1. Introduction

This term paper comes out of my first attempt trying to understand the geometric structure of the moduli stack $\mathcal{M}_g$ of smooth curves. As the title is stated, knowledge of various topics in algebraic geometry are required for that purpose. In this note, I'll try to illustrate the basic concepts and facts on those topics, based upon references listed at the end. This is a first step towards concretely understanding the notion of moduli space, whose importance is self-evident.

2. Grothendieck topology

In class, we have seen the notion of $Y$-points of a scheme. In a sense, we can understand a particular scheme via all possible morphisms of schemes to that scheme. Likewise, for an abstract category, where we cannot describe the objects as sets, we can understand an object via morphisms to that object. This is the beginning of the story where we try to extract some concrete information out of abstractness.

In class, we've also seen sheaves over topological spaces. It turns to be an extremely useful tool. A natural question would then be whether we can generalize this concept over an abstract category. Let's see a motivating example:

**Lemma 1.** Let $X$ be a scheme., $h_X$ be the functor of points. Denote $\text{Open}(Y)$ to be the collection of open subsets of $Y$. Then $h_X : \text{Open}(Y) \to \text{Set} : U \mapsto h_X(U)$ is a sheaf of sets on $Y$.

**Proof.** If $V \subset U$, morphisms from $(U,O_Y^U)$ to $X$ restrict naturally to morphisms from $(V,O_Y^V)$ to $X$. To be precise, suppose $(f,f^1)$ is a morphism from $U$ to $X$, $p_{UV}(f,f^1) = (f|_V, f^1|_V)$, where $f^1|_V : O_X(W) \to f^1_*O_Y(W) = O_Y(f^1(W) \cap V)$ is obtained by composing $f^1$ with the restriction map of the sheaf $O_Y^U$. The property of being a local morphism is a local condition and is inherited automatically.

The requirements on restriction maps are checked directly. Thus, we get $h_X$ is a presheaf.

The fact that this is indeed a sheaf, follows from gluing of morphisms. □

**Remark** A concise way to state this: $h_X(Y) \cong \prod_i h_X(U_i) \Rightarrow \prod_{i,j} h_X(U_i \cap U_j)$ is an equalizer, where $b((f_i)) = ((f_i), c((f_i)) = (f_{ij})$. This means, $(f_i)$ comes from restriction of $f \in h_X(Y)$, if and only if $f_i, f_j$ agree on $U_i \cap U_j$, i.e. $\text{Im}(a) = \{a|b(\alpha) = c(\alpha)\}$; also, $f$ is uniquely determined by such $(f_i)$, i.e. $a$ is injective. This is precisely a restatement of the sheaf condition.

This example suggests that we can consider functors as sheaves. However, in order to make sense this idea in general, one has to make sense the notion of "open set" as well as "open covering" for an abstract category. Obviously, the naive idea
of “intersection of open sets” fails in this context. The Grothendieck topology turns out to be right notion to consider.

**Definition 1.** A Grothendieck topology on a category $\mathcal{C}$ assigns to each object $U$ a collection of sets of arrows, called coverings of $U$, satisfying:

- If $V \to U$ is an isomorphism, it is a covering of $U$.
- Given a covering $\{U_i \to U\}$ and $W \to U$, then fiber products $\{U_i \times_U W\}$ exist, and $\{U_i \times_U W \to W\}$ is a covering.
- Given a covering $\{U_i \to U\}$, and for each $i$, a covering $\{U_{ij} \to U_i\}$, then $\{U_{ij} \to U_i \to U\}$ is a covering of $U$.

The intuition: one can think of arrows into $U$ as open subsets of $U$; the first axiom says (up to isomorphism) $U$ is a covering of itself; the second one is analogous to constructing covering of an open subset from a given cover, where fiber product is analogous to the notion of intersection; the last is about composition of open covers. This notion of Grothendieck topology can also be viewed as a generalization of Zariski topology on a scheme, since if we consider the category whose objects are open subschemes of $X$, and morphisms are open immersions, we basically recover the Zariski topology over $X$.

**Definition 2.** A category with a Grothendieck topology is called a site.

As was suggested in the beginning, we can now talk about sheaves in a broader sense:

**Definition 3.** Let $\mathcal{C}$ be a site, $F$ is a contravariant functor from $\mathcal{C}$ to the category of sets. Then, $F$ is a sheaf of sets on $\mathcal{C}$ if for all coverings $\{U_i \to U\}$, the diagram $F(U) \to \prod F(U_i) \xrightarrow{\varpi_{ij}^1} \prod F(U_i \times_U U_j)$ is an equalizer, where $\varpi_{ij} : U_i \times_U U_j \to U_i$, $\varpi_{ij} : U_i \times_U U_j \to U_j$ are projections.

**Remark** Since $F$ is a contravariant functor, given $\varpi_{ij} : U_i \times_U U_j \to U_i$, it makes sense to pullback $f_i \in F(U_i)$ to $\varpi_{ij}^\ast(f_i) \in F(U_i \times_U U_j)$.

In many cases, we are interested in Grothendieck topology over the category $(\text{Sch})$ or $(\text{Sch}/X)$. Usually, they are obtained by posing restriction on morphisms we allow between schemes. The following is a simple example:

**eg.** Let $X_{zar}$ be the category of schemes over $X$. Define coverings to be collections of $X$-morphisms $\{\phi_i : Y_i \to Z\}$, with $\phi_i$ being open immersions and $\bigcup \phi_i(Y_i) = Z$. Then $X_{zar}$ is a site, called the big Zariski site. The condition we posed is that the morphisms have to be open immersions.

However, we should consider more interesting examples. Here is one of them: the fpqc topology on the category $(\text{Sch}/X)$.

For that purpose, we need the following definition [10]:

**Definition 4.** Let $f : X \to Y$ be a morphism of schemes, and $\mathcal{F}$ is an $O_X$-module. Then, $\mathcal{F}$ is flat over $Y$ at $x \in X$, if $\mathcal{F}_x$ is a flat $O_{Y,y}$-module, where $y = f(x)$. We say $\mathcal{F}$ is flat over $Y$ if it is flat at every point. We also say $X$ is flat over $Y$ if $O_X$ is flat over $Y$. 

Proof. (1) ⇒ (2) is trivial, since affine set is quasi-compact. 
(2)⇒(1): Take \( V \) quasi-compact open subset of \( Y \), we want to show that \( V \) is the image of a quasi-compact open subset of \( X \). Since \( V \cap V_i \) cover \( V \), affine subsets of \( V \cap V_i \) cover \( V \), too. WLOG, take finitely many of them: \( W_1, \ldots, W_k \). Since \( V_i = f(U_i) \), with \( U_i \) quasi-compact, if \( W_i \subset V_i \), \( W_i' = f^{-1}(W_i) \cap U_i \) is a quasi-compact open subset of \( X \). This is because, a morphism from a quasi-compact scheme to an affine scheme is by definition quasi-compact. \( \bigcup W_i' \) is then quasi-compact in \( X \) and maps to \( V \) in \( Y \).
(4)⇒(3): follows from definition of quasi-compact morphism.
(3)⇒(4): just take an affine open about \( f(x) \) inside \( f(U) \).
(4)⇒(2) follows from surjectivity of \( f \).
(2)⇒(4): \( \forall x \in X, f(x) \notin V_i \) for some \( i \). Have some \( U_i \) quasi-compact and \( f(U_i) = V_i \). Since \( x \) might not be in \( U_i \), still need to take some affine open neighborhood \( U' \) of \( x \) in \( f^{-1}(V_i) \). Then, \( U_i \cup U' \) maps onto \( V_i \). Since \( U_i \) and \( U' \) are both quasi-compact, their union is also quasi-compact.
This concludes the proof. \( \square \)

This is a finiteness condition: weaker than being quasi-compact, since we don’t require \( f^{-1}(V) \) to be quasi-compact, for \( V \subset Y \) quasi-compact. It is a notion of being locally quasi-compact (local on source).

Definition 7. The fpqc topology on the category \((\text{Sch}/X)\) is the topology in which each covering \( \{U_i \rightarrow U\} \) induce an fpqc morphism \( \coprod U_i \rightarrow U \).

Why are we interested in the fpqc topology? The following theorem by Grothendieck may be one justification:

Theorem 2.1. A representable functor on \((\text{Sch}/X)\) is a sheaf in fpqc topology.

The proof of this theorem is based on a criterion saying that for a representable functor \( F \), which is a Zariski sheaf (by lemma 1), it suffices to check \( F \) satisfies the sheaf condition in fpqc topology for all faithfully flat morphisms of affine \( X \)-schemes. It is technically involved, and I shall omit it here. More importantly, the
Theorem tells us $h_X$ is still a sheaf in fpqc topology. This means, given \( \{V_i \to V\} \) an fpqc covering of $V$, and morphisms $f_i : V_i \to X$, if $\forall i, j, pr_1^*(f_i) = pr_2^*(f_j) \in Hom(V_i \times V_j, X)$, we can glue $f_i$'s to get a unique $f : V \to X$, just as we did for Zariski topology in class. In one word, fpqc topology behaves nicely with respect to glueing of morphisms.

3. Fibered category

Since stacks turn out to be categories fibered (e.g. over schemes) in groupoids satisfying some extra condition, we have to introduce the terminologies with respect to fibered categories first.

**Definition 8.** $\mathcal{F}$ is a category over $\mathcal{C}$, if there is a functor $p : \mathcal{F} \to \mathcal{C}$.

In class, we have seen the functor of points of a scheme $X$. In fact we can think of $h_X$ as a category over the category of schemes $(Sch)$: the objects are $(Y, f)$, where $Y$ is a scheme over $X$, and $f \in Hom_{Sch}(Y, X)$. A morphism from $(Y, f)$ to $(Y', f')$ is simply an $X$-morphism $g : Y \to Y'$. Denote this category as $\mathcal{X}$. The functor $p$ from $\mathcal{X}$ to $(Sch)$ sends $(Y, f)$ to $Y$, and a morphism $g$ to itself (as a morphism in the category of schemes).

**Definition 9.** Let $\mathcal{F}$ be a category over $\mathcal{C}$. A Cartesian arrow $\phi : \eta \to \zeta$ of $\mathcal{F}$ is one such that for all $f : \eta' \to \zeta$ in $\mathcal{F}$, and all $g : p(\eta') \to p(\eta)$ in $\mathcal{C}$ satisfying $p(\phi) \circ g = p(f)$, there exists a unique $\theta : \eta' \to \eta$ with $p(\theta) = g$, and $\phi \circ \theta = f$. If $\phi : \eta \to \zeta$ is a Cartesian arrow mapping to $U \to V$ of $\mathcal{C}$, say $\eta$ is a pull-back of $\zeta$ to $U$.

As we can see from the above diagram, a cartesian arrow gives rise to a notion of pull-back in the following sense: if the projection of an arrow onto the base category factorizes through some object $U$, then the arrow itself should factorize in a unique way through the pull-back of its target to $U$. The existence of such pull-backs leads to the notion of fibered categories.

**Definition 10.** A fibered category $\mathcal{F} \to \mathcal{C}$ is a category over $\mathcal{C}$ such that for any arrow $f : U \to V$ in $\mathcal{C}$ and $\eta \in Ob(\mathcal{F})$ with $p(\eta) = V$, pullback of $\eta$ with respect to $U$ through $f$ exists.

**Lemma 2.** If $\mathcal{C}$ is a category that has fiber-products, then $\text{Mor}(\mathcal{C})$ is a fibered category over $\mathcal{C}$ via the functor $p : \text{Mor}(\mathcal{C}) \to \mathcal{C} : (f : U \to V) \mapsto V$. 
Proof. Suppose we are given data \((g: V \to Y)\), then take \((pr_2: X \times_Y V \to V)\), and the morphism \((pr_2) \to (g)\) is the pair \((pr_1, \phi)\); notice that the functor \(p\) acts on \((pr_1, \phi)\) by second projection. 

Since existence of pull-back is really the main point of the definition, it is natural to require morphisms between fibered categories (over same base) to preserve cartesian arrows:

**Definition 11.** A morphism \(F\) between categories \(\mathcal{F}\) and \(\mathcal{G}\) fibered over \(\mathcal{C}\) is a functor such that:

- \(p_2 \circ F = p_1\);
- \(F\) takes cartesian arrows to cartesian arrows.

In the above notation, let \(x\) by the pull-back of \(y\) to \(U\) in \(\mathcal{F}\), then \(F(x)\) is the pull-back of \(F(y)\) to \(U\) in \(\mathcal{G}\).

In the context of fibered categories, it is natural to talk about fibers:

**Definition 12.** \(\mathcal{F}\) fibered over \(\mathcal{C}\). \(U \in \text{Ob}(\mathcal{C})\), the fiber \(\mathcal{F}(U)\) is the subcategory of \(\mathcal{F}\) whose objects are \(\xi \in \text{Ob}(\mathcal{F})\), such that \(p(\xi) = U\); and whose arrows are \(\phi\) in \(\mathcal{F}\) such that \(p(\phi) = 1d_U\).

Let \(\mathcal{F}\) be a fibered category over \(\mathcal{C}\), \(f: U \to V\) be a morphism in \(\mathcal{C}\). For each object \(\eta \in \mathcal{F}\) over \(V\), pick a pullback \(\phi_\eta: f^*\eta \to \eta\) of \(\eta\) to \(U\). Now, we can construct a covariant functor \(f^*: \mathcal{F}(V) \to \mathcal{F}(U)\) sending \(\eta\) to \(f^*\eta\) and \(\eta \to \zeta\) to \(f^*\eta \to f^*\zeta\) which comes from the commutative diagram

\[
\begin{array}{ccc}
 f^*\eta & \xrightarrow{f^*\phi} & \eta \\
 \downarrow & & \downarrow \\
 f^*\zeta & \xrightarrow{f^*\phi} & \zeta
\end{array}
\]

is a Cartesian arrow. Also, \(f^*\eta \to \eta\) and \(f^*\zeta \to \zeta\) projects to the same arrow in \(\mathcal{C}\). So there is a unique arrow in \(\mathcal{F}(U)\) that makes the above diagram commutative.) We also give a name for such a choice of unique pull-back [2].

**Definition 13.** A cleavage of a fibered category \(\mathcal{F} \to \mathcal{C}\) consists of a class \(K\) of Cartesian arrows in \(\mathcal{F}\) such that for any arrow \(f: U \to V\) in \(\mathcal{C}\) and any \(\eta \in \mathcal{F}(V)\), there exists a unique arrow with target \(\eta\) mapping to \(f\) in \(\mathcal{C}\).

**Definition 14.** Suppose \(\mathcal{F}\) is fibered over \(\mathcal{C}\), and for every object \(U \in \mathcal{C}\), \(\mathcal{F}(U)\) is a groupoid, then \(\mathcal{F}\) is said to be a category fibered in groupoids (hereafter CFG) over \(\mathcal{C}\).

4. Stack

Now we start to talk about stacks.

A quick motivational observation: If we are in the category of topological spaces, \(f: X \to U\), \(g: Y \to U\) are continuous maps, and \(\{U_i \to U\}\) is an open cover of \(U\). Suppose \(\forall i, \exists f_i: f^{-1}(U_i) \to g^{-1}(U_i)\) is a map over \(U_i\), such that \(f_i|_{f^{-1}(U_i \cap U_j)} = f_j|_{f^{-1}(U_i \cap U_j)}\). Then we can define \(F: X \to Y\) by \(F|_{f^{-1}(U_i)} = f_i\) to get a continuous...
map over $U$.

Another motivational example: suppose $\{U_i\}$ is an open covering of $U$, and $U_{ij} = U_i \cap U_j$, and $U_{ijk} = U_i \cap U_j \cap U_k$. For each $i$ there is a continuous map $f_i : X_i \to U_i$, such that $\forall i,j, \exists \phi_{ij} : f_i^{-1}(U_{ij}) \to f_j^{-1}(U_{ij})$ a homeomorphism over $U_{ij}$, and $\phi_{ijk} = \phi_{ij} \phi_{ij}$ (*), then $3$ continuous maps $f : X \to U$ together with homeomorphisms $\phi_i : f^{-1}(U_i) \to X_i$ such that $\phi_{ij} = \phi_j \circ \phi_i^{-1}$. The construction goes as follows: take $X' = \bigsqcup X_i, U' = \bigsqcup U_i$, then we get a map $X' \to U'$. Define $x_i \sim x_j$, if $\phi_{ij} x_i = x_j$. We can check this is an equivalence relation on $X'$ from the cocycle condition (*). Let $X = X'/\sim$. Since $\phi_{ij}$ is a homeomorphism over $U_{ij}$, two equivalent points in $X'$ map to same point in $U$. Hence, we obtain a continuous map $f : X \to U$, and restrict each projection $X_i \to X$ to $X_i \to f^{-1}(U_i)$, we get homeomorphisms $\phi_i$ with desired properties.

We want to consider the possibility of "gluing" together morphisms and objects in greater generality. This is done via the notion of descent data.

**Definition 15.** Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$. $U = \{U_i \to U\}$ is a covering in $\mathcal{C}$. An object with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ on $U$ is a collection of objects $\xi_i \in \mathcal{F}(U_i)$, together with isomorphisms $\phi_{ij} : pr^*_1 \xi_i \to pr^*_2 \xi_j$ in $\mathcal{F}(U_i \times_U U_j)$ satisfying: $pr^*_{13} \phi_{ik} = pr^*_{23} \phi_{ik} = pr^*_{12} \phi_{ij}$.

**Remark** $pr^*_{13} : U_i \times_U U_j \times_U U_k \to U_i \times_U U_k$ induces $pr^*_{13} : \mathcal{F}(U_i \times_U U_j \times_U U_k) \to \mathcal{F}(U_i \times_U U_k)$, so $pr^*_{13} \phi_{ik} : pr^*_{13} \xi_i \to pr^*_{13} \xi_k$. But $pr^*_{12} \phi_{ij}$ is precisely the projection $pr_3$ for the triple fibered-product. Therefore, $pr^*_{13} \phi_{ik} : pr^*_{13} \xi_i \to pr^*_{13} \xi_k$ is an arrow in the category $\mathcal{F}(U_i \times_U U_j \times_U U_k)$. The isomorphism $\phi_{ij} : pr^*_{13} \xi_i \to pr^*_{13} \xi_j$ can be thought of as identification in the fiber-product. The cocycle condition is a compatibility condition for such identification. Note: a descent data remembers those identifications.

**Definition 16.** An arrow between objects with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ and $(\{\eta_i\}, \{\psi_{ij}\})$ (with respect to same covering $U$) is a collection of arrows $\alpha_i : \xi_i \to \eta_i$ in category $\mathcal{F}(U_i)$ such that $\forall i, j$ we have the following commutative diagram:

\[
\begin{array}{ccc}
pr^*_1 \xi_i & \xrightarrow{pr^*_1 \alpha_i} & pr^*_1 \eta_1 \\
\downarrow \phi_{ij} & & \downarrow \psi_{ij} \\
pr^*_2 \xi_j & \xrightarrow{pr^*_2 \alpha_i} & pr^*_2 \eta_2 \\
\end{array}
\]

The conditions simply say that the "locally" defined morphisms are compatible with the identifications on both sides.

Since we can compose morphisms, objects with descent data form a category, denoted as $\mathcal{F}(\{U_i \to U\}) = \mathcal{F}(U)$.

For every $\xi \in \mathcal{F}(U)$, we can construct an object with descent data with respect to certain covering $\{\sigma_i : U_i \to U\}$ of the base category $\mathcal{C}$, $(\{\sigma_i^* \xi\}, \{Id\})$, namely, we identify $pr^*_1 \sigma_i^* \xi$ with $pr^*_2 \sigma_i^* \xi$, since we can think of them as pullback of $\xi$ through $U_i \times_U U_j \to U$. (When we fix the cleavage, there is only one such element.) Likewise, if we start with an arrow $\alpha : \xi \to \eta$ in $\mathcal{F}(U)$, we can get an arrow in $\mathcal{F}(U)$: $(\sigma_i^* \alpha)$. Combine these two construction, we have a functor $\mathcal{F}(U) \to \mathcal{F}(U)$.
**Definition 17.** Let \( \mathcal{F} \) be a CFG over \( \mathcal{C} \). \( \mathcal{F} \) is a stack over \( \mathcal{C} \) if for each covering \( \{ U_i \to U \} \) in \( \mathcal{C} \), the functor \( \mathcal{F}(U) \to \mathcal{F}(\{ U_i \to U \}) \) is an equivalence of categories.

**Remark** With the above description, it is very clear that \( \mathcal{F} \) is a stack means that we can always glue together arrows as well as objects via given isomorphisms, as long as certain compatibility conditions are satisfied.

The following is a concrete example [2]:

**Proposition 2.** Let \( \mathcal{C} \) be a site. \( F \) is a contravariant functor from \( \mathcal{C} \) to \( (\text{Set}) \). Then \( F \) is a stack iff it is a sheaf.

**Proof.** We can understand the functor \( F \) as a category \( \tilde{F} \) fibered over \( \mathcal{C} \) in sets. The objects of \( \tilde{F} \) are of the form \( (U, \eta) \), where \( U \) is an object of \( \mathcal{C} \), and \( \eta \in F(U) \). A morphism from \( (U, \eta) \) to \( (V, \eta') \) is a morphism \( f : U \to V \) in \( \mathcal{C} \) taking \( \eta \) to \( \eta' \). Then, \( F(U) \) is the fiber over \( U \), and it is a discrete category, since a morphism \( (U, \eta) \to (U, \xi) \) has to map to identity in \( \mathcal{C} \). Given \( f : U \to V \), the arrow \( F(f) : F(V) \to F(U) \) is the functor between \( F(V) \) and \( F(U) \). Consider a covering \( \{ \sigma_i : U_i \to U \} \) in \( \mathcal{C} \).

Elements in \( F(U) \) are \( \xi \) such that the following diagram commutes: \( \xi 
\end{equation}

In particular, if \( \mathcal{C} \) is \( (\text{Sch}/\mathcal{E}) \) or \( (\text{Sch}) \), equipped with fpqc topology, then all representable functors can be considered as stacks over \( \mathcal{C} \). For example, \( \underline{U} \) is a stack over \( (\text{Sch}) \). Hence, scheme can be thought of as a special case of stack.

## 5. DM Stack

This notion is as useful as it is technically involved; again, we need couple of new definitions. Recall from part 4 that for each scheme \( Y \), a stack \( \underline{Y} \) can be induced.

**Definition 18.** Given \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) CFGs over \( (\text{Sch}) \), \( F : \mathcal{X} \to \mathcal{Z}, G : \mathcal{Y} \to \mathcal{Z} \), the fiber product \( \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \) is defined to be the category whose objects are of the form \( (x, y, \alpha) \), where \( x \in \mathcal{X}, y \in \mathcal{Y} \) are over \( S \in (\text{Sch}) \), and \( \alpha : F(x) \to G(y) \) is an isomorphism. A morphism from \( (x', y', \alpha') \) to \( (x, y, \alpha) \) is given by morphisms \( \tau_1 : x' \to x, \tau_2 : y' \to y \)

such that the following diagram commutes: \( \alpha' \)

This definition is natural. It basically says, in the fiber product we pair elements from \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) such that their images in \( \mathcal{Z} \) (which are in a same fiber over \( (\text{Sch}) \)) can be identified via an isomorphism \( \alpha \) (and we record the isomorphism). If we forget the fact that \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) are fibered over \( (\text{Sch}) \), and assume they were discrete categories, we recover the notion of set-theoretic fiber product.

An important part of the definition of DM stack is the representability of the diagonal morphism.
Definition 19. Let $\Phi : X \to X'$ be a morphism of stacks. $\Phi$ is said to be representable if for any scheme $Y$ and any morphism $Y \to X'$, the fibered product $X \times_{X'} Y$ is equivalent to $W$ as a CFG over $(\text{Sch})$, where $W$ is a scheme.

Hereafter, we'll just say $\mathcal{Y}$ is a scheme, if $\mathcal{Y}$ is equivalent to $W$.

The importance of this property is encoded in the following proposition [1]:

Proposition 3. Let $X$ be a stack. If $X \to X \times X$ is representable, then for any $U, W \in (\text{Sch})$, $U \times W$ over $X$, $U \times X W$ is a scheme.

Proof. There is a natural equivalence between $U \times W$ and $(U \times W) \times X \times X$, sending $(u, w, \alpha)$ to $((u, w), (G(w), G(w)), \alpha \times \text{Id})$, where $G$ is the functor from $W$ to $X$. Since the diagonal is representable, and $U \times W = U \times W$, the result follows. □

Definition 20. A stack $X$ is a Delign-Mumford stack if following properties are satisfied:

- The diagonal $X \to X \times X$ is representable, quasi-compact and separated.
- There exists a scheme $U$ and a morphism $U \to X$ which is étale and surjective.

By “quasi-compact and separated” we mean the following: under base change by $\sum_{X} Y \times_{X} X = \sum_{X}$ is quasi-compact and separated (as a morphism of schemes).

By previous proposition, we see that for a DM stack $X$, for any two schemes over $X$, their fiber-product over $X$ is still a scheme. In particular, this suggests that $U \to X$ is representable in the sense of definition 19. Again, “étale and surjective” means that $\forall X \in (\text{Sch}), X \times X U \to X$ has these properties. $U \to X$ is also called an étale atlas of the DM stack. It is an important restriction in the sense that, although a DM stack does not necessarily come from a scheme, it does have some nice “geometric” behavior.

6. The moduli stack $\mathcal{M}_g$

I'll end up by relating everything we have so far to the moduli stack $\mathcal{M}_g$.

Definition 21. Let $X$ be a scheme. If $k$ is an algebraically closed field, a morphism $\text{spec}(k) \to X$ is called a geometric point of $X$. A geometric fiber is a fiber over a geometric point.

Now, we can start to describe the moduli stack $\mathcal{M}_g$.

Definition 22. The moduli stack $\mathcal{M}_g$ of curves of genus $g$ is a category over $(\text{Sch})$ whose objects are smooth projective morphisms $(C \to S)$, whose geometric fibers are connected curves of genus $g$. A morphism from $(C \to S)$ to $(C' \to S')$ consists of data of the following form: $(f, \tilde{f})$, where $f : C \to C'$ and $\tilde{f} : S \to S'$ are morphisms of schemes such that the following diagram is a Cartesian square:

$$
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow & & \downarrow \\
S & \xrightarrow{\tilde{f}} & S'
\end{array}
$$

The functor $\Phi : \mathcal{M}_g \to (\text{Sch})$ takes an object $(C \to S)$ to $S$ and a morphism $(f, \tilde{f})$ to $\tilde{f} : S \to S'$. 
Lemma 3. $M_g$ is a category over $(Sch)$ fibered in groupoids.

Proof. Take the described functor, then $M_g$ is a category over $Sch$.

First, we show $M_g$ is fibered over $(Sch)$. The pull-back of $C \to S$ by $T \to S$ to $T$ is simply the fiber-product $C \times_S T \to T$. The universal property of a fiber product implies that $(C \times_S T \to T) \to (C \to S)$ is a Cartesian arrow over $T \to S$.

Second, we show every morphism in $M_g(S)$ is an isomorphism. Suppose $C, C'$ are two families over same scheme $S$, by the Cartesian square in the definition of a morphism, it follows that morphisms in the fiber $M_g S$ are of the form $(f, Id)$, where $f$ is an isomorphism. Thus, $M_g$ is a CFG over $(Sch)$. □

Proposition 4. $M_g$ is a DM stack.

In class, we have seen the idea that some moduli spaces can be constructed as the quotient of some Hilbert scheme. This result follows from an argument of this type. As is stated in [1], inside the Hilbert scheme of $\mathbb{P}^{3g-6}$, there is a subscheme $\text{Hilb}_{g,3}$ consisting of 3-canonically embedded curves of genus $g$. The projective linear group $\text{PGL}_{5g-5}$ acts on it, with finite, reduced geometric stabilizers, and $M_g$ is isomorphic to the quotient $[\text{Hilb}_{g,3}/\text{PGL}_{5g-5}]$. It then follows from a criterion stated in [3] that the quotient is a stack. A full proof can be found in [6].

7. Reference

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