Theta Bodies Notes

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1 Background

We consider the problem of understanding the convex hull of a real variety. Let \( I \subset \mathbb{R}[x_1, \ldots, x_n] \) be an ideal, and \( V_\mathbb{R}(I) \) its real variety. Assume that \( I \) is generated by the polynomials \( \{f_1, \ldots, f_m\} \). The convex hull of \( V_\mathbb{R}(I) \) is cut out by linear functions that are non-negative on \( V_\mathbb{R}(I) \). That is,

\[
\text{conv}(V_\mathbb{R}(I)) = \{x \in \mathbb{R}^n \mid f(x) \geq 0 \ \forall f \text{ non-negative on } V_\mathbb{R}(I)\}
\]

For computational reasons, a natural relaxation of the convex hull can be achieved by lifting the non-negativity condition on linear functions defining the convex hull, to the condition that they are sums of squares in the ideal \( I \). If we further restrict the degrees of the polynomials in the sums of squares representation, then computing this relaxation can be done via semidefinite programming. Such restrictions motivate the following definitions.

**Definition 1.1.** The polynomial \( f \) is \( k \)-sos mod \( I \) if there exist \( h_1, \ldots, h_t \in \mathbb{R}[x_1, \ldots, x_n] \) all of degree at most \( k \) such that \( f \equiv \sum h_i^2 \mod I \).

**Definition 1.2.** The ideal \( I \) is \((1, k)\)-sos mod \( I \) if every linear polynomial that is non-negative on \( V_\mathbb{R}(I) \) is \( k \)-sos mod \( I \).

**Example 1.3.** Consider the ideal \( I = (x_1^2 x_2 - 1) \subset \mathbb{R}[x_1, x_2] \). Then \( \text{conv}(V_\mathbb{R}(I)) \) is the open upper half-plane. Any linear polynomial that is non-negative over \( V_\mathbb{R}(I) \) is of the form \( \alpha x_2 + \beta \) where \( \alpha, \beta \geq 0 \). Now, mod \( I \), \( \alpha x_2 + \beta \equiv (\sqrt{\alpha} x_1 x_2)^2 + (\sqrt{\beta})^2 \), and so \( I \) is \((1, 2)\)-sos.

**Definition 1.4.** For a given positive integer \( k \), the \( k \)th **theta body** of \( I \) is

\[
\text{TH}_k(I) := \{x \in \mathbb{R}^n \mid f(x) \geq 0 \text{ for every linear } f \text{ that is } k \text{ - sos mod } I\}.
\]

It is immediate that we have the chain of inclusions

\[
\text{TH}_1(I) \supseteq \text{TH}_2(I) \supseteq \cdots \supseteq \text{conv}(V_\mathbb{R}(I)).
\]

If for some positive integer \( k \), we have \( \text{TH}_k(I) = \text{conv}(V_\mathbb{R}(I)) \), then we say \( \text{TH}_k \)-**exact**. Indeed, if an ideal \( I \) is \((1, k)\)-sos, it is automatically \( \text{TH}_k \)-exact.

**Lemma 1.5.** If \( I \) is \((1,k)\)-sos, then \( I \) is \( \text{TH}_k \)-exact.

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Proof. Let \( s \in \mathbb{R}^n \) be such that \( s \notin \text{conv}(V_{\mathbb{R}}(I)) \). Then by the Separation Theorem (see standard books on convex bodies), there is a linear function \( \ell \) such that \( \ell \) is non-negative on \( \text{conv}(V_{\mathbb{R}}(I)) \) and \( \ell(s) < 0 \). Since \( I \) is \((1,k)\)-sos, \( \ell \) is \( k \)-sos, and hence \( s \notin TH_k(I) \). The result follows. \( \square \)

The reverse inclusion in fact does not hold.

Example 1.6. Let \( I = (x^2) \). All linear polynomials non-negative on \( V_{\mathbb{R}}(I) \) are of the form \( \pm a^2 x + b^2 \) for some \( a,b \in \mathbb{R} \). If \( b \neq 0 \), then \((\pm a^2 + b^2)^2 \equiv (\frac{a^2}{b^2} x \pm b)^2 \mod I \). However, when \( b = 0 \), the polynomials \( \pm x \) are not sums of squares mod \( I \), and so are not \((1,k)\)-sos for any \( k \). On the other hand, \( I \) is \( TH_1 \)-exact since it is cut out by the infinite set of inequalities

\[
\pm x + b^2 \geq 0, \ b \in \mathbb{R}, \ b \neq 0.
\]