Progress on Riemann existence via degenerations

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We fix $\bar{X}$ a smooth, proper curve of genus g over an algebraically closed field k of characteristic p, together with points

$$P_1, \ldots, P_r \in \bar{X}.$$

We then write

$$X = \bar{X} \setminus \{P_1, \ldots, P_r\}.$$

We have the following groups:

- the étale fundamental group $\pi_1(X)$;
- the tame quotient $\pi_1^t(X)$, obtained by considering only covers which extend to tamely ramified covers of $\bar{X}$;
- the prime-to-p quotient $\pi_1^{(p)}(X)$, obtained by considering covers with monodromy group prime to p (which are necessarily tame).

The Grothendieck school, beyond defining these groups, was able to produce the following relationships:

$$\begin{tikzcd}
\pi_1(g,r) \cong \widehat{\pi_1^{\text{top}}(g,r)} \arrow[r, two heads] \arrow[d, two heads] & \pi_1(g,r)^{(p)} \arrow[d, two heads, "\cong"] \\
\pi_1^t(X) \arrow[r, two heads] & \pi_1^{(p)}(X)
\end{tikzcd}$$

where we choose a curve $\tilde{X}$ of genus g over $\mathbb{C}$ with r points removed, and:

- $\pi_1^{\text{top}}(g,r)$ is the topological fundamental group of $\tilde{X}$;

- $\pi_1(g,r)$ is the étale fundamental group of $\tilde{X}$;

- both groups depend only on g and r.

We therefore understand $\pi_1(g,r)$ explicitly in terms of the usual presentation of $\pi_1^{\mathrm{top}}(g,r)$, and we have

$$\pi_1(g,r) \twoheadrightarrow \pi_1^t(X) \twoheadrightarrow \pi_1(g,r)^{(p)}.$$

Thus, we can place $\pi_1^t(X)$ between two groups which we understand explicitly, and which do not depend on the isomorphism class of X.

In particular, a usual choice of generators of $\pi_1^{\mathrm{top}}(g,r)$ gives a choice of (topological) generators of $\pi_1^t(X)$.

Although tame ramification is often easier to understand than wild ramification, $\pi^t(X)$ is a very subtle object:

- Tamagawa has shown that if $k = \bar{\mathbb{F}}_p$, no curve over k has tame fundamental group isomorphic to the (geometric) generic curve.

- Furthermore, the tame fundamental group of the geometric generic curve is also quite poorly understood.

We will discuss some progress on the latter question, obtained by degeneration arguments.

We will focus on the following question, with X generic:

Question 1 *To what degree one can understand $\pi_1^t(X)$, or more or less equivalently, the tame covers of X, by degenerating X?*

In fact, we will consider only the case that $X = \mathbb{P}^1_k$, although this can be related to the general case by considering more general degenerations.

We will first discuss two sharp results, both in the case of covers $\mathbb{P}^1_k \to \mathbb{P}^1_k$, having a single ramification point over each branch point. The cases addressed are:

- three branch points;

- all ramification indices less than p.

Both results are proved via comparison to the point of view of linear series – that is, to fixing ramification on the source rather than the target.

Second, we will consider some simple examples, in order to determine what sort of results we might hope to obtain in general, and via degenerations.

Our general setup will be as follows. We let X be the (geometric) generic r-pointed rational curve $\mathbb{P}^1_k \smallsetminus \{P_1, \ldots, P_r\}$, and because

$$\pi_1^{\text{top}}(0, r) =< \gamma_1, \ldots, \gamma_r > /(\gamma_1 \cdots \gamma_r = 1),$$

with the γ_i each local monodromy generators around the marked points, by the above we may choose generators $g_1, \ldots, g_r$ of $\pi_1^t(X)$, with each g_i a local inertia generator at P_i, and

$$g_1 \cdots g_r = 1.$$

We fix such generators, and then fix $d \in \mathbb{N}$, and choose cycles $\sigma_1, \ldots, \sigma_r \in S_d$, such that:

- $\sigma_1 \cdots \sigma_r = 1$;
- the orders e_i of σ_i are all prime to p;
- the σ_i generate a transitive subgroup G of S_d.

We also require $2d - 2 = \sum_i (e_i - 1)$ (which is equivalent to restricting to genus-0 covers).

We ask:

Question 2 *When does there exist a branched cover of X having monodromy around g_i given by σ_i for all i?*

In characteristic 0, and more generally when G has order prime to p, such a cover always exists, but it is well known that in the general case, even for general P_i a cover need not exist.

Our first result is in the case of three branch points.

We consider the point of view of linear series, and show that from this point of view, a separable map with the desired ramification exists if and only if an inseparable one doesn't. Roughly, an inseparable map exists only if the e_i are all sufficiently close to some power of p.

To give the precise statement, we introduce the following notation:

Notation 3 *Given $e \in \mathbb{N}$ prime to p, we write $\bar{e}^{[m,u]} := \lceil \frac{e}{p^m} \rceil$, and $\bar{e}^{[m,d]} := \lfloor \frac{e}{p^m} \rfloor$. Also write $e^{[m,u]} := p^m \bar{e}^{[m,u]} - e$, and $e^{[m,d]} := e - p^m \bar{e}^{[m,d]}$.*

The precise theorem we obtain is the following:

Theorem 4 *Suppose $r = 3$. Then there exists a separable map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d, branched over the P_i with monodromy cycles σ_i around each g_i, if and only if the following condition is satisfied:*

For any $m > 0$, and $S \subset \{1, 2, 3\}$ such that:

- $p^m \leq d$;
- $e_i > p^m$ *for all* $i \in S$;
- $\sum_{i \in S} \overline{e}_i^{[m,d]} + \sum_{i \notin S} \overline{e}_i^{[m,u]}$ *is odd;*

we must have:

$$\sum_{i \in S} e_i^{[m,d]} + \sum_{i \notin S} e_i^{[m,u]} \geq p^m. \qquad (1)$$

Remarks 5 *First, we note that in the case of three ramification points, $e_i + e_j > d$ for all i, j, so there can be at most one ramification point over each branch point.*

Second, it is natural that the condition be solely in terms of the e_i, as in fact under our hypotheses the σ_i are determined uniquely (up to simultaneous S_d-conjugation) by the e_i.

Example 6 *We give some simple examples of existence and non-existence:*

If $e_1 = e_2 = p - 1$, and $e_3 = 3$, with $p > 3$, then no cover exists.

If $e_1 = d, e_2 = d - 1, e_3 = 2$, with $d \not\equiv 0, 1 \pmod{p}$, then a cover exists.

Our second result is more difficult, and treats the case of more than three branch points, but with $e_i < p$ for all i.

To state our result, we need to recall that if we are given a group H, there is the natural action of the pure braid group on r-tuples h_i of H such that $h_1 \cdots h_r = 1$. The action of the full braid group B_r is generated by operations replacing the pair h_i, h_{i+1} by

$$h_{i+1}, h_{i+1}^{-1} h_i h_{i+1}.$$

The pure braid group is the kernel of the natural map $B_r \to S_r$, so sends every h_i to a conjugate of itself. In particular, if the h_i are permutations, the pure braid action preserves the combinatorial type of each permutation.

We see in particular that the pure braid group acts on our choice of fundamental group generators g_i, and also on our choice of cycles σ_i.

We show:

Theorem 7 *Suppose we have $e_i < p$ for all p. Then there exists a separable map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d, branched over the P_i with monodromy cycles σ_i around some pure braid transformation of the g_i, if and only if the following condition is satisfied:*

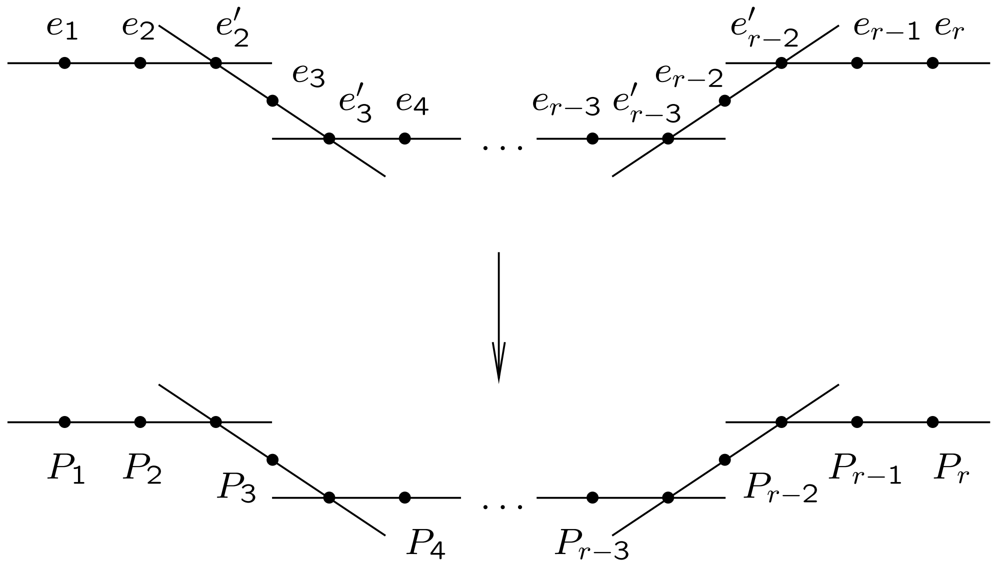
There exists some (possibly different) pure braid transformation σ'_i of the σ_i such that:

- $\sigma''_j := \prod_{i=1}^{j} \sigma'_i$ *is a cycle for* $1 < j < n$*;*

- *if we take τ, τ', τ''' to be any three consecutive cycles in the sequence*

  $$\sigma'_1, \sigma'_2, \sigma''_2, \sigma'_3, \sigma''_3, \ldots, \sigma'_{n-2}, \sigma''_{n-2}, \sigma'_{n-1}, \sigma'_n,$$

  such that τ is in an odd place in the sequence, then the orders e, e', e'' of τ, τ', τ'' satisfy the inequality $e + e' + e'' < 2p$.



The idea behind the condition of the theorem is that such covers arise by deforming totally degenerate covers, as in the figure. The condition that the σ''_i be cycles corresponds to the fact that there is a single ramification point, of index e'_i equal to the length of σ''_i, above each node.

The pure braid transformations arise in the proof of the theorem because we need fundamental group generators compatible with the degenerations. However, we will see why it is natural to allow pure braid transformations of the g_i when we consider examples.

We give some simple examples of what e_i can and cannot be obtained from tame covers, based on Theorem 7.

Example 8 *If r is odd, with $e_i = p-1$ for $i < r$, and $e_r \neq 1$, it is not possible to have a tame cover.*

Example 9 *If r is even and the e_i are all $p-1$, (as was already well known) we can obtain a cover where the partial product cycles alternate between trivial cycles and $(p-1)$-cycles.*

If the e_i alternate between $p-1$ and $p-3$, we can obtain a cover where the partial product cycles alternate between 3-cycles and $(p-3)$-cycles.

We remark that the proof of the non-existence part of Theorem 7 is far more involved than the proof of the existence part.

The proof of Theorem 7, as with the proof of Theorem 4, begins by relating the question to linear series.

However, it's not obvious that it's not possible to have branched covers existing with generic branch points, but where the ramification points are never generic.

Indeed, this happens when we allow $e_i > p$.

However, in the case $e_i < p$ we can use a finiteness result on rational functions with given ramification to prove what we want.

This is proved using a relationship to certain connections on vector bundles, and a finiteness result obtained by Mochizuki in this setting. It appears very difficult to translate the proof purely into the language of rational functions.

Once we have translated to the linear series perspective, we have to know that we can always degenerate successfully (i.e., without obtaining an inseparable map in the limit).

Again, this is a non-trivial theorem, and only true in the case that $e_i < p$ for all i.

We also have to show that the degenerations from the linear series point of view can be realized geometrically to yield admissible covers.

Putting these results together, and using the theory of admissible fundamental groups, we can show that any branched cover can be degenerated successfully to an admissible cover with a single ramification point over each branch point, and this allows us to prove the theorem.

The global monodromy of the covers we have considered are rather limited. Specifically, one can make a group-theoretic argument that:

Proposition 10 *If we have a branched cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree d with a single ramification point of order e_i over each of r branch points, and G is the monodromy group of the cover, then:*

$$G = \begin{cases} C_d : & r = 2 \\ A_d : & r > 2, e_i \text{ odd } \forall i \\ S_d : & \text{otherwise} \end{cases}.$$

One then wonders:

Question 11 *Does the pure braid group act transitively on the possibilities for σ_i?*

If so, the group-theoretic statement of Theorem 7 is equivalent to a numerical statement on the e_i, and we can replace the admissible covers and admissible fundamental group arguments in the proof by group theoretic arguments.

However, the two hardest ingredients are still necessary.

We conclude with some examples which illustrate the need for braid group transformations in the statement of Theorem 7, and which provide intriguing examples of degenerations which behave better than one might expect, although we cannot hope to always have good degenerations.

The first statement we wish to establish is the following:

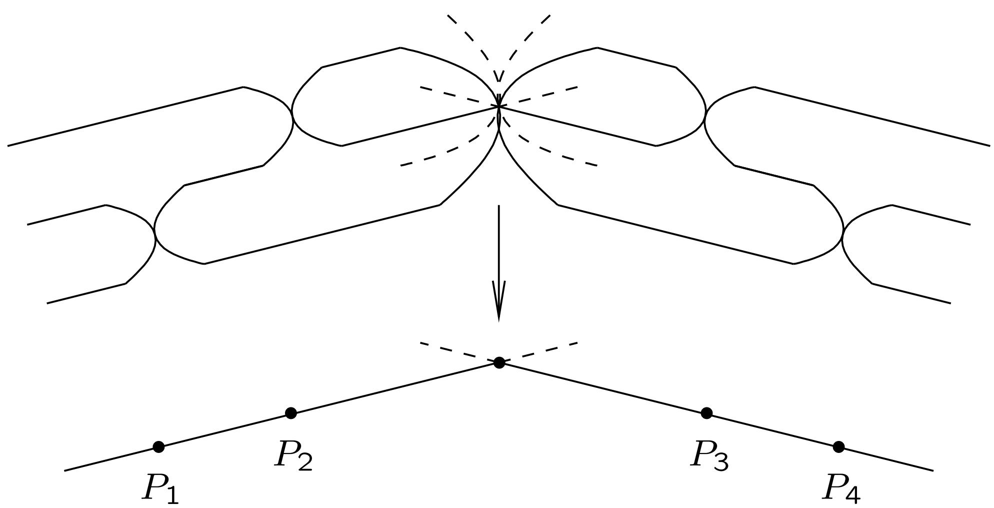
Proposition 12 *Let $r = 4$. Then the possible σ_i which occur as monodromy around the g_i are not closed under pure braid transformation. In particular:*

- *the possible σ_i will change if we apply a braid transformation to the generators g_i;*

- *there is no Riemann existence theorem giving a criterion for the σ_i to be monodromy around the g_i which works also for all pure braid transformations of the g_i.*

Since the criterion of Theorem 7 depends only on the σ_i, and since the only restriction on the g_i is that they be specializations of topological generators, we see that we necessarily have to allow pure braid transformations of the g_i in the statement.

We consider the example in which $r = 4$, $d = 3$, and every $e_i = 2$. In characteristic 3, one might expect this to provide an example of bad degeneration, by prescribing monodromy over P_1, P_2, P_3, P_4 which forces the ramification index over the node to be equal to 3.



Over $\mathbb{C}$, there are four possible covers, corresponding to Hurwitz factorizations

$$
\begin{gathered}
(12)(12)(23)(23) \\
(12)(23)(23)(12) \\
(12)(23)(31)(23) \\
(12)(23)(12)(31).
\end{gathered}
$$

However, in characteristic 3, one computes directly that there is a single cover. This is the same number that one has after degeneration, so we see that it does not give an example of a cover with bad degeneration.

Furthermore, given a choice of monodromy generators g_i, the cover must have monodromy given by one of the above four possibilities. But one checks that the pure braid group acts transitively on the four possibilities, so we conclude Proposition 10.

Another example in which degenerations behave "better than expected" is given by considering the case

$$r = 4, d = 4, e_1 = 4, e_2 = e_3 = e_4 = 2,$$

again in the case $p = 3$. In this situation, one can compute directly that there is a single cover, and one can show that from the linear series point of view, degenerations are always inseparable.

However, from the branched cover point of view, there is again a single cover after degeneration (with two ramification points over the node), so we again find that we do not obtain an example of a cover with bad degeneration.

However, Irene Bouw has an example of a higher-genus tame cover of $\mathbb{P}^1_k$ with $r = 4$ such that every degeneration is inseparable. She produces this as an étale cover Y of a higher-genus cover Z, which is produced in such a way that the p-rank of Z drops under any degeneration, so that the cover is forced to have bad specialization.

Thus, we cannot hope that degenerations are always well-behaved.

However, the earlier examples of genus 0 covers does suggest nonetheless that there is something going on to make degenerations work far more often than one might naively expect, so it seems worth investigating if there are general cases in which one can show that degeneration from the branched cover point of view is well-behaved.