LIMIT LINEAR SERIES: AN OVERVIEW

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Abstract. Linear series are fundamental to understanding many aspects of the geometry of algebraic varieties. In the 1980's, Eisenbud and Harris introduced a theory of limit linear series, allowing linear series on curves to be analyzed systematically via degeneration arguments, and producing many important consequences for classical questions. We will discuss their original theory, as well as a new construction which gives a functorial, compact moduli space of limit linear series. Finally, we will discuss the many directions of generalization that remain open.

1. Linear series

Linear series arise naturally from the study of maps of varieties into projective space. Specifically, if X is a smooth proper curve over a field k, then maps from X to $\mathbb{P}^r$ of degree d correspond to pairs $(\mathscr{L}, \{v_0, \dots, v_r\})$ of line bundles $\mathscr{L}$ on X of degree d together with sections $\{v_0, \dots, v_r\} \in H^0(X, \mathscr{L})$ such that for every $x \in X$, at least one of the v_i is non-vanishing at x. The map is non-degenerate if and only if the v_i are linearly independent, and if we wish to work up to automorphism of $\mathbb{P}^r$, we replace the $\{v_i\}$ by the $(r+1)$-dimensional vector space V which they span. Finally, it turns out we can obtain a natural compactification of this space simply by dropping the requirement that V be non-vanishing at every point of X. Thus motivated, we have:

Definition 1.1. A **linear series** of degree d and dimension r (also called a $\mathfrak{g}^r_d$) on X is a pair $(\mathscr{L}, V)$ where $\mathscr{L}$ is a line bundle of degree d on X, and V is an $(r+1)$-dimensional subspace of $H^0(X, \mathscr{L})$.

We also briefly mention the idea of ramification, which plays an important role later:

Definition 1.2. Given a linear series $(\mathscr{L}, V)$ and a point $P \in X$, the **vanishing sequence** $a_0(P), \dots, a_r(P)$ of $(\mathscr{L}, V)$ at P is defined to be the increasing sequence of orders of vanishing at P of sections of V. The non-decreasing **ramification sequence** $\alpha_0(P), \dots, \alpha_r(P)$ is defined by $\alpha_i(P) = a_i - i$. If $\alpha_i(P) = 0$ for all i, we say P is **unramified**, otherwise P is **ramified**, or a **ramification point**.

Example 1.3. A few basic cases:

1. We see that $(\mathscr{L}, V)$ corresponds to a map to $\mathbb{P}^r$ when $a_0(P) = 0$ for all P. In this case, we say $(\mathscr{L}, V)$ is **basepoint-free**.
2. If $r = 1$, and $(\mathscr{L}, V)$ is basepoint-free, then $a_1(P)$ (or $\alpha_1(P) := a_1(P) - 1$, depending on convention) corresponds to the usual ramification index at P of the map of curves.

(3) If $r = 2$ and $(\mathscr{L}, V)$ is basepoint-free, then a point with $a_1(P) > 1$ corresponds to a cusp-type singularity in the plane curve image of X, while a point with $a_1(P) = 1$ but $a_2(P) > 2$ corresponds to a flex point.
(4) If $r = g - 1$, and $d = 2g - 2$, the only $\mathfrak{g}^r_d$ on X is $(\Omega^1_X, H^0(X, \Omega^1_X))$, and ramification points correspond precisely to the Weierstrass points of X.

The study of linear series is a natural way to try to better understand X. For instance, the classical theorems that every curve of genus 1 can be imbedded as a cubic plane curve, and that every curve of genus 2 can be represented as a 2-to-1 cover of $\mathbb{P}^1$ branched over 6 points, are powerful tools for understanding such curves. One is thus led to Brill-Noether theory, which can be roughly summarized as the study of the following question:

Question 1.4. Given a curve X, and r, d, does X have a linear series of degree d and dimension r? If so, what is the dimension of the space of the $\mathfrak{g}^r_d$'s?

This question is answered for general curves by the famous Brill-Noether theorem, which was ultimately proved by Kempf, Kleiman, Laksov and Griffiths-Harris (based also work of Severi and Castelnuovo):

Theorem 1.5. *Given g, r, d, let $\rho = (r+1)(d-r)-rg$. If $\rho \geqslant 0$, then for all smooth, proper curves X of genus g, the space of $\mathfrak{g}^r_d$'s on X is non-empty of dimension at least ρ. On a general curve X of genus g, the space of $\mathfrak{g}^r_d$'s has dimension exactly ρ, and is empty if $\rho < 0$.*

One also has a version of the same theorem for $\mathfrak{g}^r_d$'s with prescribed ramification sequences $\{\alpha_i(P_j)\}$ at points $P_1, \dots, P_n$ on X. In this case, the non-emptyness statement does not hold (although one can give an explicit criterion for when it does), and ρ is replaced by $\rho = (r+1)(d-r) - rg - \sum_{i,j} \alpha_i(P_j)$. Also, as with the results discussed below, the result is known only in characteristic 0.

A less immediately obvious application of the theory of linear series is to the study of Weierstrass points. For instance, Eisenbud and Harris proved in [2]:

Theorem 1.6. *Fix $g > 1$, and let ω be a Weierstrass semigroup having weight at most $g/2$. Then there exist smooth curves X of genus g with a Weierstrass point having semigroup ω.*

In fact, there are far subtler ways to apply linear series. One such application is as follows: in the cases of genus 1 and genus 2 curves above, we have that any curve of genus 1 can be written as $y^2 = x^3 - ax + b$ (when $\operatorname{char} k$ doesn't divide 6), and any curve of genus 2 can be written as $y^2 = x^6 + a_5x^5 + \dots a_1x + a_0$ (where we have to normalize to get rid of the singularity at infinity). In particular, in both cases we see that we can write down very explicit families of curves with rational parameters which contain the general curve of genus 1 and 2. It is natural to ask whether it is possible to do this more generally, or equivalently:

Question 1.7. Is the moduli space $\mathscr{M}_g$ of curves of genus g always unirational?

This question leads to the study of divisors on $\mathscr{M}_g$, and it turns out that divisors described in terms of Brill-Noether theory play a vital role, and allowed Eisenbud and Harris [3] to prove:

Theorem 1.8. *$\mathscr{M}_g$ is not unirational for $g \geqslant 23$ (and is in fact of general type for $g \geqslant 24$).*

Given such a range of applications, Eisenbud and Harris were perhaps justified in asserting in [1] that "most problems of interest about curves are, or can be, formulated in terms of (families of) linear series."

2. Limit linear series

The proofs of the Brill-Noether theorem and the theorems of Eisenbud and Harris have a key point in common: in order to give the desired analysis of linear series on smooth curves, they rely on analysis of degenerations to singular curves. The most powerful and general such technique is the theory of limit linear series, introduced by Eisenbud and Harris in [1] in order to prove the above-mentioned theorem. We briefly outline the issues involved, and the idea of their construction.

We first note that the definition of a linear series still makes perfect sense on a singular curve. However, let us suppose we have a (flat, proper) one-parameter family X/B of curves, smooth over the generic point $\eta \in B$, but degenerating to a singular curve over a point $b_0 \in B$. We see that if we want to study the linear series on X_η in terms of those on $X_0 := X|_{b_0}$, we immediately run into problems. The subspaces V of $H^0(X, \mathscr{L})$ are not the issue: given a line bundle $\mathscr{L}$ on the whole family, and a $V_\eta \subset H^0(X_\eta, \mathscr{L}|_{X_\eta})$, we can obtain a unique extension V to the whole family. The problem is rather with the line bundles $\mathscr{L}$ themselves. There are essentially two cases to consider, and in both we assume that the singular fiber X_0 remains smooth except for a single node:

(1) If the curve degenerates to an irreducible nodal curve, then a line bundle $\mathscr{L}_\eta$ on X_η may not extend to any line bundle on all of X: i.e., the relative Picard scheme of X/B is not universally closed.

(2) If the curve degenerates to reducible curve, with two smooth components Y and Z glued at a node, then any $\mathscr{L}_\eta$ extends to X/B, but does not do so uniquely. If the total space of X is regular, we may see this by noting that Y and Z are Cartier divisors on X, so given an extension $\mathscr{L}$ to all of X, we can also consider $\mathscr{L}(iY)$ for $i \in \mathbb{Z}$, which does not affect $\mathscr{L}_\eta$, but does change the line bundle over b_0. Thus, the relative Picard scheme of X/B is not separated.

It turns out that the second problem is more easily fixed than the first, and Eisenbud and Harris focused on that case. It was well-known that by also specifying the degree of $\mathscr{L}$ on Y and Z, one in fact obtains a separated, hence proper Picard scheme. Thus, given a linear series $(\mathscr{L}_\eta, V_\eta)$ for each i, we obtain a unique family of extensions $(\mathscr{L}^i, V_i)$ to the whole family, characterized by the condition that $\mathscr{L}^i$ has degrees $d-i$ and i when restricted to Y and Z, respectively.

The main insight of Eisenbud and Harris was that for many purposes, it is enough to consider the two linear series $(\mathscr{L}^0, V_0), (\mathscr{L}^d, V_d)$. Because $\mathscr{L}^0$ has degree 0 on Z, and $\mathscr{L}^d$ has degree 0 on Y, we don't lose any information by restricting to $(\mathscr{L}^Y, V^Y) := (\mathscr{L}^0|_Y, V_0|_Y)$ and $(\mathscr{L}^Z, V^Z) := (\mathscr{L}^d|_Z, V_d|_Z)$, giving us a pair of $\mathfrak{g}^r_d$'s on Y and Z. Thus, for every linear series $(\mathscr{L}_\eta, V_\eta)$, we obtain a pair $(\mathscr{L}^Y, V^Y), (\mathscr{L}^Z, V^Z)$, and the question becomes which such pairs arise in this way. Eisenbud and Harris showed:

Proposition 2.1. *If $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ arise as the limit of a $(\mathscr{L}_\eta, V_\eta)$, a $\mathfrak{g}^r_d$ on X_η, and if we write $P := Y \cap Z$, and a^Y_i, a^Z_i for the vanishing sequences at P of $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ respectively, then we have:*

$$a^Y_i + a^Z_{r-i} \geqslant d, \forall i : 0 \leqslant i \leqslant r. \tag{2.1.1}$$

Eisenbud and Harris are thus motivated to define:

Definition 2.2. A pair of $\mathfrak{g}^r_d$'s $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ on Y and Z is called a **limit linear series** on X_0 if it satisfies (2.1.1).

If further (2.1.1) is an equality for all i, the pair is called a **refined** limit series, and otherwise it is called a **crude** limit series.

The question then becomes whether all limit series arise as limits of linear series on X_η. The answer is in general no, but a complicated construction of a scheme parametrizing linear series and refined limit linear for a family X/B, together with a relatively simple dimension-counting argument, allowed Eisenbud and Harris to prove:

Theorem 2.3. *Given X/B, smooth sections P_i, and integers $r, d, \alpha_i(P_j)$, there is a scheme $G^{r,\mathrm{EH}}_d$ parametrizing linear series on smooth fibers of X, and refined limit series on singular fibers of X, both of degree d and dimension r, and having ramification at least $\alpha_i(P_j)$ at each P_j. The dimension of any fiber of $G^{r,\mathrm{EH}}_d$ is at least ρ.*

If the dimension of the space of refined limit series on a singular fiber X_0 is exactly ρ, then every refined limit series can be smoothed to linear series on nearby fibers.

3. The new construction

There are two major drawbacks to Eisenbud and Harris' construction. The first is that the construction itself is rather ad-hoc, and does not obviously represent any natural functor. The second is that it does not include the crude limit series, so the space constructed is not proper, and although Eisenbud and Harris have a number of tools for dealing with this, certain arguments become more awkward than they should be.

In [5], it is observed that the way around this problem is to remember all intermediate $(\mathscr{L}^i, V_i)$, and not simply $(\mathscr{L}^0, V_0)$ and $(\mathscr{L}^d, V_d)$. Of course, since $\mathscr{L}^i := \mathscr{L}^0(iY)$, all the $\mathscr{L}^i$ are determined by $\mathscr{L}^0$. On X, we have natural maps $\mathscr{L}^i \to \mathscr{L}^{i+1} = \mathscr{L}^i(Y)$, and if we fix an isomorphism (which will exist locally on the base) $\mathscr{O}_X \cong \mathscr{O}_X(Y+Z) = \mathscr{O}_X(X_0)$, we get maps $\mathscr{L}^{i+1} \to \mathscr{L}^i$ as well. The natural condition on a collection $(\mathscr{L}^i, V_i)$ is then that V_i map into V_{i+1} and V_{i+1} map into V_i under these maps for all i. Considering also the situation after restriction to fibers, we are able to define a functor $\mathscr{G}^r_d$ of limit series on X/B, and prove along lines similar to Eisenbud and Harris:

Theorem 3.1. *Given X/B, smooth sections P_i, and integers $r, d, \alpha_i(P_j)$, the functor $\mathscr{G}^r_d$ of limit series on X/B having ramification at least $\alpha_i(P_j)$ at each P_j is compatible with base change, and representable by a scheme G^r_d proper over B. The dimension of any fiber of G^r_d is at least ρ.*

If the dimension of a fiber of G^r_d is exactly ρ, then every limit series in that fiber can be smoothed to (limit) linear series on nearby fibers.

The functor is defined in such a way to clearly correspond to classical linear series on smooth fibers. If we restrict the picture to a reducible fiber X_0, with components Y and Z as before, we can compare the situation of our functor to Eisenbud and Harris' construction. After restriction to X_0, the maps $\mathscr{L}^i \to \mathscr{L}^{i+1}$ are given by inclusion on Z (on which we are simply tensoring with $\mathscr{O}_Z(P)$, where

$P := Y \cap Z$), and the zero map on Y, and vice versa for $\mathscr{L}^{i+1} \to \mathscr{L}^i$. Thus the condition that V^i maps into V^{i+1} means simply that the spaces V_Z^i may be considered as an increasing filtration of $V^Z := V^d|_Z$, and similarly we have that the V_Y^i are a decreasing filtration of V^Y.

To summarize, if we remember all the intermediate $(\mathscr{L}^i, V_i)$, the data that we expect to remember on X_0 is the line bundle $\mathscr{L}^0$ (which is uniquely determined by the pair $\mathscr{L}^Y$ and $\mathscr{L}^Z$), together with a collection of spaces $V_0, \dots, V_d$ of dimension $r+1$, glued together from increasing and decreasing sequences of subspaces of V^Y and V^Z. This is, in principle and in practice, strictly more information than the $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ composing the Eisenbud-Harris limit series.

In fact, one checks that the compatibility conditions on the V_i of a limit linear series imply Eisenbud and Harris' condition (2.1.1), and moreover, we can prove:

Theorem 3.2. *There is a set-theoretic surjective map $G_d^r \to G_d^{r,\mathrm{EH}}$, which induces an isomorphism on the open subschemes corresponding to refined limit series.*

The essential content of the second statement is that given $(\mathscr{L}^0, V_0)$ and $(\mathscr{L}^d, V_d)$ determining a refined limit series, there is a unique way to fill in the intermediate V_i.

However, this is certainly not true in general, and the map $G_d^r \to D_d^{r,\mathrm{EH}}$ has higher-dimensional fibers in general. This poses the problem that, unlike in the Eisenbud-Harris setting, it is not clear how crude limit series can be understood in terms of $\mathfrak{g}_d^r$'s on Y and Z, ruining inductive arguments in cases where the crude series are included.

However, a careful analysis of the situation in [6] allows us to prove:

Theorem 3.3. *The dimension of the space of limit series corresponding to a given Eisenbud-Harris limit series $((\mathscr{L}^Y, V^Y), (\mathscr{L}^Z, V^Z))$ is bounded above by*

$$\sum_{i=0}^{r} (a_i^Y + a_{r-i}^Z - d).$$

This is exactly the bound needed to make inductive statements on dimensions of spaces of linear series, and together with Theorem 3.1, yields a new, completely straightforward proof of Theorem 1.5 by inductively breaking off elliptic tails to reduce the genus.

In particular, we conclude:

Corollary 3.4. *On a general reducible curve, every limit series, and not only refined limit series, may be smoothed to nearby smooth curves.*

We thus see that the compactification afforded by G_d^r is really as well-behaved as one might hope, and doesn't include any extraneous components.

In a different direction, one may also show [4]:

Theorem 3.5. *(joint with D. Helm) If G_d^r has the expected dimension, then it is flat and Cohen-Macaulay.*

4. Generalization

There are various directions of generalization that one might hope to pursue, and we briefly discuss these.

First, the construction discussed has only been carried out in the case of curves of 2 components. Although it seems reasonable to expect that the generalization to more components is straightforward, it is also somewhat combinatorially complicated, and should not be taken entirely for granted.

Next, the construction generalizes almost immediately to at least certain cases of higher-dimensional varieties, but runs afoul of a rather different problem: the "expected dimension" of our analysis is no longer necessarily the actual dimension, and inductive arguments can become far more complicated. Indeed, the interpolation problem in the plane is essentially the $r = 0$ case of linear series on $\mathbb{P}^2$, and the problem of when these have the expected dimension is still open.

Two other directions of generalization run into similar problems: to generalize either to curves not of compact type, or to vector bundles of higher rank, the underlying moduli space of vector bundles is no longer proper, and one has to find a compactification which behaves well with respect to tracking subspaces of sections. The case of higher rank further runs into problems of not having any obvious inductive interpretation.

However, in most of these cases, the basic construction discussed still goes through in at least some interesting cases, and it then becomes a question of finding a good interpretation of the objects on the special fiber, so one might consider that some modest degree of progress has been made.

References

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