COUNTING CURVES ON TORIC SURFACES

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Abstract. A few years ago, Tzeng settled a remarkable conjecture of Goettsche on counting nodal curves on smooth surfaces, proving that the formulas are given by certain universal polynomials. At the same time, Ardila and Block used the tropical approach of Brugalle, Mihalkin and Fomin to count nodal curves on a certain class of (not necessarily smooth) toric surfaces, and obtained similar polynomiality behavior. In light of recent work of Block, Colley and Kennedy, we revisit the work of Ardila and Block, giving simpler proofs of stronger results, and in particular expressing the formulas in a manner compatible with Tzeng's theorem for smooth surfaces. In this way, we obtain explicit combinatorial formulas for the coefficients arising in Tzeng's theorem, and also give correction terms arising from singularities. Although the motivation and main theorems are expressed in terms of algebraic geometry, the work itself is largely combinatorial.

This is joint work with Fu Liu.

1. Background

The most classical form of Severi degree is the answer to the following question:

Question 1.1. Given d and δ, fix also $\binom{d+2}{2} - 1 - \delta$ general points P_i in $\mathbb{P}^2$. Then how many curves[1] of degree d having δ nodes pass through all the P_i?

This was completely answered (in the sense of giving elementary recursive formulas) by work of Ran, Kontsevich, and Caporaso-Harris in the 1990's, and then extended by Vakil to rational ruled surfaces.

During the same period, the same question was studied from a different perspective: first, Di Francesco and Itzykson conjectured that for a fixed δ, the Severi degrees considered in Question 1.1 are given by polynomials in d (for d sufficiently large). Vainsencher, Kleiman-Piene and Goettsche considered what happens for arbitrary smooth projective surfaces, where Question 1.1 is rephrased as follows:

Question 1.2. Let S be a smooth projective surface, $\delta \geqslant 0$, and $\mathscr{L}$ a sufficiently ample line bundle on S, set $n = |\mathscr{L}| - \delta$, and fix n general points P_i on S. Then how many curves are there in $|\mathscr{L}|$ having δ nodes and passing through all the P_i?

Based on having worked out the answer to Question 1.2 for small values of δ, they were led to the remarkable observation that, for fixed δ, not only is the answer to Question 1.2 given by polynomials for a fixed surface S, but there is a universal polynomial "in the surface S" which works for all surfaces! Goettsche made a precise conjecture on the generating function for these numbers, which we state a part of as follows.

Conjecture 1.3. *Denote the answer to Question 1.2 by $N^\delta(S, \mathscr{L})$. Let $\mathscr{K}$ be the canonical class on S, and c_2 the second Chern class of S. Then for fixed δ, there is a polynomial $T_\delta(w, x, y, z)$ such that*

$$N^\delta(S, \mathscr{L}) = T_\delta(\mathscr{L}^2, \mathscr{L} \cdot \mathscr{K}, \mathscr{K}^2, c_2).$$

Furthermore, if we form the generating function

$$\mathcal{N}(t) = \sum_{\delta \geqslant 0} T_\delta(w, x, y, z) t^\delta,$$

and set $\mathcal{Q}(t) := \log \mathcal{N}(t)$, then

$$\mathcal{Q}(t) = w f_1(t) + x f_2(t) + y f_3(t) + z f_4(t)$$

for some $f_1, \ldots, f_4 \in \mathbb{Q}[[t]]$.

[1] Classically, only irreducible curves were considered, but going back at least 15 years it was found useful to consider also reducible curves.

The last statement can be understood as follows: if we define $Q^\delta(S,\mathscr{L})$ to be the coefficient of t^δ in

$$\log\left(\sum_{\delta\geqslant 0} N^\delta(S,\mathscr{L})t^\delta\right),$$

then for $\mathscr{L}$ sufficiently ample relative to δ, we have that $Q^\delta(S,\mathscr{L})$ is given by a *linear* function in the variables $\mathscr{L}^2, \mathscr{L}\cdot\mathscr{K}, \mathscr{K}^2, c_2$. Goettsche's conjecture went further, giving a partial closed formula for the f_i in terms of quasimodular forms, but there is not yet a fully closed formula.

In 2009, Fomin-Mikhalkin proved the polynomiality conjecture for the case of $\mathbb{P}^2$, and then in 2010, it was proved for arbitrary smooth surfaces independently by Tzeng and by Kool-Shende-Thomas, with Tzeng's approach yielding a proof of Goettsche's full conjecture. Meanwhile, building on the work of Fomin-Mikhalkin (and earlier work of Brugalle-Mikhalkin), Ardila and Block studied the same question for a certain family of (not necessarily smooth) toric surfaces, and discovered similar polynomiality behavior.

Our work (joint with Fu Liu) revisits the work of Ardila and Block, rederiving their results in a simpler form, and exploring the comparison with the formula proved by Tzeng.

2. Curves on toric surfaces

For us, the basic setting will be a toric surface S with a very ample line bundle $\mathscr{L}$. The data of such a pair is encoded simply by a polygon Δ in the plane, having integer coordinates for its vertices (the surface itself is encoded by the normal fan to Δ, while the polygon determines also the extra data of the line bundle). For such a polygon Δ, denote by $Y(\Delta)$ the associated toric surface, and $Y(\Delta)$ the line bundle, and set

$$N^\delta(\Delta) := N^\delta(Y(\Delta),\mathscr{L}(\Delta)), \text{ and } Q^\delta(\Delta) := Q^\delta(Y(\Delta),\mathscr{L}(\Delta)).$$

We do not consider arbitrary polygons, but rather those satisfying an additional condition as follows:

Definition 2.1. A polygon Δ is ***h*-transverse** if all its normal vectors have (infinite or) integer slope.

This condition is odd from the point of view of toric geometry, since it is not at all invariant under isomorphism, but it comes up naturally in the work of Brugalle and Mikhalkin, who used tropical geometry to give explicit combinatorial formulas for $N^\delta(\Delta)$ when Δ is h-transverse (in fact, Brugalle-Mikhalkin considered irreducible curves, and it was Ardila and Block who reformulated their results to treat $N^\delta(\Delta)$).

Ardila and Block then carried out a detailed further combinatorial analysis in order to prove that $N^\delta(\Delta)$ is given by universal polynomials in their case as well. Specifically, they parametrized Δ by vector $\vec{c},\vec{d}$ corresponding to the slopes of the normals of the edges, and the (lattice) lengths of the edges respectively. Roughly stated, they proved the following:

Theorem 2.2 (Ardila-Block)**.** *If the vertices of Δ are sufficiently spread out relative to δ, then for fixed δ and $\vec{c}$, the numbers $N^\delta(\Delta)$ are given by a polynomial in $\vec{d}$.*

If also $\vec{c}$ is sufficiently spread out, then $N^\delta(\Delta)$ is given by a polynomial in $\vec{c}$ and $\vec{d}$.

Compared to Tzeng's/Kool-Shende-Thomas' theorem, the main advantage of the above is that it treats many singular surfaces. On the other hand, the universality is not nearly as strong: one has to fix the number of vertices of Δ, infinite slopes are treated separately, and the number of variables grows with the number of edges of Δ.

In order to state our theorem, we need a couple more definitions:

Definition 2.3. If v is a vertex of a lattice polygon Δ, we define $\det(v)$ to be $\det|w_1,w_2|$, where w_1 and w_2 are primitive integer normal vectors to the edges adjacent to v.

Then the singularites of $Y(\Delta)$ correspond precisely to vertices v of Δ with $\det(v) > 1$.

Definition 2.4. We say an h-transverse polygon Δ is **strongly** h-transverse if either there is a non-zero horizontal edge at the top of Δ, or the vertex v at the top has $\det(v)\in\{1,2\}$, and the same holds for the bottom of Δ.

It turns out that an h-transverse polygon Δ is strongly h-transverse if and only if $Y(\Delta)$ is Gorenstein, or equivalently, if and only if all the singularities of $Y(\Delta)$ are rational double points.

Our main result is then as follows.

Theorem 2.5. *Fix $\delta > 0$. Then there exist constants $A(\delta), B(\delta), C(\delta), D(\delta), E(\delta)$ and $E_i(\delta)$ for $i = 1, \dots, \delta - 1$ such that if Δ is a strongly h-transverse polygon with all edges having length at least δ, then*

$$Q^\delta(Y(\Delta), \mathscr{L}(\Delta)) = A(\delta) \cdot \mathscr{L}(\Delta)^2 + B(\delta) \cdot (\mathscr{L}(\Delta) \cdot \mathscr{K}) + C(\delta)\mathscr{K}^2 + D(\delta)\tilde{c}_2 + E(\delta)S + \sum_{i=1}^{\delta-1} E_i(\delta) S_i,$$

where $\mathscr{K}$ is the canonical line bundle on $Y(\Delta)$, S_i is the number of singularities of $Y(\Delta)$ of Milnor number i, $\tilde{c}_2 = c_2(Y(\Delta)) + \sum_{i \geqslant 1} i S_i$, and $S = \sum_{i \geqslant 1}(i+1)S_i$.

Moreover, our coefficients for $\mathscr{L}^2, \mathscr{L} \cdot \mathscr{K}, \mathscr{K}^2$ and $\tilde{c}_2$ must agree with those of Tzeng's theorem.

Note that $\tilde{c}_2 = c_2$ in the nonsingular case. In general, $\tilde{c}_2$ is the second Chern class of a minimal desingularization of $Y(\Delta)$.

In fact, our calculation treats arbitrary h-transverse polygons, with additional correction terms for non-Gorenstein singularities. However, we have yet to find an equally satisfactory statement in this case. We also give an adjusted version of the Goettsche-Yau-Zaslow formula. The original formula is as follows.

Theorem 2.6 (The Goettsche-Yau-Zaslow formula)**.** *There exist universal power series $B_1(q)$ and $B_2(q)$ such that*

$$\sum_{\delta \geqslant 0} T_\delta(\mathscr{L}^2, \mathscr{L} \cdot \mathscr{K}, \mathscr{K}^2, c_2)(DG_2(\tau))^\delta = \frac{(DG_2(\tau)/q)^{\chi(\mathscr{L})} B_1(q)^{\mathscr{K}^2} B_2(q)^{\mathscr{L} \cdot \mathscr{K}}}{(\Delta(\tau) D^2 G_2(\tau)/q^2)^{\chi(\mathscr{O}_S)/2}},$$

where $q = e^{2\pi i \tau}$, $G_2(\tau)$ is the second Eisenstein series $-\frac{1}{24} + \sum_{n>0} \left(\sum_{d|n} d \right) q^n$, $D = q \frac{d}{dq}$ and $\Delta(\tau) = q \prod_{k>0}(1-q^k)^{24}$.

Our adjusted version is then the following.

Theorem 2.7.

$$\sum_{\delta \geqslant 0} T_\delta(\mathscr{L}^2, \mathscr{L} \cdot \mathscr{K}, \mathscr{K}^2, \tilde{c}_2; S, S_1, S_2, \dots)(DG_2(\tau))^\delta = \frac{(DG_2(\tau)/q)^{\chi(\mathscr{L})} B_1(q)^{\mathscr{K}^2} B_2(q)^{\mathscr{L} \cdot \mathscr{K}}}{(\Delta(\tau) D^2 G_2(\tau)/q^2)^{\chi(\mathscr{O}_S)/2}} \mathscr{P}(q)^{-S} \prod_{i \geq 2} \mathscr{P}\left(q^i\right)^{S_{i-1}},$$

where $\mathscr{P}(x) = \sum_{n \geq 0} p(n) x^n$ is the generating function for partitions.

3. Combinatorics of curve counting

Both our work and the previous work of Ardila and Block built on the work of Brugalle and Mikhalkin, which combinatorialized Severi degrees for (some) toric surfaces using tropical curves. We briefly summarize the relevant results. The definitions given here have been refined further by Block, Colley and Kennedy, as well as Liu.

Definition 3.1. A **long-edge graph** G is a graph (V, E) with a weight function ρ satisfying the following conditions:

a) The vertex set $V = \mathbb{N} = \{0, 1, 2, \dots\}$, and the edge set E is finite.
b) Multiple edges are allowed, but loops are not.
c) The weight function $\rho : E \to \mathbb{Z}_{>0}$ assigns a positive integer to each edge.
d) There is no "short edge," i.e., there is no edge connecting i and $i+1$ with weight 1.

We define the **multiplicity of** G to be

$$\mu(G) = \prod_{e \in E} (\rho(e))^2,$$

and the **cogenus of** G to be

$$\delta(G) = \sum_{e \in E} \left(l(e)\rho(e) - 1 \right),$$

where for any $e = \{i, j\} \in E$ with $i < j$, we define $l(e) = j - i$.

Suppose that $\boldsymbol{\beta} = (\beta_0, \dots, \beta_M)$ is a nonnegative integer sequence, and G is a long-edge graph whose edges are all supported between 0 and M. Then one defines an integer $P^s_{\boldsymbol{\beta}}(G)$, given explicitly in terms of $\boldsymbol{\beta}$ and G, which can be used to calculate Severi degrees. We first have to give one more definition.

Definition 3.2. Given a lattice polygon Δ, a **reordering** Δ' of Δ is a (non-convex) polygon obtained by repeatedly swapping adjacent slopes on the left or right of Δ.

Denote by $\boldsymbol{\beta}(\Delta')$ the sequence of widths of Δ'. Set $\delta(\Delta')$ to be the sum of the terms of the sequence $\boldsymbol{\beta}(\Delta) - \boldsymbol{\beta}(\Delta')$.

That is, we can swap adjacent slopes on the left, and adjacent slopes on the right, but not slopes which are adjacent to a horizontal edge, or a pair of adjacent slopes crossing from left to right. In particular, the height of Δ' must be the same as that of Δ.

Theorem 3.3 (Brugalle-Mikhalkin, Ardila-Block)**.** *For any h-transverse polygon Δ and any $\delta \geq 0$, the Severi degree $N^{\Delta,\delta}$ is given by*

$$N^\delta(\Delta) = \sum_{\Delta'} \sum_{G} \mu(G) P^s_{\boldsymbol{\beta}(\Delta')}(G). \tag{3.3.1}$$

where the first summation is over all reorderings Δ' of Δ satisfying $\delta(\Delta') \leq \delta$, and the second summation is over all long-edge graphs G with $\delta(G) = \delta - \delta(\Delta')$.

Thus, the problem of computing $N^\delta(\Delta)$ is completely combinatorialized. The argument uses the correspondence theorem for tropical curves to reduce the count to the tropical case, and then considers a point configuration in which the marked points are very far apart from one another vertically in order to simplify the tropical curves which can occur, and combinatorialize the resulting counting problem.

The counting problem can be broken down further as follows: if G can be partitioned into subgraphs G_1, G_2 with no overlap in the support of their edges, then

$$P^s_{\boldsymbol{\beta}}(G) = P^s_{\boldsymbol{\beta}}(G_1) P^s_{\boldsymbol{\beta}}(G_2).$$

It is thus enough to study long-edge graphs for which no such decomposition exists, which we call **shifted templates** (a template is such a graph with at least one edge adjacent to 0).

Consistent with the general approach of considering the logarithm of the generating function, in place of $P^s_{\boldsymbol{\beta}}(G)$ we will consider a function $\Phi^s_{\boldsymbol{\beta}}(G)$ which is essentially a formal logarithm of $P^s_{\boldsymbol{\beta}}(G)$: specifically, we have

$$Q^{\beta,\delta} = \sum_{G:\delta(G)=\delta} \mu(G) \Phi^s_{\boldsymbol{\beta}}(G).$$

This turns out to have much simpler formal properties.

Lemma 3.4 (Block-Colley-Kennedy, Liu)**.** *Suppose G is not a shifted template. Then $\Phi^s_{\boldsymbol{\beta}}(G) = 0$.*

In particular,

$$Q^{\boldsymbol{\beta},\delta} = \sum_{\Gamma} \mu(\Gamma) \sum_{k \geqslant 0} \Phi^s_{\boldsymbol{\beta}}(\Gamma_{(k)}),$$

where the outer sum runs over templates Γ with $\delta(\Gamma) = \delta$, and $\Gamma_{(k)}$ denotes the kth shift of Γ.

The key background result for us is the following.

Theorem 3.5 (Liu)**.** *Let G be a long-edge graph. Then there exists a multivariable linear function $\Phi(G, \boldsymbol{\alpha})$ such that for any $\boldsymbol{\beta}$ with G being strictly $\boldsymbol{\beta}$-semiallowable, we have*

$$\Phi^s_{\boldsymbol{\beta}}(G) = \Phi(G, \boldsymbol{\beta}).$$

In the above, strict $\boldsymbol{\beta}$-semiallowability is an explicit condition which is in particular always satisfied for $\boldsymbol{\beta}$ sufficiently large.

4. Techniques

Given the above-described ingredients, our proof proceeds in a few steps.

- We first give a formula for the sum of $\Phi^s_{\boldsymbol{\beta}}(G)$ as G varies over shifts of a fixed template, and $\boldsymbol{\beta} = \boldsymbol{\beta}(\Delta)$ does not involve any reorderings. A useful trick here is to simultaneously consider Γ and its flip, which leads to substantial cancellation in formulas.
- We next analyze what happens when we have to sum over reorderings. Under mild hypotheses, we are able to obtain an explicit formula for $Q^{\boldsymbol{\beta}(\Delta'),\delta}$ in terms of $Q^{\boldsymbol{\beta}(\Delta),\delta}$ for a fixed Δ', and further manipulations lead to formulas for $Q^{\Delta,\delta}$ in terms of $Q^{\boldsymbol{\beta}(\Delta),\delta}$.
- To simplify and strengthen our formulas, we use the trick of considering Δ_1 and Δ_2, where Δ_2 is obtained from Δ_1 by rotating 90 degrees. Since we necessarily have

$$N^\delta(\Delta_1) = N^\delta(\Delta_2),$$

  we can obtain very simple proofs of certain helpful but non-obvious combinatorial facts.

A persistent difficulty is to keep track of the condition of strict $\boldsymbol{\beta}$-semiallowability and its effect on the relevant formulas. A careful accounting leads to our threshold value of all edges of Δ having length at least δ, which agrees with the condition obtained separately by Block in the special case of $\mathbb{P}^2$ (this has since been improved by Kleiman and Shende to $\delta/2$ in the same special case).

Finally, our formulas for the numbers $A(\delta), B(\delta), C(\delta), D(\delta), E(\delta)$ and $E_i(\delta)$ involve understanding the coefficients of the linear functions $\Phi(\Gamma, \boldsymbol{\beta})$ as Γ ranges over templates of cogenus δ. The expressions in terms of these linear coefficients are relatively simple, but calculating the coefficients themselves is not so easy. Thus, further refinements would likely involve improving our understanding of the functions $\Phi(\Gamma, \boldsymbol{\beta})$.