NOTES ON COHOMOLOGY AND BASE CHANGE

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The theory of cohomology and base change is fundamental to much of algebraic geometry, but more often than not, the standard references provide statements which are, in one way or another, unsuitable for a specific desired application. With the focus on balancing generality with recognizability, we present the basic statements as derived from the far more general (and correspondingly unrecognizable) statements in the presentation of EGA, with further development out of Mumford's Abelian Varieties, and finally an explicit example. We note that here we are concerned with the behavior of arbitrary base change; it is of course a theorem that flat base change always has good behavior, see EGA III.1, 1.4.15.

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1. The Statements of EGA

We begin with two statements that can be drawn directly from EGA, after appropriate translation. The main points of the translation are the equation indentifying higher direct image functors for sheaves as special cases of hypercohomology functors for complexes in EGA III.2, 6.2.1, and the notation for §7.7 defined in 7.7.1.1. Both statements deal with a morphism $f : X \to S$ and a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, flat over S; the question is when the operation of taking ith higher direct images (and in particular, pushforward) of $\mathscr{F}$ commutes with pullback of $\mathscr{F}$ under arbitrary base change $S' \to S$. When this is the case, we say (with apologies to the English language) that **cohomology and base change commute for** $\mathscr{F}$ in degree i. We will see that this property is also related to the question of when the pushforward of $\mathscr{F}$ is locally free.

We have:

Theorem 1.1. *Let $f : X \to S$ be a separated, quasi-compact morphism, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module which is flat over S, and suppose further that all the $R^i f_*(\mathscr{F})$ are flat over S for all i. Then cohomology and base change commute for $\mathscr{F}$ in all degrees.*

Proof. This is simply a special case of EGA III.2, 6.9.9.2, in the case that $Y = S$, and $\mathscr{P}_\bullet$ is the single-term complex with $\mathscr{F}$ in degree zero. $\square$

Theorem 1.2. *Let $f : X \to S$ be a proper morphism to a locally Noetherian scheme, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module which is flat over S. Then given an integer i, the following are equivalent:*

a) *Cohomology and base change commute for $\mathscr{F}$ in degree i.*
b) *Cohomology commutes with base change to any point $s \in S$ for $\mathscr{F}$ in degree i.*
c) *For all $s \in S$, the canonical map $R^i f_*(\mathscr{F}) \to H^i(X_s, \mathscr{F}_s)$ is surjective.*

Additionally, of the following conditions, a) implies b) and c), and when S is reduced, we also have that b) implies a) (and hence c)).

a) *Cohomology and base change commute for $\mathscr{F}$ in degrees i and $i+1$.*
b) *The function $s \mapsto \dim H^{i+1}(X_s, \mathscr{F}_s)$ is constant with value r.*
c) *$R^{i+1}f_*(\mathscr{F})$ is locally free of rank r.*

Proof. This simply translates and pieces together several results from EGA III.2, §7, with S replacing Y, and $\mathscr{P}_\bullet$ the single-term complex consisting of $\mathscr{F}$ in degree zero. For the equivalence of the first three conditions, cohomology and base change commuting in degree i is simply condition d) of Theorem 7.7.5; it is clear that our a) implies our b), and our b) implies our c), and the last statement of Proposition 7.7.10, combined with Theorem 7.7.5, implies that our c) is equivalent to our a).

For the next three conditions, we combine the exactness conditions of Theorem 7.7.5, with the local-to-global statement of Proposition 7.7.10, at which point the desired assertions are obtained directly from parts b), d), and e) of Proposition 7.8.4. □

Before stating corollaries, we have a couple of statements which are useful and frequently arise in relation to cohomology and base change, even though their statements appear unrelated:

Theorem 1.3. *Let $f : X \to S$ be a proper morphism to a locally Noetherian scheme, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module which is flat over S. Then we have:*

(i) *For any i, the function $s \mapsto \dim H^i(X_s, \mathscr{F}_s)$ is upper semi-continuous.*
(ii) *The function $s \mapsto \sum_i (-1)^i \dim H^i(X_s, \mathscr{F}_s)$ is locally constant.*

Proof. The first statement is EGA III.2, Theorem 7.7.5, I). The second is EGA III.2, Theorem 7.9.4. □

Notation 1.4. When S is connected, we denote the value of the second function of the preceding theorem by $\chi_S(\mathscr{F})$, and call it the **Euler characteristic** of $\mathscr{F}$.

Finally, we have the following useful corollary:

Corollary 1.5. *Let $f : X \to S$ be a proper morphism to a locally Noetherian scheme, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module, flat over S. If for some i_0, we have $H^i(X_s, \mathscr{F}_s) = 0$ for all $s \in S$ and all $i \neq i_0$, then $R^{i_0}f_*(\mathscr{F})$ is locally free, of rank equal to $(-1)^{i_0}\chi_S(\mathscr{F})$.*

In the special case that $i_0 = 0$, the same conclusions hold under the weaker hypothesis that $R^i f_*(\mathscr{F}) = 0$ for all $i \neq i_0$.

Proof. These are simply Corollaries 7.9.9 and 7.9.10 of EGA III.2. □

2. Further Development Courtesy of Mumford and Harris

It is worth noting that Mumford's Abelian Varieties includes a substantially more readable proof of a major portion of the theory of cohomology and base change. Specifically, the following is II.5, Corollary 2, on p. 50, with Y replaced by S:

Theorem 2.1. *Let $f : X \to S$ be a proper morphism of Noetherian schemes, with S reduced and connected, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module, flat over S. Then for all i, the following are equivalent:*

(i) *$s \mapsto \dim_{k(s)} H^i(X_s, \mathscr{F}_s)$ is a constant function,*

(ii) $R^i f_*(\mathscr{F})$ *is a locally free sheaf* $\mathscr{E}$ *on* S*, and for all* $s \in S$*, the natural map*

$$\mathscr{E} \otimes_{\mathscr{O}_S} k(s) \to H^i(X_s, \mathscr{F}_s)$$

is an isomorphism.

If these conditions are fulfilled, we have further that

$$R^{i-1} f_*(\mathscr{F}) \otimes_{\mathscr{O}_S} k(s) \to H^{i-1}(X_s, \mathscr{F}_s)$$

is an isomorphism for all $s \in S$*.*

Looking at the proof in more detail, one finds a corollary which greatly clarifies the behavior of the $\dim_{k(s)} H^i(X_s, \mathscr{F}_s)$ when it is not in fact constant as the point s varies. It follows from the upper semicontinuity theorem and the local constancy of the Euler characteristic that any jump must be upward, and the total jump must be the same in even and odd indices. However, J. Harris pointed out in a course that one can in fact see:

Theorem 2.2. *Let* $f : X \to S$ *be a proper morphism of Noetherian schemes, and* $\mathscr{F}$ *a coherent* $\mathscr{O}_X$*-module, flat over* S*. Then any jumps in the function*

$$s \mapsto \dim_{k(s)} H^i(X_s, \mathscr{F}_s)$$

are necessarily paired in consecutive indices i *and* $i+1$*. Given any point* $s \in S$*, we can thus inductively determine for which pairs* $i, i+1$ *such a jump occurs, and by how much.*

Moreover, in the case that S *is regular of dimension* 1*, we find that* $R^{i+1} f_* \mathscr{F}$ *has torsion if and only if such a jump occurs for* $i, i+1$ *(in particular,* $R^i f_* \mathscr{F}$ *remains locally free, unless there is also a jump for* $i-1, i$*).*

Proof. The first assertion follows immediately from the proof of II.5, Corollary, on p. 50 on Mumford. Specifically, in the notation of that proof (with our S again corresponding to his Y), $\dim_{k(y)} H^i(X_y, \mathscr{F}_y)$ jumps up by the total amount that $\dim_{k(y)}[\ker(d^j \otimes_A k(y))]$ for $j = i-1, i$ jumps down; this is then also reflected in $\dim_{k(y)} H^{j'}(X_y, \mathscr{F}_y)$ for $j' = i-1, i+1$ respectively, as desired. Thus, a jump at index $i, i+1$ can be measured by the drop in rank of d_i; but we can also determine it intrinsically simply by looking at the amount that $\dim_{k(s)} H^j(X_s, \mathscr{F}_s)$ jumps for all j, and working up inductively from $j = 0$.

For the case in which S is regular of dimension 1, in the same notation as before (and in particular, having reduced to the case Y is affine), note that by the structure theorem for maps of modules over PIDs, applied to the local rings of Y, we have that $d^i : K^i \to K^{i+1}$ drops rank at a point if and only if $K^{i+1}/\operatorname{im}(d^i)$ has torsion. Considering the exact sequence

$$0 \to \ker(d^{i+1})/\operatorname{im}(d^i) \to K^{i+1}/\operatorname{im}(d^i) \to K^{i+1}/\ker(d^{i+1}) \to 0,$$

we have the last term injecting into K^{i+2}, so it is necessarily torsion-free, and the torsion in $K^{i+1}/\operatorname{im}(d^i)$ must come from torsion in $\ker(d^{i+1})/\operatorname{im}(d^i)$, which by construction (see II.5, Theorem 2, p. 46) is isomorphic to $R^{i+1} f_* \mathscr{F}$, giving the desired result. □

We obtain in particular:

Corollary 2.3. *In the situation of the previous theorem, with S reduced, if $H^1(X_s, \mathscr{F}_s)$ is locally constant over all $s \in S$, or if S is Dedekind and $R^1 f_* \mathscr{F}$ is locally free, then we conclude that the function*

$$s \mapsto \dim_{k(s)} H^0(X_s, \mathscr{F}_s)$$

is locally constant, $\pi_* \mathscr{F}$ is locally free, and pushforward commutes with base change for $\mathscr{F}$.

Proof. By the previous theorem, if the H^1s are locally constant, the H^0s must also be; similarly, if S is Dedekind and $R^1 f_* \mathscr{F}$ is locally free, it is in particular torsion free, and there cannot be jumps in the H^0s. Either of our earlier Theorems 1.2 or 2.1 then give that since S is reduced, the H^0 being locally constant implies that the pushforward is locally free and commutes with base change. □

3. An Example

We conclude with an example which illuminates the situation further. This example was borrowed from a course of J. Harris.

Example 3.1. Let E be an elliptic curve, and set $S = E$, $X = E \times E$, and $\mathscr{F}$ the line bundle on X obtained via the isomorphism $E \cong \mathrm{Pic}^0(E)$. Then the line bundle above $0 \in E$ is the trivial bundle, and has a non-zero global section, but over every other point of E, we get vanishing H^0. As required by Theorem 2.2, the H^1 also jumps at the fiber over 0. Now, $f_* \mathscr{F}$ is in fact locally free in this example; indeed, there is no open subset of the base over whose preimage $\mathscr{F}$ has a non-zero section, so $f_* \mathscr{F} = 0$. We therefore see explicitly that pushforward does not commute with base change, as when we base change to the fiber over 0, we get a non-zero H^0. In particular, we cannot have $R^1 f_* \mathscr{F} = 0$, as then all $R^i f_* \mathscr{F}$ would be flat over S, and Theorem 1.1 above would imply that cohomology and base change commute. In fact, $R^1 f_* \mathscr{F}$ is a skyscraper sheaf supported above 0.

Remark 3.2. This example highlights an important distinction between f_* and $R^i f_*$ for $i > 0$: despite the symmetry between $i = 0$ and $i > 0$ in all the general theorems above, and despite the fact that if the H^i on fibers have constant dimension, $R^i f_*$ will be locally free of the corresponding rank, when $i > 0$, $R^i f_*$ has the key distinction from f_* that it cannot be represented as the pushforward of a sheaf on X whose global sections on fibers X_s would be $H^i(X_s, \mathscr{F})$. Indeed, if it were, then the same argument from the example which showed that $f_* \mathscr{F} = 0$ would have also implied that $R^1 f_* \mathscr{F} = 0$.