BEYOND SCHLESSINGER: DEFORMATION STACKS

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Abstract. Many basic examples of deformation theory fall into certain predictable patterns, but have not been studied systematically in an elementary way. Certain statements are intuitively clear: for instance, that for a deformation problem on a scheme X with locally trivial deformations and obstructions, the tangent space is given by $H^1(X,T)$ and obstructions lie in $H^2(X,T)$, where T is the sheaf of infinitesimal automorphisms of the problem. However, even making such statements precise requires a language that goes beyond the deformation functors studied by Schlessinger and many others. In this report on ongoing work, we will begin by reviewing the basics of deformation theory and stacks, and then we will discuss a new stack-based framework for deformation theory which allows a systematic and elementary treatment of the passage from local to global in deformation problems.

1. Deformation theory

We begin by reviewing the basics of classical infinitesimal deformation theory.

1.1. Formal deformations.

We begin by recalling classical infinitesimal deformation theory, as developed by Grothendieck and Schlessinger. The starting point for such a theory is the following basic observation:

Proposition 1.1. *Let X be a scheme over field k, and $x \in X$ a point with residue field k. Then the tangent space to X at x can be naturally described as the set of maps* $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to X$ *with set-theoretic image x. In particular, this set of maps has a natural structure of a vector space over k, and the dimension of X is bounded by the dimension of this vector space.*

Now let us suppose that X is in fact a moduli space, so that for any scheme T over k, maps from T to X correspond to some families over T. For instance, we could have (flat) families of smooth, proper curves of genus g, corresponding to the moduli space of curves of genus g, or, if we fix a scheme Y over k, (flat) families of lines bundles on $Y \times T$, corresponding to the Picard scheme of Y. If we wish to obtain an upper bound on the dimension of the moduli space, frequently the easiest approach is to compute the dimension of the tangent space at a point.

To understand the tangent space of the moduli space at a point x (which correspond to an object over $\operatorname{Spec} k$), we are thus naturally led to consider families of the appropriate type over $\operatorname{Spec} k[\epsilon]/\epsilon^2$; from now on, we will always assume that ϵ is a square-zero element, and simply write $k[\epsilon]$.

Similarly, we can understand more about the local structure of X near a point by looking at maps of $\operatorname{Spec} A \to X$ with image x, where A is an arbitrary Artinian ring with residue field k; for instance, from the formal criterion for smoothness one can deduce that X is smooth at x if and only if for all surjections $A' \to A$

of Artinian local rings with residue field k, every map $\operatorname{Spec} A \to X$ with image x factors through some map $\operatorname{Spec} A' \to X$.

In the case that X is a moduli space, we can therefore test whether it is smooth by asking whether every family over $\operatorname{Spec} A$ is the restriction of some family over $\operatorname{Spec} A'$.

In analogy to studying moduli problems by considering their functors, we are naturally led to the following sort of functors that will arise in studying formal deformations:

Definition 1.2. Given a field k, denote by $\operatorname{Art}(k)$ the category of Artinian k-algebras with residue field k. A (covariant) functor $\mathscr{F} : \operatorname{Art}(k) \to \operatorname{Set}$ is called a **pre-deformation functor** if $\mathscr{F}(k)$ consists of a single object.

Given a pre-deformation functor $\mathscr{F}$, we call $\mathscr{F}(k[\epsilon])$ the **tangent space** of $\mathscr{F}$. We say that $\mathscr{F}$ is **unobstructed** if for each surjection $A' \to A$ in $\operatorname{Art}(k)$, and each object $\eta \in \mathscr{F}(A)$, there exists an object $\eta' \in \mathscr{F}(A')$ restricting to η.

We remark that in this generality, the tangent space to $\mathscr{F}$ does not carry a vector space structure, but it will have such a structure under mild hypotheses discussed below.

In particular, if we are given a scheme X over $\operatorname{Spec} k$ and a point $x \in X$ with residue field k, we obtain a pre-deformation functor simply by setting $\mathscr{F}(A)$ to be the maps $\operatorname{Spec} A \to X$ with image x. By the above comments, the tangent space to $\mathscr{F}$ is the tangent space to X at x, and $\mathscr{F}$ is unobstructed if and only if X is smooth at x.

Basic deformation theory follows two complementary branches: one is to ask how close $\mathscr{F}$ is to being representable, and the other is to try to obtain nice descriptions of the tangent space of $\mathscr{F}$, and to understand obstructions to lifting objects over extensions of Artin rings.

1.2. **Schlessinger's criteria.** In [3] (1968), Schlessinger studied the question of representability of a pre-deformation functor in detail. The stronger possibility is called **pro-representability**. We note that given $\mathscr{F}$, if $\widehat{\operatorname{Art}}(k)$ denotes the category of complete local Noetherian k-algebras R with residue field k, we can construct a functor $\hat{\mathscr{F}}$ from $\widehat{\operatorname{Art}}(k)$ to Set by taking projective limits over $\mathscr{F}(R/\mathfrak{m}^n)$. We say that $\mathscr{F}$ is pro-representable if there exists an $R \in \widehat{\operatorname{Art}}(k)$ and a $\xi \in \hat{\mathscr{F}}(R)$ such that the pair (R, ξ) represents the functor $\hat{\mathscr{F}}$.

There is also a weaker possibility, which we do not define precisely, called having a **hull**. This is somewhat similar to having a coarse moduli space, and Schlessinger shows [3, Prop. 2.9] that if a hull exists, it is unique up to isomorphism.

We say that $A' \to A$ is a **small extension** if it is surjective, and the kernel is a non-zero principal ideal killed by the maximal ideal of A'; equivalently if the kernel is a one-dimensional k-vector space. For instance, the map $k[x]/x^{n+1} \to k[x]/x^n$ is a small extension for all n.

We also note that if we are given morphisms $A' \to A$, $A'' \to A$ in $\operatorname{Art}(k)$, there is a natural map

$$\mathscr{F}(A' \times_A A'') \to \mathscr{F}(A') \times_{\mathscr{F}(A)} \mathscr{F}(A''). \tag{1.2.1}$$

Observe that this map is always a bijection if $\mathscr{F}$ is prorepresentable.

Schlessinger showed:

Theorem 1.3. *Consider the following conditions on a pre-deformation functor $\mathscr{F}$:*

(H1) *The map (1.2.1) is surjective whenever* $A'' \to A$ *is a small extension.*
(H2) *The map (1.2.1) is a bijection when* $A = k$*, and* $A'' = k[\epsilon]$*.*
(H3) *The tangent space of* $\mathscr{F}$ *is finite-dimensional over* k*.*
(H4) *The map (1.2.1) is a bijection whenever* $A'' = A'$ *and* $A' \to A$ *is a small extension.*

Then $\mathscr{F}$ *has a hull if and only if conditions (H1)-(H3) are satisfied, and is pro-representable if and only if (H1)-(H4) are satisfied.*

Of course, to make sense of this, one needs the tangent space of $\mathscr{F}$ to be a vector space over k. Schlessinger shows:

Lemma 1.4. *Suppose a pre-deformation functor* $\mathscr{F}$ *satisfies (H2). Then the tangent space of* $\mathscr{F}$ *naturally carries the structure of a* k*-vector space.*

In practice, all natural deformation problems seem to satisfy conditions (H1) and (H2). Problems with any reasonable finiteness hypotheses will also satisfy (H3), but (H3) will not be satisfied, for instance, by deformations of a non-proper scheme. Motivated by this, we now define:

Definition 1.5. A pre-deformation functor $\mathscr{F}$ is a **deformation functor** if it satisfies conditions (H1) and (H2).

Schlessinger's theorem gives a workable and satisfyingly general criterion for existence of hulls and for pro-representability. The only caveat is that the conditions may seem somewhat strange: for instance, the particular sorts of restrictions on the ring maps imposed in (H1), (H2), and (H4) may appear capricious, and the appearance of the fiber product of rings is somewhat odd in the context of moduli problems (or in algebraic geometry more generally).

We remark that even here, automorphisms arise naturally: Schlessinger comments that in "geometric situations", the obstruction to (H4) is determined by whether every infinitesimal automorphism extends over every small extension.

1.3. **Examples of tangent spaces and obstructions.** For a deformation functor $\mathscr{F}$, we have already seen that the tangent space of $\mathscr{F}$ is a naturally-defined vector space over k, and is frequently worth studying. It turns out that for many deformation functors $\mathscr{F}$, there is also some natural **obstruction space** for $\mathscr{F}$: specifically, there is some vector space V, such that for any small extension $A' \to A$, and any object $\eta \in \mathscr{F}(A)$, one obtains naturally an element $o_\eta \in V$, such that $o_\eta = 0$ if and only if there exists an object $\eta' \in \mathscr{F}(A')$ restricting to η; thus, o_η measures the obstruction to lifting η to an object over A'. If one has an obstruction space which is 0, the deformation functor will be unobstructed.

We briefly mention some examples of deformation problems, and their tangent and obstruction spaces.

Example 1.6. If $f : X \to Y$ is a morphism of smooth varieties, we can consider the deformations of f, with X and Y fixed: i.e., over $A \in \mathrm{Art}(k)$, we consider maps $f_A : X \times A \to Y \times A$ which recover f if we restrict to X and Y. In this case, it is not hard to see that the tangent space is given by $H^0(X, f^*T_Y)$, and one can also check that there is a natural obstruction space given by $H^1(X, f^*T_Y)$. Here T_Y denotes the tangent sheaf of Y.

Example 1.7. If X is a smooth variety, we can consider the deformations of X itself: for $A \in \mathrm{Art}(k)$, we consider schemes X_A, flat over A, with maps $X \to X_A$

inducing an isomorphism after base change to k. Here, we note that the infinitesimal automorphisms of X are given by $H^0(X, T_X)$, and one can show that if X is affine, every deformation of X over $k[\epsilon]$ is isomorphic to the trivial deformation $X \times \operatorname{Spec} k[\epsilon]$. It then follows that for any X, deformations over $k[\epsilon]$ are obtained by gluing together copies of the trivial deformation along infinitesimal automorphisms, and therefore the tangent space is given by $H^1(X, T_X)$. By a similar argument, one shows that obstructions lie naturally in $H^2(X, T_X)$.

Over and over, in deformation theory one finds that there is a sheaf (or complex) associated to a deformation problem, and the tangent space and obstruction space are given by H^0 and H^1, or H^1 and H^2, or hypercohomology $\mathbb{H}^1$ and $\mathbb{H}^2$ of the sheaf/complex. This arises in studying deformations of pointed covers of curves, in deformations of vector bundles with connection, and in deformations of subschemes of a given scheme.

Much of this is explained in Illusie's definitive work [2], but although this is very general, it does not naturally encompass the range of deformation problems which one runs across in practice, and it is also so technical that few people have read it. Our goal is to set up a framework in which one can prove, in an elementary and intuitive way, precise statements along the following lines:

"If a deformation problem has locally trivial deformations and obstructions, then its tangent space is given by H^1 of the sheaf of infinitesimal automorphisms, and obstructions lie in H^2 of the same sheaf."

2. Stacks

We next discuss motivation for and the definition of abstract stacks.

2.1. **Functors and sheaf conditions.**

Stacks mean different things to different people. There are two quite different points of view on stacks: abstract stacks, and algebraic stacks. Although algebraic stacks have generally arisen more frequently in algebraic geometry, our point of view will be to focus on abstract stacks, which can be thought of as generalizing a certain well-behaved class of functors.

As with deformation theory, abstract stacks may be motivated by considering moduli problems. Grothendieck's initial approach was to consider moduli spaces by describing a functor of points $\mathscr{F}$ from the category of schemes to the category of sets, and then to prove that in particularly good cases (such as the Grassmannian, the Picard scheme, and the Hilbert scheme), this functor is representable by a scheme X together with an object $\eta \in \mathscr{F}(X)$. Precisely, for any scheme T and any map $T \to X$, we obtain an object of $\mathscr{F}(T)$ by pulling back η, so there is a map $\operatorname{Mor}(T, X) \to \mathscr{F}(T)$, and (X, η) represents $\mathscr{F}$ if this map is a bijection for all T.

We note that being representable places some immediate restrictions on a functor $\mathscr{F}$. The restrictions we will focus on are **sheaf conditions**: for instance, if we fix X, and consider a scheme T with an open cover $\{U_i\}$, it is clear that the set $\operatorname{Mor}(T, X)$ is in bijection with the subset of $\prod_i \operatorname{Mor}(U_i, X)$ of tuples of φ_i such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. A more formal way of saying this is that $\operatorname{Mor}(-, X)$ is a sheaf of sets for the Zariski topology on the category of schemes. In fact, Grothendieck showed that the same statement remains true if we replace the Zariski open sets with étale morphisms.

Thus, from the point of view of considering potential representable functors, it makes sense to restrict to functors which form sheaves in the Zariski or even étale topologies; if we do this, we in some sense understand how to glue together local data to obtain global data. However, we immediately see that for many natural moduli problems, this is not a reasonable restriction, and the reason is the existence of automorphisms of the objects being parametrized. Specifically, let's consider the case of moduli of curves, and suppose a scheme T is covered by open sets U_1 and U_2. Suppose further we have families of curves $C_1 \to U_1$ and $C_2 \to U_2$ such that there exists an isomorphism

$$C_1|_{U_1 \cap U_2} \xrightarrow{\sim} C_2|_{U_1 \cap U_2}.$$

The sheaf condition would say that there should be a unique curve $C \to T$ such that $C|_{U_i} \cong C_i$ for $i = 1, 2$. However, if $C_1|_{U_1 \cap U_2}$ has non-trivial automorphisms, then the above isomorphism is not unique, and we know that we can glue C_1 and C_2 along different isomorphisms, and that if we do so, we will at least sometimes obtain non-isomorphic curves C over T.

The solution is in fact suggested by the problem: although the sheaf condition is not satisfied for isomorphism classes of curves, a sheaf-type condition is nonetheless satisfied, if we say that, roughly speaking, curves C over T correspond to curves C_i over T_i, for $i = 1, 2$, *together with the additional data* of an isomorphism on $U_1 \cap U_2$ as above.

2.2. **The definition.** The notion of a stack is thus motivated by the idea that for a large class of moduli problems, there is a natural sheaf-type condition which makes sense, and which is helpful in describing how to go from local data to global data, but which involves taking isomorphisms of objects into account. We therefore want to replace functors to sets by functors to **groupoids**, which can be thought of as categories in which every morphism is an isomorphism. The objects will be the objects of our moduli problem, and the morphisms the isomorphisms between objects. Although this is the intuition, there are technical problems in doing this, so we instead introduce the notion of categories fibered in groupoids: if C is our starting category (in our case, the category of schemes), instead of a functor from C to the category of groupoids, we will have some category $\mathscr{S}$ with a functor to C.

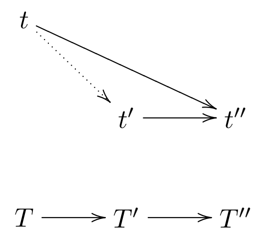
Definition 2.1. Given a functor $\mathscr{S} \to C$, and an object $T \in C$, we can form a category, called the **fiber** of $\mathscr{S} \to C$ over T, by taking the objects of $\mathscr{S}$ lying over T, and the morphisms of these objects which lie over the identity map $T \overset{\text{id}}{\to} T$.

Categories fibered in groupoids over C will roughly be functors $\mathscr{S} \to C$ such that every fiber is a groupoid. The actual definition we will use is the following:

Definition 2.2. Fix a category C. We say that a category S, together with a (contravariant) functor to C, is a **category fibered in groupoids** over C, if:

(i) ("pullbacks exist...") given a morphism $T \to T'$ in C, and an object t' in S over T', there exists t in S over T and a morphism $t \to t'$ over the given morphism $T \to T'$.

(ii) ("... and are unique up to unique isomorphism") Given a diagram with the solid arrows below:



the dotted arrow, making the triangle commute, exists and is unique.

Remark 2.3. Note that (ii) implies that the fiber category over any object is a groupoid, if we set $T = T' = T''$ with the identity morphism between them, and $t = t''$ with the identity morphism, then we see that an arbitrary morphism $t' \to t''$ in the fiber has an inverse. This justifies the terminology.

Also note that (ii) allows us to pull back morphisms as well as objects: if we have $T' \to T$ in C, objects s, t in $\mathscr{S}$ over C with a morphism $s \to t$ over id_T, and pullbacks s', t' to T', we obtain a unique pullback morphism $s' \to t'$ by applying (ii), to the morphisms $t' \to t$ and $s' \to t$, where the latter obtained by composing $s' \to s \to t$.

Finally, we note that as advertised, (ii) implies that any two pullbacks are canonically isomorphic (in the fiber category), if we set $T = T'$.

In our case, C will be the category of schemes, and each fiber of S over C will be the groupoid that we want to associate to the given scheme. We note that every category fibered in groupoids has an associated functor from C to Set, simply by taking the set of isomorphism classes of the groupoid associated to each scheme.

In this setting, we are now able to impose a sheaf condition which mimics the situation described above for curves, taking the isomorphism data into account. In order to do so, we need to have some notion of a topology on C; for intuition, one can picture simply the Zariski topology, but in fact we will typically want to consider the étale topology as defined by Grothendieck. We will assume in particular that C has fiber products. Before giving the definition, we make some observations, supposing we have a category $\mathscr{S}$ fibered in groupoids over C, and a topology on C.

Suppose we are given $T \in C$, and an open cover $\{U_i\} \to T$. For all i, j (we must allow $i = j$ for general topologies on C), we denote by $U_{i,j}$ the fiber product $U_i \times_T U_j$. For all i, j, k we similarly set $U_{i,j,k} := U_i \times_T U_j \times_T U_k$.

The first situation we want to consider involves pulling back morphisms along the open cover. Hence, we suppose we have two objects s, t in S over T, and fix pullbacks $\{s_i\}, \{t_i\}$ of s and t to the U_i, and pullbacks $\{s_{i,j}\}$, $\{s'_{i,j}\}$ and $\{t_{i,j}\}$, $\{t'_{i,j}\}$ of s_i, s_j, t_i, and t_j to $U_{i,j}$. Note that $s_{i,j}$ is canonically isomorphic to $s'_{i,j}$, since they are both pullbacks of s to $U_{i,j}$, and similarly for $t_{i,j}$ and $t'_{i,j}$. In particular, we can replace $s'_{i,j}$ and $t'_{i,j}$ by $s_{i,j}$ and $t_{i,j}$, so that we have a single choice of pullback for each $U_{i,j}$. In this situation, the remarks above on pulling back morphisms mean that we have a natural map

$$\mathrm{Mor}_{\mathrm{id}_T}(s,t) \to \left\{ \{\varphi_i \in \mathrm{Mor}_{\mathrm{id}\, U_i}(s_i, t_i)\}_i : \varphi_i|_{U_{i,j}} = \varphi_j|_{U_{i,j}} \forall i, j \right\}, \tag{2.3.1}$$

where $\mathrm{Mor}_\varphi(a, b)$ denotes morphisms between objects a and b, lying over a fixed morphism φ in C.

The second situation we want to consider is that of pulling back objects themselves along the open cover. Here, we don't have the luxury of specifying choices of pullbacks in advance, so everything we do should be taken to mean "up to canonical isomorphism" In any case, we have a natural map

$$\mathrm{Obj}_T \to \left\{ \begin{array}{c} (\{s_i \in \mathrm{Obj}_T\}_i, \{\varphi_{i,j} \in \mathrm{Mor}_{\mathrm{id}_T}(s_i|_{U_{i,j}}, s_j|_{U_{i,j}})\}_{i,j} : \\ \forall i,j,k, \varphi_{j,k}|_{U_{i,j,k}} \circ \varphi_{i,j}|_{U_{i,j,k}} = \varphi_{i,k}|_{U_{i,j,k}} \end{array} \right\}, \tag{2.3.2}$$

where Obj_T denotes objects of $\mathscr{S}$ over T.

Definition 2.4. A category fibered in groupoids S/C, given a topology on C, is called a **stack** (for the given topology) if the following sheaf conditions hold:

(i) ("morphisms form a sheaf") The map (2.3.1) is always a bijection.
(ii) ("objects form a 2-sheaf") The map (2.3.2) is always a bijection.

Remark 2.5. In fact, one checks that condition (i) implies that (2.3.2) is always injective, as compatible isomorphisms of the pullback objects on the U_i glue to give an isomorphism of two objects on T. Thus, (ii) may be stately equivalently (and with less confusion about being up to isomorphism) as a surjectivity statement, which is to say that descent is effective for objects of $\mathscr{S}$; this is how it is phrased in Deligne and Mumford [1, Def. 4.1].

We thus see that although the definition of a stack is abstract and somewhat intimidating, it is motivated quite directly by considering classical moduli problems, and is a direct generalization of the notion of functors which form sheaves. Indeed, one sees that if one has a stack in which every object has trivial automorphisms, then the sheaf-type conditions of the stack imply that the associated functor is a sheaf, since there is no ambiguity about gluing along isomorphisms.

3. A new framework

Our primary motivation is to place examples of deformation theory into a more coherent framework, which will ultimately mean a systematic approach to passing from local statements to global ones. We will do this in two steps: first, we will add a "second variable" to our deformation problem, so that instead of associating a set of deformations to every Artin ring A, we will associate a set of deformations to every pair (A, U), where A is an Artin ring, and U is a Zariski open subset of some fixed scheme X (here, X could be the scheme we are deforming, or a scheme with a vector bundle on it which we are deforming, or a scheme with a morphism to some Y, where we want to deform the morphism, etc.). Next, following the transition from functors to stacks, we will work with categories fibered over $\mathrm{Art}(k) \times \mathrm{Zar}(X)$, and impose some natural stack conditions. Together, this will allow us to accomplish two things: we will be able to systematically describe how to go from local to global in deformation problems, and we will also shed new light on Schlessinger's criteria.

Our main definition is the following:

Definition 3.1. A category $\mathscr{S}$ fibered in groupoids over $\mathrm{Art}(k) \times \mathrm{Zar}(X)$ is a **geometric deformation stack** if:

(i) for each $U \in \mathrm{Zar}(X)$, the groupoid over $\{k\} \times \{U\}$ is "trivial": i.e., there exists a unique morphism between any two objects;
(ii) for each $A \in \mathrm{Art}(k)$, if we restrict $\mathscr{S}$ to $\{A\} \times \mathrm{Zar}(X)$, we obtain a stack over $\mathrm{Zar}(X)$;

(iii) for each $U \in \mathrm{Zar}(X)$, if we restrict $\mathscr{S}$ to $\mathrm{Art}(k) \times \{U\}$, we obtain a stack over $\mathrm{Art}(k)$, where the diagrams which we consider to be "open covers" are of the form

$$\begin{array}{ccc} A' \otimes_A A'' & \longleftarrow & A' \\ \uparrow & & \uparrow \\ A'' & \longleftarrow & A \end{array}$$

with $A \to A' \times A''$ injective, $A \to A''$ surjective, and the image in A' of $\ker(A \to A'')$ closed under multiplication.

Remark 3.2. The condition that $A \to A' \times A''$ be injective is simply the condition that the "cover" be scheme-theoretically surjective, i.e., $\mathrm{Spec}\, A' \amalg \mathrm{Spec}\, A'' \twoheadrightarrow \mathrm{Spec}\, A$.

In fact, one checks that the diagrams which we use as "open covers" are completely equivalent to the diagrams that Schlessinger considers; we have simply rephrased them in a way more obviously compatible with standard stack axioms.

Given a geometric deformation stack, we have the naturally associated functor of isomorphism classes $\mathscr{F}_{\mathscr{S}} : \mathrm{Art}(k) \times \mathrm{Zar}(X) \to \mathrm{Set}$. It follows trivially from (i) above that for any $U \in \mathrm{Zar}(X)$, the functor $\mathscr{F}_{\mathscr{S}}(-, U)$ is a pre-deformation functor; in fact, condition (iii) will ensure that it is a deformation functor.

Remark 3.3. We expect that in all examples, it will be no harder to prove that a deformation problem is a geometric deformation stack than to show that it satisfies Schlessinger's conditions (H1) and (H2); indeed, we expect that checking condition (iii) will essentially be implicit in any proof of (H1) and (H2). Because condition (ii) is only stated in the Zariski topology, it should always be close to tautological.

We also have two natural sheaves on X: the sheaf $\mathscr{D}$ of germs of first-order deformations, and the sheaf $\mathscr{A}$ of infinitesimal automorphisms:

Definition 3.4. Let $\mathscr{S}$ be a geometric deformation stack: we define $\mathscr{D}$ to be the sheafification of the presheaf on X given by $\mathscr{F}_{\mathscr{S}}(k[\epsilon], -)$. For $U \in \mathrm{Zar}(X)$, denote by η_U the trivial deformation in $\mathscr{S}$ over $(k[\epsilon], U)$ (i.e., a pullback of an object over (k, U)). Then we define $\mathscr{A}$ to be the sheaf on X given by $\mathrm{Aut}(\eta_U)$.

Note that $\mathscr{A}$ is already a sheaf, by condition (ii) for a geometric deformation stack.

We already know that the tangent space of a deformation functor automatically inherits the structure of a k-vector space, and it turns out that, by a similar construction, under our hypotheses the space of infinitesimal automorphisms is naturally a sheaf not only of groups, but of abelian groups, with a k-vector space structure. We thus have that in fact $\mathscr{D}$ and $\mathscr{A}$ are sheaves of k-vector spaces on X.

We can summarize our results as follows:

Theorem 3.5. *Let $\mathscr{S}$ be a geometric deformation stack.*

Denote by $T_{\mathscr{S}}$ the tangent space of $\mathscr{F}_{\mathscr{S}}(-, X)$. Then we have an exact sequence of k-vector spaces

$$0 \to H^1(X, \mathscr{A}) \to T_{\mathscr{S}} \to H^0(X, \mathscr{D}) \to H^2(X, \mathscr{A}),$$

and if $\mathscr{F}_{\mathscr{S}}$ is locally unobstructed, we have successive obstructions for $\mathscr{F}_{\mathscr{S}}(-, X)$ lying in $H^1(X, \mathscr{D})$, and $H^2(X, \mathscr{A})/H^0(X, \mathscr{D})$.

Furthermore, $\mathscr{F}_{\mathscr{S}}(-,X)$ is a deformation functor. It satisfies (H4) if and only if every automorphism of an object in $\mathscr{S}$ extends over any small extension. If X is proper and $\mathscr{D},\mathscr{T}$ are coherent $\mathscr{O}_X$-modules, it also satisfies (H3), and hence has a hull R, and if $\mathscr{S}$ is locally unobstructed, we have:

$$h^0(X,\mathscr{D})+h^1(X,\mathscr{A})-h^1(X,\mathscr{D})-h^2(X,\mathscr{A})\leqslant \dim R\leqslant \dim T_{\mathscr{S}}.$$

In particular, if a geometric deformation stack $\mathscr{S}$ has locally trivial deformations and locally trivial obstructions, we have that $T_{\mathscr{S}}=H^1(X,\mathscr{A})$, and obstructions lie in $H^2(X,\mathscr{A})$. If instead $\mathscr{A}=0$ and $\mathscr{S}$ has locally trivial obstructions, then $T_{\mathscr{S}}=H^0(X,\mathscr{D})$, and obstructions lie in $H^1(X,\mathscr{D})$.

As mentioned, this is a report on ongoing work. The "ongoing" part, aside from the fact that nothing has been written out in full detail, consists mainly of the following two (possibly related) investigations:

First, there should be a well-behaved notion of a short exact sequence of geometric deformation stacks, which would correspond to situations such as

$$\operatorname{Def}(f)\to\operatorname{Def}(X,f)\to\operatorname{Def}(X),$$

where $f:X\to Y$ is a morphism. This should yield a canonical complex based on the first and last terms whose hypercohomology computes the tangent space and obstructions to the middle term. Perhaps, it will also give a simple criterion for checking that the middle term is a geometric deformation stack in terms of the first and last terms being geometric deformation stacks.

Second, the structure of the description of the tangent space and obstruction space in the theorem strongly suggests that, even in this generality, there is a natural two-term complex with kernel $\mathscr{A}$ and cokernel $\mathscr{D}$, whose hypercohomology computes the tangent space and obstruction space to the problem. If so, this would be extremely interesting, as it would give the existence of a natural obstruction space for any deformation problem which is locally unobstructed.

References

1. Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Institut Des Hautes Etudes Scientifiques Publications Mathematiques (1969), no. 36, 75–109.
2. Luc Illusie, *Complexe contangent et déformations. I*, Lecture Notes in Mathematics, no. 239, Springer-Verlag, 1971.
3. Michael Schlessinger, *Functors of Artin rings*, Transactions of the AMS **130** (1968), 208–222.