Transversality of non-general Schubert cycles

Brian Osserman

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Fix Schubert classes

$$\sigma_{\alpha^1}, \ldots, \sigma_{\alpha^n}$$

in the Grassmannian $\mathbb{G}(r, d)$ of projective r-planes in $\mathbb{P}^d$, such that the intersection class is 0-dimensional. If one chooses cycles Σ_{α^i} representing each of the σ_{α^i} by choosing general flags in the ambient $\mathbb{P}^d$, it is known that the Σ_{α^i} intersect transversely.

Question 1 *Is the intersection of the Σ_{α^i} still transverse if instead one chooses flags to be osculating flags of general points $P_1, \ldots, P_n$ on the rational normal curve in $\mathbb{P}^d$?*

In this situation, the Schubert cycles are no longer general. Although their intersection is conjectured to be transverse, the problem remains open.

Why do this? One reason is that this corresponds to solving the problem of the number of maps from $\mathbb{P}^1$ to $\mathbb{P}^r$ of degree d, with ramification prescribed at the P_i.

This suggests a natural generalization, by replacing $\mathbb{P}^1$ with higher-genus curves:

Question 2 *Choose g, n, r, d, and ramification sequences $\alpha^1, \ldots, \alpha^n$, such that there is a finite expected number of maps from a curve C of genus g to $\mathbb{P}^r$ of degree d, with ramification sequence at least α^i at points P_i (up to automorphism of $\mathbb{P}^r$). Then how many such maps are there for C and the P_i general?*

Since maps from C to $\mathbb{P}^r$ up to automorphism of $\mathbb{P}^r$ are essentially equivalent to linear series of degree d and dimension r (denoted $\mathfrak{g}^r_d$'s), this question may be thought of as a version of Schubert calculus for the spaces $G^r_d(C)$ of $\mathfrak{g}^r_d$'s on C.

These spaces are known to be smooth, and generalize the Grassmannian, which is the case $C = \mathbb{P}^1$.

While much work has been carried out on the cohomology ring of the flag manifold, not so much is known about G^r_d spaces, so we ask:

Question 3 *What is the cohomology ring of $G^r_d(C)$ for a general curve C? Is it generated by "Schubert cycles" obtained by imposing ramification conditions? Is the ring structure determined by an answer to Question 2?*

Since for $g > 0$ there are no obvious complementary cycles, the last question is not immediately answered even by a positive answer to the previous one.

Unlike the case $g = 0$, where we know the space and its cohomology *a priori*, and the question of ramification conditions on a curve is harder, for $g > 0$ we have no independent description of the G^r_d spaces, so it appears our best hope for understanding the cohomology ring is in terms of intersection theory of ramification conditions.

As a simplest base case, one can ask:

Question 4 *Does Question 1 have a positive answer for $n = 3$, when any three distinct P_1, P_2, P_3 are general?*

In the case $r = 1$, Pieri's formula implies that the intersection product of any three cycles is always 0 or 1, which gives transversality. However, despite extensive experimental data suggesting a positive answer for $r > 1$, the general case remains open.

Theorem 5 *For a positive answer to Question 1 for all n, it suffices to handle the case $n = 3$ (i.e., Question 4).*

In fact, a positive answer to Question 4 would go further.

Theorem 6 *A positive answer in the* $n = 3$, $g = 0$ *case (i.e., Question 4) implies that the ramification conditions of Question 2 intersect transversely for any* n *and* g, *and there is a formula for the number of maps from* C *to* $\mathbb{P}^r$ *with the desired ramification in terms of Schubert calculus.*

Specifically, one finds that the number of maps from C to $\mathbb{P}^r$ is equal to the number of maps from $\mathbb{P}^1$ to $\mathbb{P}^r$ of the same degree, with ramification α^i at n general points P_i, and ramification sequence $0, 1, \ldots, 1$ at g additional general points $Q_1, \ldots, Q_g$.

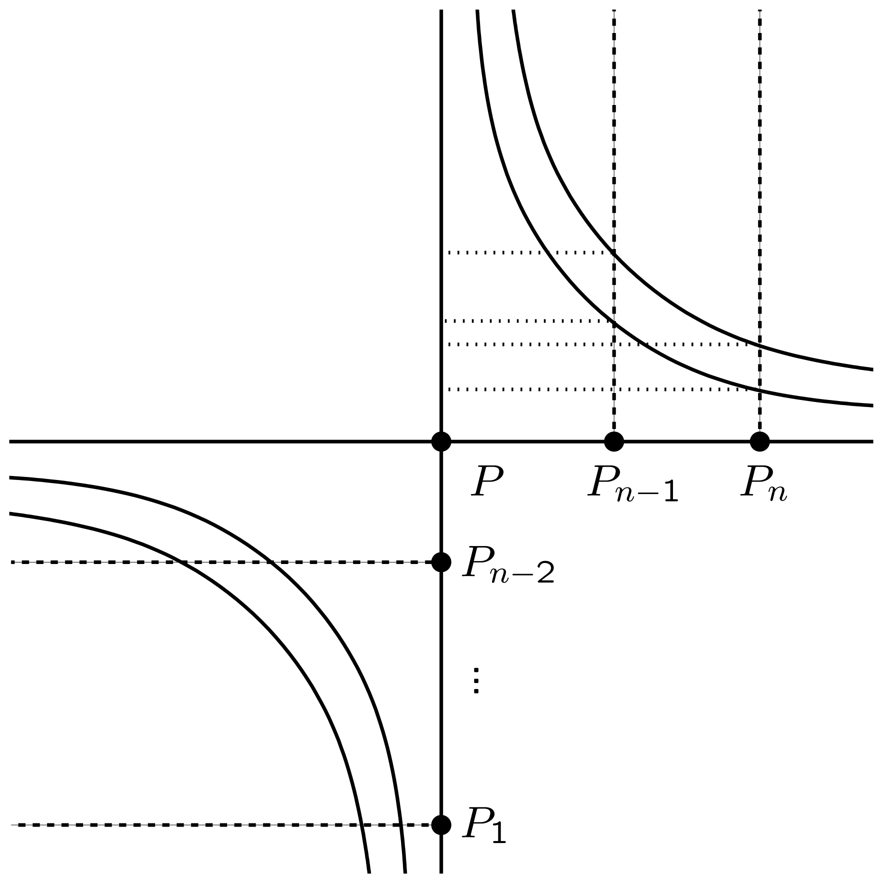
Since both theorems are essentially in terms of linear series, it is natural that they are both proved with degeneration arguments making use of Eisenbud and Harris' theory of limit linear series.

To make such an argument, we start with a family of curves degenerating to a reducible curve, say with m components. The theory of limit linear series sets up a relationship between

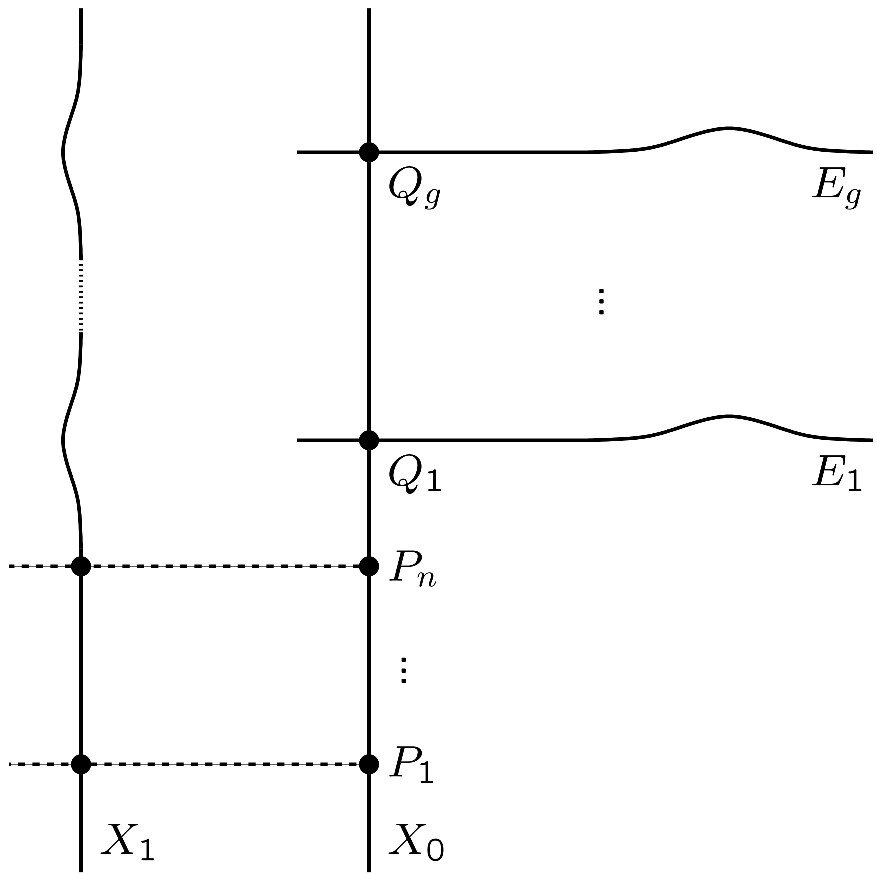
$\mathfrak{g}^r_d$'s on the general curve on the family

and

m-tuples of $\mathfrak{g}^r_d$'s on each component of the reducible fiber with enough new ramification at the nodes to keep the expected dimension the same.



In the $g = 0$ case, we consider a family of genus-0 curves as in the figure. Thus, the reducible fiber has $n-1$ ramification points on one component, and 3 on the other, and an induction argument lets us conclude that the base case of $n = 3$ is enough to deduce the case of arbitrary n.



In the case of $g > 0$, we degenerate as in the figure. Because we need enough ramification on each E_i at Q_i to get finitely many expected series, one shows that in fact there is a single possibility for the ramification and the linear series on each E_i, reducing the problem down to the $\mathbb{P}^1$ case, with g additional ramification points at the Q_i.

We do not expect that limit linear series will be able to answer Question 4. However, degeneration techniques such as those used by Vakil and Coskun to obtain formulas for Littlewood-Richardson coefficients could plausibly work, as they are specifically designed to handle triple intersections of Schubert cycles.

Another application is to reality questions, and in particular the Shapiro-Shapiro conjecture, and the corresponding question for higher genus. We cannot hope to use degenerations to prove the full version of this conjecture, but our argument does yield the following:

Theorem 7 *A positive answer to Question 4, in the strengthened form that all solutions are real if the ramification points are real, would imply that for any g, n, there exist real curves C of genus g with real points $P_1, \ldots, P_n$ such that all the maps of Question 2 are real.*