in Y \) and \( Z \). Moreover, it follows that if \( \{\sigma_i\} \) is a basis adapted to \( P \in Y \) and \( Z \), where \( P = Y \cap Z \), then \( (\text{ord}_P \sigma^Y_1, \ldots, \text{ord}_P \sigma^Y_r) \) is precisely the vanishing sequence of \( (L_Z, V_Z) \) at \( P \). Note that the order is reversed.

**Lemma** (2.4). Suppose that \( a_0 < \ldots < a_r \) are integers. If \( \{b_i\} \) are distinct integers with \( a_i \leq b_i \) for each \( i \), and \( \sigma \) is a permutation such that

\[
b_{\sigma(i)} < \cdots < b_{\sigma(r)}
\]

then \( a_i \leq b_{\sigma(i)} \). Further, if \( a_j = b_{\sigma(j)} \) for some \( j \) then \( \sigma(j) = j \) and hence \( a_j = b_j \).
Proof. This is a typical induction argument. Clearly it is true for \( r = 1 \). Now suppose that \( b_j \) is the largest integer, so that \( \sigma(r) = j \). Then we have \( a_i \leq b_{i-1} \) for all \( i \geq j \). Swapping \( b_j \) with \( b_r \) allows us to apply induction to the first \( r \) terms, giving the result.

**Proposition (2.1).** \( L \) is a crude linear series.

Proof. Let \( P = Y \cap Z \) be a node of \( X_0 \) and \( \{ \sigma_i \} \) be a basis of \( V_0 \) adapted to \( P \in Y \) and \( Z \). Then the sequence \( (\text{ord}_P \sigma_i^Z) \) is, by Proposition 2.2, at least \( d - \text{ord}_P \sigma_i^Y \). Because we are adapted, the vanishing sequence of \( (\mathcal{L}_Z|_Z, V_Z) \) is simply a rearrangement of \( \text{ord}_P \sigma_i^Z \). Hence, by Lemma 2.4, we obtain

\[
\text{ord}_P \sigma_{r-i}^Z \geq d - \text{ord}_P \sigma_i^Y.
\]

**Proposition (2.5).** For a limit \( L \) of \( (\mathcal{L}, V) \), the following are equivalent.

1. \( L \) is a limit series.
2. For each node \( P = Y \cap Z \) of \( X_0 \), each free basis \( \check{\sigma}_i^Y \) of \( \check{V}_Y \) adapted to \( P \in Y \) and \( Z \), and each \( \sigma \in \text{span}_{\kappa(0)} \{ \check{\sigma}_i^Y \} \), we have \( \text{ord}_P \sigma^Y + \text{ord}_P \sigma^Z = d \).
3. No ramification points of \( (\mathcal{L}, V) \) specialize to a node of \( X_0 \).

Remark. It may be confusing to speak of the \( \kappa(0) \)-span of \( \check{\sigma}^Y \), as the \( \check{\sigma}^Y \in \check{V} \), which is not a \( \kappa(0) \)-vector space. But remember we have \( \check{V} = \pi_*(\mathcal{L}) \cap V \), which we can view as a coherent sheaf coming from a free \( \mathcal{O} \)-module. With this interpretation, the sections \( \check{\sigma}^Y \) are lattice points in the \( \kappa(0) = \mathcal{O}/t \)-vector space \( V = \check{V} \otimes_{\mathcal{O}} \mathcal{O}/t \).

Proof. \( 2 \Rightarrow 1 \) Assume \( L \) is a limit series. Then we have \( a_i^L = a_j^L(P) + a_{r-i}^L(P) = d \). Now let \( \sigma_i \) be a basis of \( V_0 \) adapted to \( P \in Y \) and \( Z \). Then, as in the proof of 2.1, the extensions \( \check{\sigma}_i^Y \) and \( \check{\sigma}_i^Z \) form bases of \( V_Y \) and \( V_Z \) such that

\[
\text{ord}_P \sigma_i^Y + \text{ord}_P \sigma_i^Z = d,
\]

for the vanishing sequence of \( V_Z \) at \( P \) is given, in descending order, by \( \text{ord}_P \sigma_i^Z \).

Now fix \( \sigma \in \text{span}_{\kappa(0)} \{ \check{\sigma}_i^Y \} \). Then \( \text{ord}_P \sigma = \text{ord}_P \sigma_i^Y \) for some fixed \( i \). Then we can write

\[
\sigma = s_i \check{\sigma}_i^Y + \sum_{j>i} s_j \check{\sigma}_j^Y
\]

with \( s_i \neq 0 \) and \( s_j \in \kappa(0) \) for \( j \geq i \), since lower order terms (i.e., terms with \( \check{\sigma}_k^Y \) with \( k < i \)) would make \( \text{ord}_P \sigma < \text{ord}_P \check{\sigma}_i^Y \).

Viewing \( \sigma \) now in the \( \kappa(0) \)-span of \( \{ \check{\sigma}_i^Z \} \), recalling that the vanishing orders are reversed as in our remark following Proposition 2.3, we can conclude that \( \sigma \) can be written as a \( \kappa(0) \)-combination of \( \check{\sigma}_j^Z \) for \( j \leq i \). Hence we can conclude \( \text{ord}_P \sigma^Z = \text{ord}_P \sigma_i^Z \). Therefore \( \text{ord}_P \sigma^Y + \text{ord}_P \sigma^Z = \text{ord}_P \sigma_i^Y + \text{ord}_P \sigma_i^Z = d \).

1 \( \iff \) 3 Follows from a semicontinuity argument, namely, a ramification point \( Q \) of \( (\mathcal{L}, V) \) having weight \( w \) which specializes to a smooth point of \( P \in Y \) of \( X_0 \) will immediately give the inequality \( w^L(P) \geq w \).

**Proposition (2.6).** Let \( (\mathcal{L}, V) \) be a line bundle on \( X_n \). After blowing up nodes of \( X_0 \) sufficiently often, making a finite base change, and resolving the resulting singularities of \( X \), we obtain a family \( X''/B' \) such that \( X'' \rightarrow B' \) satisfies situation 2.0, having generic fiber \( X''_n = X_n \), special fiber \( X'_0 \) derived from \( X_0 \) by inserting
chains of rational curves at nodes of $X_0$, and such that the limit of $(\mathcal{L}_\eta, V_\eta)$ on $X'_0$ is a limit series.

Proof. Omitted. \hfill \square

§3: EQUATIONS AND DEFORMATIONS

Let $X_0$ be a curve of compact type with a limit linear series $g^r_d$. We say this limit can be smoothed if it is the limit of a linear series $(\mathcal{L}_\eta, V_\eta)$ on a family $X \to B$ satisfying situation 2.0. There are three main results in this section; roughly stated they are:

1. Every $g^1_d$ can be smoothed;
2. Not every $g^r_d$ can be smoothed;
3. The family of limit $g^r_d$'s over $X_0$ which can be smoothed is a fine moduli space (i.e., scheme).

More technically, the content of 1 is that not only can limit $g^1_d$'s be smoothed, but the smoothing can be done such that ramification indices away from the nodes of $X_0$ are preserved. This is Proposition 3.1.

To specify more precisely how the ramification occurs in the smoothing, we introduce the following notation. A sequence of integers $b = (b_0, \ldots, b_r)$ with $0 \leq b_0 \leq \cdots \leq b_r \leq d - r$ is a ramification index (or sequence) of type $(r, d)$. For $P \in X_0$ a smooth point, a limit $g^r_d$ satisfies the ramification condition $(P, b)$ if the ramification sequence of $g^r_d$ at $P$ is termwise $\geq b_i$. For $b^1, \ldots, b^s$ ramification indices of type $(r, d)$, the adjusted Brill-Noether number $\rho$ is

$$\rho(g, r, d; b^1, \ldots, b^s) = (r + 1)(d - r) - rg - \sum_{i,j} b^i_j.$$

A smoothing family $X/B$ is a family of curves such that

- $\pi: X \to B$ is a flat and proper map having genus $g$ curves of compact type as fibers;
- There are sections $p_1, \ldots, p_s: B \to X$ such that there exists a relatively ample divisor on $X$ with support disjoint from the images of all the $p_i$; [What is a relatively ample divisor?]
- $B$ is irreducible;
- The irreducible components of the singular locus of $\pi$ map isomorphically onto their image in $B$;
- The images of the sections $p_i$ are disjoint and lie in the smooth locus of $\pi$.

[Added by Advisor] $X$ is nonsingular, $B$ is regular, and $X$ is regular at nodes smooth generically on $B$ [I'm not sure what this last statement means].

We can actually strengthen the statement of 3. Not only is the family of smoothable $g^r_d$'s a fine moduli space, but the family of smoothable $g^r_d$'s having proscribed ramification at marked points is also a fine moduli space. The statement, in all its technical glory, is

**Theorem (3.3).** Let $\pi: X \to B, p_1, \ldots, p_s: B \to X$ be a smoothing family, with $b^1, \ldots, b^s$ ramification indices of type $(r, d)$. Then there exists a scheme $G = G^r_d(X/B; (p_i, b^i))$, quasiprojective over $B$ and compatible with base change such that the fibers $G_Q$ are families of $g^r_d$'s over $X_Q$ satisfying ramification conditions
Additionally, every component of $G$ has dimension at least $\dim B + \rho$. Finally, if
\[ \sum_{i,j} b_{ij} = (r + 1)d + \binom{r + 1}{2}(2g - 2), \]
the maximum possible, then $G$ is proper over $B$.

With this theorem, one can now say something about which refined $g^r_d$’s are smoothable. A sufficient condition is, roughly, for the smoothing family as above to have only reducible fibers, and the dimension of the component of $G$ in which $g^r_d$ lies has dimension $\dim B + \rho$.

**Theorem (3.4).** Consider a smoothing family and ramification indices as above, such that all fibers $X_Q$ are reducible curves, and $L$ a refined $g^r_d$ in a fiber $Q$ such that the component of $G$ in which $Q$ lies has dimension $\dim B + \rho$. Then $L$ smooths to desired ramification: there is a 1-parameter family $X_t$ of curves in $X$ with $X_0 = X_Q$ and $X_t$ smooth for $t \neq 0$ and a family of linear series $L_t$ satisfying the ramification conditions $((p_i, b^i))$ having limit $L_0 = L$.

One special case of Theorem 3.4 is when $B$ is zero dimensional. If that is the case, not only can we smooth every $g^r_d$, but we can smooth through any nearby curve, not just a 1-paramenter family in $G$.

[What does nearby curve mean, precisely? That $X_0$ is near $X_1$ if there is some family $X_t$ of schemes with $X_1$ deforming to $X_0$?]

**Corollary (3.5).** If all irreducible components of $G = G^r_d(X/B; ((p_i, b^i)))$ have dimension $\rho$, then every $g^r_d$ on $X$ satisfying ramification conditions $((p_i, b^i))$ can be smoothed to any nearby curve, with smoothing done so as to maintain ramification conditions at nearby points.

**Remark.** If we specialize this theorem to the fiber over a closed point of $B$, we obtain a powerful result about curves. In the scheme $G$ as above with 0 a closed point on $B$, observe that $G_0$ is a family of limit linear series on $X_0$, a curve. Then if $G_0$ has dimension exactly $\rho$, every limit linear series (having the correct ramification) can be smoothed to nearby curves $X_\eta$ of $X$.

**The Brill-Noether number $\rho$ and its generalization**

The formula for $\rho$ given in §3 currently lacks motivation. This section attempts to correct that by providing a derivation of $\rho$ where ramification conditions are absent.

Since we are looking at $g^r_d$’s, it is natural to step back and ask if the parameter space $G^r_d(C/k)$ of all linear series having degree $d$ and rank $r + 1$ is a scheme. Here $C$ is a smooth projective curve over a field $k$ of genus $g$. We might first begin by looking at the degree $d$ line bundles on $C$, $\text{Pic}^d(C/k)$. At this point we are using $\text{Pic}^d(C/k)$ to simply denote the set of line bundles having degree $d$ on $k$. However, we can equip this set with more structure, viewing $\text{Pic}^d(\cdot/k)$ as a contravariant functor from schemes over $k$ to sets given by
\[ T \mapsto \text{Pic}^d(C \times_k T/k)/\text{Pic}(T/k). \]

Under mild hypothesis, [what they are I don’t know], this functor is representable. Hence it is given by a pair, which we abuse notation and denote by $(\text{Pic}^d(C/k), \mathcal{L})$. 
Thus \( \text{Pic}^d(C/k) \) is a \( k \)-scheme whose points correspond to degree \( d \) line bundles on \( C \) and \( L \in \text{Pic}^d(\text{Pic}^d(C/k)) \). Let us be clear \( L \) is not a line bundle on \( C \), but a line bundle on \( \text{Pic}^d(C/k) \times_k C \). We will use later the not-trivial fact that the Picard scheme \( \text{Pic}^d(C/k) \) has dimension \( g \). Also, let \( p_1 \) denote the morphism \( \text{Pic}^d(C/k) \times C \to \text{Pic}^d(C/k) \) and \( p_2 \) the morphism to \( C \).

We introduced the Picard scheme because we would like to view this part: is \( \text{Pic}^d(C/k) \times_k C \) universal in the sense of the representation of the grassmannian? Notice [I think] for each degree line bundle \( L \) on \( C \), the stalk of \( (p_1)_*L \) on the scheme \( \text{Pic}^d(C/k) \) are all rank \( r \) sub-bundles of the pushforward \( \left( (p_1)_*L \right)_\rho \). Notice \( \left( (p_1)_*L \right)_\rho \) are precisely the global sections of \( E \) on \( C \).

Now if we took \( d \) large (specifically, \( d > 2g - 2 \), the degree of the canonical divisor on \( C \)), then for any degree \( d \) line bundle \( E \) on \( C \), we see the stalk of the pushforward \( \left( (p_1)_*L \right)_E \) are precisely the global sections of \( E \) on \( C \). Then by Riemann-Roch, we have that the dimension of the space of global sections of \( (p_1)_*L \) is \( d - g + 1 \).

Now we can compute the dimension of \( \text{Gr}_d(C/k) \). It is another fact that the formula for the relative dimension of the grassmannian of rank \( r + 1 \) subbundles of \( \left( (p_1)_*L \right)_\rho \) is \( (r + 1)(d + 1 - g - (r + 1)) \). Then as this is a sheaf over \( \text{Pic}^d(C/k) \), we must add in the dimension of the Picard group to conclude

\[
\dim \text{Gr}_d(C/k) = g + (r + 1)(d + 1 - g - (r + 1)) = (r + 1)(d - r) - rg = \rho.
\]

Now we do a more general case. Letting \( D \) denote an effective divisor on \( C \) of degree greater than \( 2g - 2 \). Let \( D' = p_2^*D \), a divisor on \( \text{Pic}^d(C/k) \times C \). That is, if \( D = \sum_P n_PP \) on \( C \), then \( D' = \sum_P n_P p_2^{-1}(P) \). Then \( (p_1)_*(L(D')) \) is a vector bundle of rank \( d + \deg D + 1 - g \). [I have no idea why this is true].

Recall that \( \text{Gr}(r + 1, (p_1)_*(L(D'))) \) parametrizes line bundles \( L_0 \) of degree \( d \) on \( C \) together with an \( r + 1 \) dimensional subspace of global sections of \( L_0(D) \). Hence it contains our space of interest, \( \text{Gr}_d(C/k) \).

Now let \( \pi: \text{Gr}(r + 1, (p_1)_*(L(D'))) \to \text{Pic}^d(C/k) \) be the forgetful map, taking a \( g_d \) to its defining sheaf. Let \( \mathcal{F} \subseteq \pi^*(p_1)_*(L(D')) \) be the universal subbundle [I don’t get this part: is \( \mathcal{F} \) universal in the sense of of the representation of the grassmannian?]

[Okay, so I don’t understand any of what follows: I simply typed what I had written down]

Then consider the composition

\[
\mathcal{F} \to \pi^*(p_1)_*(L(D')) \to \pi^*(p_1)_*(L(D'|D'))
\]

induced from the short exact sequence

\[
0 \to L \to L(D') \xrightarrow{f} L(D'|D') \to 0.
\]

We are interested in global sections of \( L \), so we now consider \( \ker f \).

We have that \( \text{Gr}_d(C/k) \) is therefore a closed subscheme of \( \text{Gr}(r + 1, (p_1)_*(L(D'))) \) on which the above composed map is zero. This is a map from a rank \( r + 1 \) bundle, \( \mathcal{F} \), to a bundle of rank \( \deg D \). Thus the imposed constraints force the codimension to not exceed \( (r + 1)(\deg D) \). Therefore

\[
\dim \text{Gr}_d(C/k) \geq g + (r + 1)(d + \deg D + 1 - g - (r + 1)) - (r + 1)(\deg D) = \rho.
\]

This is the classic statement of the Brill-Noether theorem: For all smooth curves \( C \), \( \dim \text{Gr}_d(C/k) \geq \rho \). In fact, for general curves (an open dense subset of \( \mathcal{M}_g \)), one has equality.
§4: References

§5: References


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