LIMIT LINEAR SERIES: AN UPDATED SURVEY

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Abstract. Linear series are fundamental to the study of algebraic curves, and the most powerful technique to date for studying linear series is the theory of limit linear series, a degeneration technique introduced by Eisenbud and Harris. After reviewing their original theory, we will discuss how an alternative point of view sheds new light both on the original construction, and on various generalizations.

1. Linear series

Linear series arise naturally from the study of maps of varieties into projective space. Let $X$ be a smooth proper curve over a field $k$. Then we define:

Definition 1.1. A $g_d^r$ on $X$ is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle of degree $d$ on $X$, and $V$ is an $(r+1)$-dimensional subspace of $H^0(X, \mathcal{L})$.

The idea of ramification will also be important:

Definition 1.2. Given a linear series $(\mathcal{L}, V)$ and a point $P \in X$, the vanishing sequence $a_0(P) < \cdots < a_r(P)$ of $(\mathcal{L}, V)$ at $P$ is defined to be the increasing sequence of orders of vanishing at $P$ of sections of $V$. The non-decreasing ramification sequence $\alpha_0(P), \ldots, \alpha_r(P)$ is defined by $\alpha_i(P) = a_i - i$.

Example 1.3. A few basic cases:

1. The pair $(\mathcal{L}, V)$ corresponds to a map to $\mathbb{P}^r$ when $a_0(P) = 0$ for all $P$. In this case, we say $(\mathcal{L}, V)$ is basepoint-free.
2. If $r = 1$, and $(\mathcal{L}, V)$ is basepoint-free, then $a_1(P)$ (or $\alpha_1(P) := a_1(P) - 1$, depending on convention) corresponds to the usual ramification index at $P$ of the map of curves.
3. If $r = 2$ and $(\mathcal{L}, V)$ defines a birational map onto its image, then a point with $a_1(P) > 1$ corresponds to a cusp-type singularity in the plane curve image of $X$, while a point with $a_1(P) = 1$ but $a_2(P) > 2$ corresponds to an inflection point.

The following question is then quite natural:

Question 1.4. Given a curve $X$, and $r, d$, does $X$ have a $g_d^r$? If so, what is the dimension of the space of $g_d^r$’s? What about if we also impose ramification conditions?

This question is answered for general curves by the famous Brill-Noether theorem, and its generalization due to Eisenbud and Harris:

Theorem 1.5. Given $g, r, d, n$, and nondecreasing sequences $\alpha^j := \alpha^j_0 \leq \cdots \leq \alpha^j_r$ of nonnegative integers bounded by $d - r$, let

$$\rho = (r + 1)(d - r) - rg - \sum_{j=1}^{n} \sum_{i=0}^{r} \alpha^j_i.$$

Suppose that one of the following holds:

1. $\text{char } k = 0$ or $d > \text{char } k$;
2. $n \leq 2$.

If $X$ together with $P_1, \ldots, P_n$ is a general (smooth) proper $n$-marked curve of genus $g$, the space of $g_d^r$’s on $X$ with ramification sequence at least $\alpha^j$ at every $P_j$ has dimension exactly $\rho$ if it is nonempty.
2. Limit linear series

Theorem 1.5 was proved via degeneration techniques, using the Eisenbud-Harris theory of limit linear series. We describe their theory and briefly sketch this proof. As further motivation, we mention in passing that Eisenbud and Harris gave a range of other applications, including proving the existence of a range of types of Weierstrass points, and proving that the moduli space of curves of genus at least 24 is of general type. For the latter argument, the key point is the explicit computation of the classes in the Picard group of $\mathcal{M}_g$ of some effective divisors defined in terms of Brill-Noether theory.

As the name suggests, the theory of limit linear series attempts to understand what happens to a linear series as a smooth curve degenerates to a singular one. Eisenbud and Harris considered the case of curves of compact type, of which we will focus on the simplest case: a curve $X_0$ consisting of two smooth components $Y$ and $Z$ glued at a single simple node $Q$. The idea here is relatively simple: imagine a family $\mathcal{X}$ of curves over the spectrum of a DVR, with regular total space, $X_0$ as the special fiber, and smooth generic fiber $X_\eta$.

then we want to think about how a $g^r_d$ on $X_\eta$ extends to $X_0$. First, any line bundle $\mathcal{L}$ of degree $d$ on $X_\eta$ can be extended over $X_0$, but not uniquely: $Y$ and $Z$ are divisors on $\mathcal{X}$, and twisting by them doesn’t change anything away from $X_\eta$. However, it turns out that this is the only ambiguity, and it follows that if we specify a pair of degrees $(i, d-i)$ for the restrictions to $Y$ and $Z$ there is a unique extension which we denote by $\mathcal{L}_{(i,d-i)}$. If we have a linear series $(\mathcal{L}, V)$ on $X_\eta$, then given a choice of the extension of $\mathcal{L}$, the space $V$ of global sections extends uniquely. Thus, we find that we have an infinite family of extensions $(\mathcal{L}_{(i,d-i)}, V_{(i,d-i)})$ of $(\mathcal{L}, V)$ to $X_0$.

Eisenbud and Harris consider the extensions $(\mathcal{L}_{(d,0)}(0), V_{(d,0)}(0))$ and $(\mathcal{L}_{(0,d)}(0), V_{(0,d)}(0))$ and observe that in this case, no information is lost by restricting to $Y$ and $Z$ respectively. Thus, from a $g^r_d$ on $X_\eta$ defined away from $X_0$, one obtains a pair $(\mathcal{L}^Y, V^Y)$ and $(\mathcal{L}^Z, V^Z)$ of $g^r_s$ on $Y$ and $Z$ respectively. One then wants to determine what additional condition best captures that they arose as limits of a $g^r_d$ on $X_\eta$. Their definition is as follows:

**Definition 2.1.** Let $(\mathcal{L}^Y, V^Y)$ and $(\mathcal{L}^Z, V^Z)$ be $g^r_s$ on $Y$ and $Z$ respectively. Then they constitute an (Eisenbud-Harris) limit $g^r_d$ on $X_0$ if they satisfy the following condition:

\[(2.1.1) \quad a^Y_i + a^Z_{r-i} \geq d, \quad \forall i : 0 \leq i \leq r,\]

where $a^Y_i$, $a^Z_i$ denote the vanishing sequences at $P$ of $(\mathcal{L}^Y, V^Y)$ and $(\mathcal{L}^Z, V^Z)$ respectively. If further (2.1.1) is an equality for all $i$, we say the limit $g^r_d$ is **refined**.

The same definition works for more complicated curves of compact type, working with a collection of $g^r_d$'s for each component, with the condition (2.1.1) imposed independently at each node.

Eisenbud and Harris showed that if $(\mathcal{L}^Y, V^Y)$ and $(\mathcal{L}^Z, V^Z)$ are obtained as the limit of a $g^r_d$ on $X_\eta$, then they satisfy (2.1.1). They also showed that in characteristic 0, after base change and blowing up the special fiber (introducing new rational components to $X_0$) we can arrange so that the resulting limit $g^r_d$ is refined. On the other hand, they showed that if the space of refined limit $g^r_d$s on $X_0$ has the expected dimension $\rho$, then every refined limit $g^r_d$ arises as the limit of $g^r_s$'s on $X_\eta$. This last result is proved by constructing a single scheme parametrizing $g^r_s$'s on $X_\eta$ and refined limit $g^r_d$'s on $X_0$. One then carries out a dimension count to prove the desired statement.

Given these results, to study limits of linear series on $X_\eta$, it more or less suffices to study linear series on $Y$ and $Z$ (which can be chosen to have smaller genus), at the price of considering also imposed ramification conditions. Thus, the context of Theorem 1.5 is the natural setting to apply the theory.

The first case of Theorem 1.5 can be proved via a relatively straightforward inductive argument using limit linear series. This proof encapsulates the power of the Eisenbud-Harris approach: limit linear series on the reducible curve can be described in terms of independent linear series on each component. However, it also points to the weakness of the approach: it is not at all obvious how to construct a moduli space in families as discussed above. Even the construction of Eisenbud and Harris, which only contains refined limit $g^r_s$'s on $X_0$, is difficult and technical, and it has remained open until now to produce a proper moduli space which contains not necessarily refined limit $g^r_s$ as well. The fact that the space constructed only contains refined limit $g^r_s$ means that we need an alternate construction to handle the second case of the theorem.
3. A New Construction

In my thesis, I introduced an alternate construction of limit linear series, which has recently been applied in joint work with Eduardo Esteves to fibers of Abel map for reducible curves, and in joint work with Montserrat Teixidor i Bigas to Brill-Noether theory for vector bundles of rank 2 with special determinant. Inspired by this construction, I have recently realized how to construct a proper space of (limit) linear series in families, so that fibers parametrize linear series for smooth curves, and (not necessarily refined) limit linear series for reducible curves. The first observation is that if we have a \( g^r_d \) on the smooth generic fiber \( X_0 \) of \( \mathcal{X} \) as above, and \((L^Y, V^Y), (L^Z, V^Z)\) is the resulting limit \( g^r_d \), if we consider the \( g^r_d \) on \( X_0 \) given by \((L^{i,d-i}, V_{(i,d-i)})\) for some \( i \) between 0 and \( d \), the sections of \( V_{(i,d-i)} \) are glued together from sections of \( V^Y \) vanishing to order at least \( d - i \) at the node \( Q \), and sections of \( V^Z \) vanishing to order at least \( i \) at \( Q \).

The next observation is that in fact the Eisenbud-Harris conditions (2.1.1) are equivalent to the condition that for every nonnegative degree pair \((i,d-i)\), it is possible to construct an \((r+1)\)-dimensional space of global sections of \( L^{i,d-i} \) by gluing together sections from \( V^Y \) and \( V^Z \). This is more or less expressible as a determinantal condition (there are subtleties because the pushforward need not commute with base change), and can be stated equivalently in terms of \( V_{(d,0)} \) and \( V_{(0,d)} \). The latter version can easily be applied in families for which some fibers are singular and some are smooth.

From this point of view, we see that one may think of Eisenbud-Harris limit linear series spaces as something of a hybrid between classical \( G^r_d \) and \( W^r_d \) spaces. It is not yet obvious how to describe an intrinsic functor represented by this construction, but this should not be too difficult to work out. More substantively, ideally one would be able to carry out the Eisenbud-Harris dimension count for the compactified space provided by this construction, This may well be non-trivial, but should be approachable. However, it is not necessary for most arguments: for instance, for the non-existence portion of the Brill-Noether theorem it is enough to know that the space is proper, while for the existence portion, it is enough to have the dimension lower bound on the refined locus, which certainly holds.

4. Curves Not of Compact Type

Developing a robust theory of limit linear series for (reducible) curves not of compact type has been a long-standing open problem. One hopes that such a theory would help compute cohomology classes of effective cycles on \( \overline{M}_g \) having codimension higher than 1. Also, the flexibility to degenerate to curves not of compact type may prove useful in dealing with remaining open questions on linear series on curves, such as the maximal rank conjecture. In any event, the first test of a robust theory would be to provide a new proof of the Brill-Noether via degeneration to a reducible curve not of compact type.

I have recently started to work with Eduardo Esteves on developing a theory of limit linear series for curves not of compact type. For concreteness, we will focus on the case that \( X_0 \) is a “banana curve,” with components \( Y \) and \( Z \) glued at nodes \( Q_1 \) and \( Q_2 \). Complications include that line bundles are no longer determined by their restrictions to components, that the effect of twisting by \( Y \) or \( Z \) depends on the family \( \mathcal{X} \), and that the moduli space of line bundles is no longer proper. However, these issues appear surmountable, so we will instead focus on the question of what is the right generalization of the Eisenbud-Harris conditions for limit linear series. As in the case of a single node, given a family of curves \( \mathcal{X} \) degenerating to \( X_0 \), there are infinitely many extensions to \( X_0 \) of line bundles on \( X_\eta \). Again, an extension is uniquely determined by its degrees on \( Y \) and \( Z \), but not every degree can be achieved: the parity is predetermined. Again for simplicity, let’s consider the case that \( d \) is even and \((d,0)\) (equivalently) \((0,d)\) can be achieved as degrees on \( Y \) and \( Z \). Then as in the Eisenbud-Harris case, from a family of \( g^r_d \)'s on smooth fibers we obtain a pair of \( g^r_d \) on \( Y \) and \( Z \) respectively. The natural question is what additional conditions this pair will satisfy, analogous to (2.1.1).

Here, the philosophy espoused above has been useful: the conditions should ensure that for any intermediate degree pair \((i,d-i)\) (with \( i \) even), we should have an \((r+1)\)-dimensional space of sections of the corresponding extended line bundle, obtained by gluing together sections of \( V^Y \) and \( V^Z \). Following through this idea, we are led to the following sequence giving orders of simultaneous vanishing at a collection of points:

**Definition 4.1.** Given points \( P_1, \ldots, P_m \) on a smooth curve \( X \), and \((L^r, V)\) a \( g^r_d \) on \( X \), the multivanishing sequence \( a_0 \leq a_1 \leq \ldots \leq a_r \) of \((L^r, V)\) at the \( P_i \) is the sequence in which the number of times \( j \) appears is
equal to the dimension of 

\[ V(\sum_{j=1}^{m} (P_j + \cdots + P_m)) / V(\sum_{j=1}^{m+1} (P_j + \cdots + P_m)), \]

where for an effective divisor \( D, V(-D) \) denotes \( V \cap \Gamma(kL(-D)). \)

Thus, the sequence is allowed to have up to \( m \) repetitions of any given number. The \( m = 1 \) case is simply the usual vanishing sequence at a point. It then turns out that the correct generalization of the Eisenbud-Harris condition (2.1.1) to the case of the banana curve is (still in the case that \( d \) is even and \( (d,0) \) can be achieved as an extension of \( L \)) given by

\[ a_Y^i + a_Z^{r-i} \geq \frac{d}{2}, \quad \forall i : 0 \leq i \leq r, \]

where the \( a_Y^i \) and \( a_Z^{r-i} \) are now the multivanishing sequences of \( (L_Y, \nu_Y) \) and \( (L_Z, \nu_Z) \) at \( P_1 \) and \( P_2 \). However, this condition is not enough: in indices where (4.1.1) achieves equality and a particular value shows up without repetition in the vanishing sequence, we also need to impose a gluing condition on the sections with this vanishing. Imposing both (4.1.1) and the gluing condition appears to produce the correct expected dimension, although some work remains to show that the gluing conditions are actually independent for sufficiently general curves.

Pleasantly, the background theory developed to work with the limit linear series construction from my thesis for the compact type case is sufficiently general that it can be used as-is to apply also to the case of curves with two components and more than one node, so it seems quite likely that we have not only produced a definition of limit linear series for such curves which gives the expected dimension, but that furthermore a new proof of the Brill-Noether theorem via these methods is within reach.

5. Multivanishing indices on smooth curves

Limit linear series for curves not of compact type motivates a systematic look at multivanishing sequences of linear series on smooth curves. But in fact, these sequences are quite natural to consider in and of themselves. Not only do they generalize vanishing sequences at a point, but they also contain many other classical concepts: for instance, for 2-dimensional linear series and pairs of points, the multivanishing sequence specifies whether a given pair of points maps to a node in the image, or to a bitangent. As a first direction for investigation, it is natural to look for a generalized Brill-Noether theorem.

The first observation is that, just as with imposing vanishing conditions at a point, imposing at least a certain multivanishing sequence at a collection of points is described by a relative Schubert condition inside the space of \( G_{g,r}^{d} \). This gives an expected codimension (which is slightly more complicated than the case of vanishing sequences, but can be described explicitly), and it is natural to wonder whether Theorem 1.5 generalizes further to this setting. The answer is that it does:

**Theorem 5.1.** Given \( g, r, d, n, m_1, \ldots, m_n \), and \( n \) nondecreasing sequences \( a^j := a^j_0 \leq \cdots \leq a^j_r \) of non-negative integers bounded by \( d \), with each entry of \( a^j \) repeating at most \( m_j \) times, suppose that one of the following holds:

1. \( \text{char } k = 0 \) or \( d > \text{char } k \);
2. \( n \leq 2 \).

Let \( X \) be a general curve of genus \( g \), and \( (P_i^j)_{1 \leq j \leq n, 1 \leq i \leq m_j} \) general points of \( X \). Then the space of \( G_{d}^{g} \)s on \( X \) with multivanishing sequence at least \( a^j \) at the \( P_i^j \) has the expected dimension, if it is nonempty.

The proof uses limit linear series in a way quite similar to that of Theorem 1.5, by placing all the imposed multivanishing onto genus-0 components, and arguing directly for the case of genus-0 and \( n = 2 \).