Recent progress on limit linear series

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Limit linear series

In the 1980's, Eisenbud and Harris developed a theory of "limit linear series" on curves of **compact type** – that is, nodal curves with dual graph a tree. They applied this to give simpler proofs of the Brill-Noether and Gieseker-Petri theorems, and to prove a variety of new results, including on existence of Weierstrass points and on the geometry of moduli spaces of curves (specifically, that they are of general type in genus at least 24).

The idea behind limit linear series is that they keep track of what happens to a family of linear series on a families of nonsingular curves degenerating to a curve of compact type.

We first review the Eisenbud-Harris notion of limit linear series, beginning with some background definitions relating to smooth curves.

Recall that if X is a nonsingular (projective) curve, a $\mathfrak{g}^r_d$ on X is a pair $(\mathscr{L}, V)$ of a line bundle $\mathscr{L}$ of degree d, and an $(r+1)$-dimensional vector space V of global sections of $\mathscr{L}$.

Given a $\mathfrak{g}^r_d$ on X and a point $P \in X$, the **vanishing sequence** at P is defined to be the increasing sequence

$$a_0 < \cdots < a_r$$

of orders of vanishing at P of nonzero sections in V.

Throughout this talk, X will be a (possibly nodal, possibly reducible) connected projective curve, with dual graph Γ. We let Z_v be the component corresponding to a vertex $v \in V(\Gamma)$, and P_e be the node corresponding to an edge e.

Now suppose that X is of compact type. Then the definition of Eisenbud-Harris limit linear series is as follows.

Definition 1 *Given $r, d > 0$, a **limit** $\mathfrak{g}^r_d$ on X consists of a tuple $(\mathscr{L}^v, V^v)_{v \in V(\Gamma)}$ of $\mathfrak{g}^r_d$s on the components Z_v, satisfying the condition:*

$$a_j^{(e,v)} + a_{r-j}^{(e,v')} \geq d \quad \text{for } j = 0, \ldots, r,$$

where v, v' are vertices of Γ connected by e, and $a^{(e,v)}$ denotes the vanishing sequence of $(\mathscr{L}^v, V^v)$ at P_e.

Curves not of compact type

For almost 30 years, it has been an open problem to develop a generalization of the theory of limit linear series to nodal curves not of compact type. Various papers have studied aspects of this question, but until recently no authors had proposed a general definition, let alone developed the corresponding theory (proving for instance specialization and smoothing results).

The more complicated gluings that occur for curves not of compact type manifest themselves in various complications, which make it far from obvious how one should modify the definition of Eisenbud and Harris for a more general setting.

Recent progress

Recently, the theory of divisors on graphs has renewed interest in this subject, and culminated in the work of Amini and Baker (2012) proposing a general definition of limit linear series for curves not of compact type, and proving a specialization result for their definition.

Independently, I discovered a new definition of Eisenbud-Harris limit linear series, which I have applied to new constructions in the compact type case (including in the higher-rank situation), and which has also led to a definition of limit linear series on curves not of compact type. This definition has some common elements with the Amini-Baker approach, but is quite distinct. I have proved specialization results and constructed moduli spaces in full generality. I have also proved smoothing theorems and evaluated expected dimensions in the context of curves of "pseudocompact type," discussed below.

In this talk I will focus on how my theory sheds new light on the divisor theory of graphs.

Basic ideas

For the rest of the talk, we will assume that the curve X is of **pseudocompact type**, meaning that if we collapse all multiple edges in Γ, we obtain a tree. Thus, such curves include both all 2-component curves, and all curves of compact type.

One idea that goes into the definition in this context is the following:

Definition 2 *Given a smooth projective curve Z, a $\mathfrak{g}^r_d$ $(\mathscr{L}, V)$ on Z, and a non-decreasing sequence of effective divisors $0 = D_0 \leq D_1 \leq \dots D_{b+1}$ such that $\deg D_{b+1} > d$, the **multivanishing sequence** of $(\mathscr{L}, V)$ along $D_\bullet$ is the sequence $a_0 \leq \cdots \leq a_r$ such that a appears m times in the sequence if $a = \deg D_j$ for some j and*

$$\dim V(-D_j) - \dim V(-D_{j+1}) = m.$$

For instance, given distinct points $P_1, \dots, P_n$ on Z, we could set $D_i = i(P_1 + \cdots + P_n)$, and different multivanishing sequences would tell us when all the P_i are mapped to the point, or have the same tangent lines, etc.

In order to define limit linear series, we first put extra structures on the curve X.

The first structure is a **chain structure**, meaning simply a positive integer at each node of X. This is intended to keep track of singularities in one-parameter families, and corresponds precisely to the metric on Γ considered in the tropical theory, except that we only allow integer edge lengths.

The second structure is an **enriched structure**, consisting of a collection of twisting line bundles which allow us to make certain global constructions, and have no analogue on the tropical side.

Given X with a chain structure and an enriched structure, let $\widetilde{X}$ be obtained by inserting chains of rational curves at each node of X, as dictated by the chain structure. Define an **admissible multidegree** on $\widetilde{X}$ to be an assignment of integers to each component such that on each inserted chain of rational curves, the assigned integer is always 0 or 1, with 1 occurring at most once on each chain. (Thus, an admissible multidegree is in particular a divisor on the dual graph $\widetilde{\Gamma}$ of $\widetilde{X}$)

Fixing an admissible multidegree w_0, for each component Z_v of X, we can twist w_0 to concentrate degree on Z_v, and let w_v be the resulting admissible multidegree, and d_v its value at v. (The numerical effect of twisting is precisely a certain kind of chip firing move, and w_v is in essence a v-reduced divisor)

Definition 3 *With the above data, a **limit** $\mathfrak{g}^r_d$ on X consists of the data of a $\mathfrak{g}^r_{d_v}$ $(\mathscr{L}^v, V^v)$ on Z_v for each v, together with a choice of gluing of (suitable twists of) the $\mathscr{L}^v$ to obtain a global line bundle on X, such that for each pair of components $Z_v, Z_{v'}$ of X which intersect nontrivially:*

(I) the multivanishing sequences at $Z_v \cap Z_{v'}$ satisfy a generalization of the Eisenbud-Harris condition, and

(II) the sections of V^v and $V^{v'}$ satisfy certain gluing conditions along $Z_v \cap Z_{v'}$.

In fact, the divisor sequences considered for the multivanishing sequence in (I) depend on the chain structure, with the trivial chain structure corresponding to the divisor sequence $i(Z_v \cap Z_{v'})$, and general chain structures giving refinements of this sequence.

The gluing conditions in (II) depend on both the chain structure and the enriched structure; we will shortly sketch the dependence on chain structure, as it will be important.

Expected dimension

As evidence that our definition isn't missing anything, we show that it has the correct "expected dimension." For simplicity, we restrict to characteristic 0, although there are natural positive-characteristic versions as well.

Theorem 4 *If every component of X is general, then any nonempty limit linear series space on X has dimension*

$$\rho := g - (r+1)(r+g-d),$$

provided that the gluing conditions impose maximal codimension.

We also show that there are certain specific cases where we can replace the generality condition on the components of X with something more explicit. One such case is when the components are rational, and each meets at most two other components.

Thus, the main remaining task is to analyze the gluing conditions.

Gluing analysis

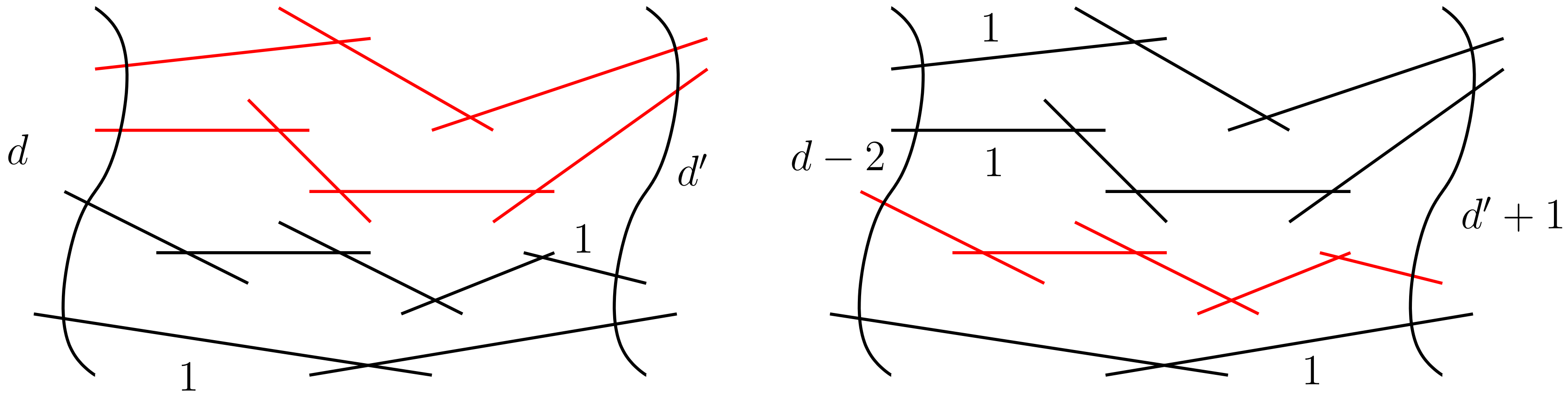

Figure 1: Twisting to shift degree from the lefthand component to the righthand component; gluing conditions are imposed along rational chains in red.

Any gluing condition along $Z_v \cap Z_{v'}$ occurs somewhere between the multidegrees w_v and $w_{v'}$. To get from w_v to $w_{v'}$, we keep shifting 1's along each rational chain of $\widetilde{X}$ connected Z_v to $Z_{v'}$, and sections have gluing conditions in a given multidegree only along chains with no 1 on them. Because gluing conditions only matter up to scaling, they are only nontrivial when they are imposed at two or more chains at a time.

Now, suppose that Z_v meets $Z_{v'}$ in only two or three nodes. Then any single nontrivial gluing condition can be achieved (with the expected codimension) by imposing it on the gluing of the line bundles between Z_v and $Z_{v'}$. Thus, one way to show that the gluing conditions imposes the maximal codimension is to cook up the numerology so that we can have at most one nontrivial gluing condition between each Z_v and $Z_{v'}$.

In the case that Z_v and $Z_{v'}$ meet at only two nodes, with chain lengths $n_1 - 1$ and $n_2 - 1$, we quickly see that we can achieve this by having $\mathrm{lcm}(n_1, n_2)$ be large relative to n_1 and n_2, since it will take $\mathrm{lcm}(n_1, n_2)$ twists to go from one multidegree with a nontrivial gluing condition to the next, and the number of twists between w_v and $w_{v'}$ can be bounded in terms of d and the n_i.

This leads to the following result:

Proposition 5 *If each component of X meets every other component in at most three nodes, suppose further that we have the following numerical condition on the chain structure:*

for any adjacent vertices v, v' of Γ, if v, v' are connected by edges $(e_i)_i$, with associated chain lengths $n_i - 1$, then for any integers $(x_i)_i$ with $\sum_i x_i n_i = 0$, if there is a unique j with $x_j > 0$, then we have

$$\sum_i \lfloor x_j n_j / n_i \rfloor > d.$$

Then the gluing conditions on limit linear series on X always impose the maximal codimension.

Thus, if we assume that the components of X are general, then by Theorem 4 the limit linear spaces all have expected dimension.

In the case that Z_v and $Z_{v'}$ are connected by a single edge, the inequality of Proposition 5 is tautologically satisfied, so we recover the usual expected dimension statements for limit linear series spaces in the compact type case.

If we consider the case of two connecting edges, the inequality of Proposition 5 becomes

$$\frac{\operatorname{lcm}(n_1, n_2)}{n_1} + \frac{\operatorname{lcm}(n_1, n_2)}{n_2} > d,$$

so setting $d = 2g - 2$ we recover precisely the "genericity" condition on edge lengths arising in the work of Cools, Draisma, Payne and Robeva.

Thus, we can think of our work as saying that the reason the Brill-Noether theory is good for the graphs considered in Cools-Draisma-Payne-Robeva is that gluing conditions (which do not appear to be detected in the tropical setting) automatically impose maximal codimension, and do so on the level of line bundle gluing (which is likewise not a part of the tropical setting).

This naturally leads to further speculation about the Brill-Noether theory of graphs. For instance, if we consider chains as in Cools-Draisma-Payne-Robeva, but allow components to meet at three nodes instead of two, then one now expects that the numerical condition imposed in Proposition 5 would be enough to obtain Brill-Noether generality for such graphs.

In the opposite direction, one might speculate that as soon as gluing conditions become subtler, then the associated graphs should not typically be Brill-Noether general. This is corroborated by the observation that a graph consisting of two vertices connected by m edges always carries a divisor of degree 2 and rank 1, irrespective of edge lengths, so is never Brill-Noether general when $m \geq 4$.

It would be interesting to study such examples more systematically from this perspective.