RATIONAL FUNCTIONS IN ONE VARIABLE WITH GIVEN RAMIFICATION

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Abstract. We sketch the solution to a simple enumerative problem in linear series in characteristic p: given points P_i on the projective line, integers $e_i < p$ and d satisfying $\sum_i (e_i - 1) = 2d - 2$, how many maps from $\mathbb{P}^1$ to itself are there of degree d and having ramification index precisely e_i at the P_i? Limit linear series are a tool for solving this problem. I'll explain what they are, and some of the special issues that come up in characteristic p.

1. The main question and background

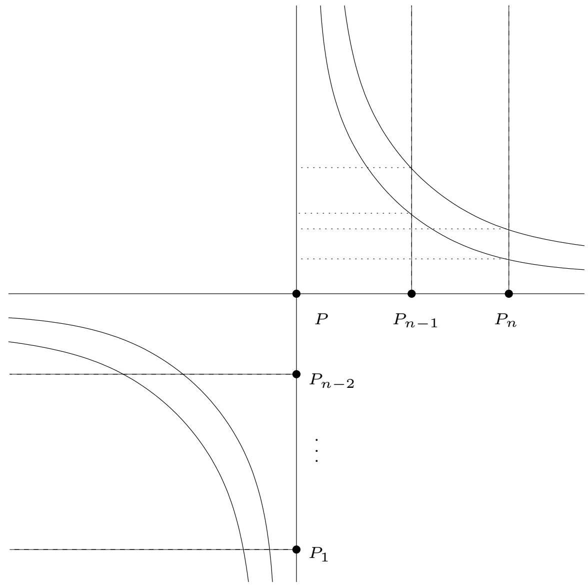
We work throughout over an algebraically closed field k. The material discussed is taken from [2].

The theory of limit linear series has provided a powerful tool for approaching a range of problems in characteristic 0, but people have typically avoided the situation of positive characteristic, on the grounds that inseparability is too difficult to control. We will demonstrate that while this apprehension is reasonably well-justified, it is not an absolute obstruction to obtaining results. Our guiding question is:

Question 1.1. Given n general points P_i on $\mathbb{P}^1$ and integers $e_i \geq 2$, with $\sum_i (e_i - 1) = 2d - 2$, and $e_i \leq d$ for all i, counted modulo automorphism of the image, how many separable maps are there from $\mathbb{P}^1$ to $\mathbb{P}^1$ of degree d which ramify to order e_i at the P_i?

Notation 1.2. When the answer to the above question is finite, we denote it by $N(\{e_i\}_i)$.

The answer to our question can be rephrased in terms of intersections of Schubert cycles on Grassmannians. Indeed, rational maps of degree d, taken up to automorphism, correspond to 2-dimensional spaces of polynomials of degree d without common factors, so are points in $\mathbb{G}(1, d)$. And one checks that each intersection condition cuts out a Schubert cycle in this Grassmannian. In the case of characteristic 0, these Schubert cycles intersect transversely, so the answer to our question is given by Schubert calculus. This was first proved by Eisenbud and Harris [1, Thm. 9.1], as a key part of an argument for the Brill-Noether theorem using degeneration to rational cuspidal curves. Originally, motivation for the characteristic-p version was provided by a relationship with certain logarithmic connections with vanishing p-curvature on vector bundles on $\mathbb{P}^1$, and from there to vector bundles on higher-genus curves which are semistable, but pull back to unstable bundles under the Frobenius map. Additionally, one can use the work discussed here to derive new existence and nonexistence results on branched covers of $\mathbb{P}^1$ in positive characteristic.



One may answer our main question in characteristic 0 via a simple degeneration argument using the Eisenbud-Harris theory of limit linear series. Specifically, one considers the degeneration pictured in the figure. If one has a family of maps with the prescribed ramification on the smooth curves, one can obtain a limit on an axis by projecting the whole family of curves to that axis, considering the family of maps as a family on a fixed $\mathbb{P}^1$, and taking the limit. One may obtain common factors at the node this way (consider for instance $\frac{x+t}{x^2}$ at $t = 0$), but if one factors these out, the Eisenbud-Harris theory (which is of course far more general) sets up a correspondence between maps of degree d on the general smooth curve in the family having the prescribed ramification at the P_i, and pairs of maps of degree $d_1, d_2 \leq d$ on each axis, with prescribed ramification at the P_i, and ramification index e at the node (on both components), where $d_1 + d_2 = d + e$. This sets up a natural induction argument on the dimension and reducedness of the linear series in question, with the base case being that of three ramification points; in this case, the relevant intersection number is always 0 or 1, so transversality (given the correct dimension, which is easy to show more generally) is automatic.

In characteristic p, the fundamental idea behind the argument is still valid, but a number of difficulties arise, and we see a number of themes:

- there can be infinitely many separable maps with the desired ramification, even for tame ramification indices (consider $x^{p+2} + tx^p + x$);
- even when separable maps have the expected dimension, an excess intersection corresponding to inseparable maps will typically occur, so the answer is no longer an intersection number;
- in order to make a degeneration argument as above work, the base case of three points requires some extra work, and more importantly, one must be able to argue that for general ramification points, separable maps cannot degenerate to inseparable maps;
- a ramified Brill-Noether theorem is required for the argument, and while true, it requires a substantial new idea.

These obstacles can all be overcome, and we are able to solve the problem when all e_i are less than p:

Theorem 1.3. *When all $e_i < p$, we have the following complete solution to our main question:*

$$N(e_1, e_2, e_3) = \begin{cases} 1 & p > d \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

$$N(\{e_i\}_{i \le n}) = \sum_{\substack{d-e_{n-1}+1 \\ d-e_n+1} \le d' \le \substack{d \\ p+d-e_{n-1}-e_n}} N(\{e_i\}_{i \le n-2}, e), \text{ with } e = 2d' - 2d + e_{n-1} + e_n - 1 \tag{1.2}$$

Equivalently, for $n > 3$, $N_{\text{gen}}(\{e_i\}_i)$ is given as the number of $(n-3)$-tuples of positive integers $e'_2, \ldots, e'_{n-2}$ such that any consecutive triple e, e', e'' of the sequence

$$e_1, e_2, e'_2, e_3, e'_3, \ldots, e_{n-2}, e'_{n-2}, e_{n-1}, e_n,$$

with $e = e_1$ or some e'_i, satisfies the following properties:

- *The sum $e + e' + e''$ is odd, and less than $2p$;*
- *The triple e, e', e'' satisfies the triangle inequality: i.e., $e \le e' + e''$, $e' \le e + e''$, and $e'' \le e + e'$.*

2. The argument

One may prove the following ramified Brill-Noether statement in our situation by considering an incidence correspondence which compares to the situation covers with fixed branching, and using standard results in that setting.

Theorem 2.1. *Let e_i be prime to p, and suppose $\sum_i (e_i - 1) = 2d - 2$. Then for a general choice of points P_i, we have that there are only finitely many separable maps from $\mathbb{P}^1$ to $\mathbb{P}^1$ ramified to order e_i at P_i, modulo automorphism of the image.*

From this, it is essentially an exercise to deduce the three-point case of Theorem 1.3, by showing that a separable map exists with the desired ramification conditions if and only if an inseparable one doesn't. One can check combinatorially that to prove the general asserted formula via the same degeneration argument as outlined in characteristic 0, one needs to show that in the degeneration used, for general configuration of the P_i, there will not be separable maps on the general fiber specializing to inseparable maps on the special fiber. Strangely, a key step in the proof is the following observation:

Proposition 2.2. *Suppose $e_1 > p$ but still prime to p, and $e_i < p$ for all $i > 1$. Then if one map exists with ramification e_i at P_i, infinitely many do.*

Proof. One simply places P_1 at infinity, and can then check that adding tx^p for any $t \in k$ does not change the ramification. □

Note that the example given earlier shows that such infinite families really do exist.

In particular, by Theorem 2.1 we see:

Corollary 2.3. *In the situation of the proposition, if the P_i are general, no maps exist with the specified ramification.*

The main result for controlling degeneration, although self-contained, was discovered by calculation of examples of certain degenerations of the logarithmic connections on $\mathbb{P}^1$ mentioned earlier, and then translating into self-maps of $\mathbb{P}^1$. It states:

Theorem 2.4. *Let A be a DVR containing its residue field k and with uniformizer t, and f_t be a family of maps of degree d from $\mathbb{P}^1$ to $\mathbb{P}^1$ over $\operatorname{Spec} A$ whose generic fiber is tamely ramified along sections P_i with all $e_i < p$, and whose special fiber is inseparable. We further assume that the P_i stay away from infinity. Then if the limit of the P_i in the special fiber is denoted by $\bar{P}_i$, we have:*

- (i) *If the $\bar{P}_i$ are distinct, they are in a special configuration allowing the existence of separable maps of degree $d+m-1$ ramified to order e_i at $\bar{P}_i$, and $2m-1$ at infinity;*
- (ii) *If $\bar{P}_j = \bar{P}_{j'}$ with $e_j + e_{j'} < p$, and the other $\bar{P}_i$ distinct, then the $\bar{P}_i$ are in a special configuration allowing separable maps of degree $d+m-1-b$, ramified to order e_i at the $\bar{P}_i$ for $i \neq j, j'$, $e_j + e_{j'} - 2b - 1$ at $\bar{P}_j = \bar{P}_{j'}$, and $2m-1$ at infinity;*

in either case, m is some integer with $p \leq m \leq d$, and in the second case b is a non-negative integer less than $(e_j + e_{j'} - 1)/2$.

Proof. The main idea of the proof is not dissimilar to the basic operation of applying fractional linear transformations to be able to factor out a power of the uniformizer if one is given a family of maps degenerating to a constant map. However, in this case we will apply a fractional linear transformation with inseparable coefficients; this will behave similarly, but will not preserve the degree of the map, and also does not appear to work readily in nearly the generality of the constant case.

We work for the most part explicitly with pairs of polynomials and their differents, only dealing with common factors at the end to translate to rational functions and ramification indices. We can write f_t as F/G, where $F, G \in A[x]$, and have no common factors. We denote by F_0 and G_0 the polynomials obtained from F and G by setting $t = 0$, and by $\bar{F}_0$ and $\bar{G}_0$ the inseparable polynomials obtained by canceling the common factors of F_0 and G_0; since F, G represent a linear series of dimension 1, we may further assume that they were chosen so that F_0, G_0 defines a non-constant function. Then let H_1 and H_2 be inseparable polynomials of degree strictly less than $\bar{F}_0$ and $\bar{G}_0$ respectively, such that $\bar{F}_0 H_2 - \bar{G}_0 H_1 = 1$ (this is possible by dividing the exponents $\bar{F}_0$ and $\bar{G}_0$ by p, applying Euclid's algorithm in $k[x]$, and multiplying all exponents by p). We now construct a new family $\tilde{F}/\tilde{G}$ over $\operatorname{Spec} A$ as follows: if we denote by ν the map from $A[x]$ to itself which simply factors out common powers of t, then $\tilde{F} := \nu(F\bar{G}_0 - G\bar{F}_0)$, and $\tilde{G} := FH_2 - GH_1$. It is easy to check that applying an inseparable fractional linear transformation to F/G will change $(dF)G - F(dG)$ by the determinant of the transformation; in our case, by construction the determinant is 1, and it follows that $(d\tilde{F})\tilde{G} - \tilde{F}(d\tilde{G})$ is the same as $(dF)G - F(dG)$, but with a positive power of t factored out.

At $t = 0$, we note that since we had $\bar{F}_0 H_2 - \bar{G}_0 H_1 = 1$, $\tilde{G}$ is made up precisely of the common factors of F_0 and G_0, of which there can be at most $d - \deg f_0 \leq d - p$. Since we removed a positive power of t from $(dF)G - F(dG)$, if we still have an inseparable limit, we can repeat the process as many times as necessary to remove all the powers of t and obtain a separable limit. Each time we do, the degree of the denominator at $t = 0$ clearly remains at most $d - p$. We thus end up with a family $\tilde{F}/\tilde{G}$ which over the generic fiber has the same different as F/G away from infinity. If we let K be the fraction field of A, we also note that we must have that the ideal generated by $\tilde{F}, \tilde{G}$ in $K[x]$ is the same as that generated by F, G. Since F, G had no common factors over K, it follows that $\tilde{F}, \tilde{G}$ have no common factors either. Now, since we have no common factors, we find by considering differents that away from $t = 0$ (that is, at the generic fiber), $\tilde{F}/\tilde{G}$ has the same ramification as F/G except possibly at infinity, since all the e_i were specified to be less than p.

The main idea of the rest of the proof is to show that by reducing the degree of the denominator, our construction creates enough new ramification at infinity to bound the degree of the new map strongly from below, determining the situation as what is described in the statement of the theorem. □

Since we just observed that functions of this type can occur only for special configurations of P_i, it follows that separable maps can degenerate to inseparable maps only for special configurations, which is exactly what we needed to complete our degeneration argument for Theorem 1.3.

References

1. David Eisenbud and Joe Harris, *Divisors on general curves and cuspidal rational curves*, Inventiones Mathematicae **74** (1983), 371–418.
2. Brian Osserman, *Rational functions with given ramification in characteristic p*, Compositio Mathematica **142** (2006), no. 2, 433–450.