NOTES ON DIMENSION THEORY OF SCHEMES

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In this expository note, we discuss various aspects of the theory of dimension of schemes, in particular focusing on which hypotheses are necessary in order to insure good behavior, and what sort of pathologies can otherwise arise. References are frequently exercises in disguise (or not in disguise, as the case may be). In particular, because the intended applications are geometric, we will state everything in terms of schemes rather than rings. The primary situation which will guide our discussion will be that of a scheme X of finite type over a Noetherian base scheme B, and manipulation of various closed subschemes Z of X. Dimension refers always to Krull dimension, and recall that in this situation, the codimension of Z in X is defined to be the maximal length of a chain of integral closed subschemes containing Z inside X.

1. The Basic Issues

We will work throughout with Noetherian schemes, to maintain some degree of good behavior.

We begin by discussing three types of pathologies: first, that dimension need not be finite; second, that maximal chains of subschemes can have different lengths depending on where they start and finish; and third, that even specifying the ends of such a chain may not determine its length.

An example of Nagata (see [1, Exercise 9.6, p. 230]) gives a Noetherian (indeed, regular) affine scheme which has infinite dimension, so there is no reasonably general hypothesis which gives finite-dimensionality. Of course, it is true that:

Theorem 1.1. *Noetherian local schemes are finite-dimensional* [1, Cor. 10.7].

and

Theorem 1.2. *Schemes of finite type over a field are finite-dimensional* [1, Easy from Cor. 10.13 a].

Typically, although the Noetherian hypothesis is reasonable for dimension-theory arguments, one does not want to restrict to either local schemes or schemes over a field (let alone those of finite type). One solution would be to add a finite-dimension hypothesis, but as we will see, a cleaner approach is to work with dimensions of local rings, at least on the base B. Another frequently-effective alternative is to substitute codimension for dimension whenever possible; indeed, we will see that these are often essentially equivalent, although circumstance will often dictate that one is more appropriate than the other.

Slightly more subtle issues of dimension behavior involve how lengths of chain of subschemes behave with certain restrictions. If a scheme has more than one component, they can have unrelated dimensions. Another way of saying this is that in this case, the length of a maximal chain of integral closed subschemes depends on which is the largest subscheme in the chain. This situation is easily avoided, by restricting to irreducible schemes. Luckily, this turns out to be not unreasonably burdensome in practice. However, it turns out that the length of maximal chains can also depend on which closed point is the smallest subscheme of the chain. We can rephrase this by saying that the following is not automatically satisfied (see, e.g. [1, Exer. 13.1]).

$$\dim \mathscr{O}_{X,x} = \dim X \text{ for all } x \in X \text{ closed.} \tag{1.2.1}$$

It is difficult to impose this condition via extra hypotheses, but working with local rings or codimensions often provides an effective work-around.

More generally, one might very reasonably ask for the following.

$$\operatorname{codim} Z + \dim Z = \dim X. \tag{1.2.2}$$

Once again, this turns out to be true only very rarely, as the case that Z is a closed point simply recovers (1.2.1).

However, we observe that if chains between a fixed pair of integral subschemes have well-behaved lengths, we will have that (1.2.1) implies (1.2.2); this is not automatic, but is a mild hypothesis known as the catenary condition, and is described in the next section.

In effect, we see that dimension is very poorly-behaved globally. On the other hand, as localization won't affect the chains of integral schemes containing a given one, we see:

Proposition 1.3. *Given Z in X an irreducible closed subscheme, and $x \in Z$ any closed point,* $\operatorname{codim} Z$ *is the same in X and in* $\operatorname{Spec} \mathscr{O}_{X,x}$.

As a result, as suggested earlier, one can frequently substitute codimension for dimension in arguments. We will see in concrete situations how this will often suffice in practice, but for the moment we have (with apologies to Körner):

Slogan 1.4. *Dimension is a local property; codimension is a global property.*

We briefly conclude with a set of axioms with taken together, characterize the dimension of Noetherian schemes. They are:

Let X be a Noetherian scheme.

(i) $\dim X = \max\{\dim \mathscr{O}_{X,x}\}_{x \in X}$, and further, $\forall x \in X$, $\dim \mathscr{O}_{X,x} = \dim \hat{\mathscr{O}}_{X,x}$, where the latter denotes the complete local ring.
(ii) Let $Z \subset X$ be any closed subscheme containing X_{red}. Then $\dim Z = \dim X$.
(iii) Let $Y \to X$ be a finite, scheme-theoretically surjective morphism. Then $\dim Y = \dim X$.
(iv) If R is a complete discrete valuation ring, then $\dim R[[x_2, \dots, x_n]] = n$.

Note that the last condition contains as a special case that if k is a field, $\dim k[[x_1, \dots, x_n]] = n$. See [1, §8.1] for further discussion of these axioms.

2. Catenary Schemes

The technical condition which turns out to be necessary for codimension and dimension to behave manageably is relatively mild, but not automatic.

Definition 2.1. We say that a scheme X is **catenary** if, given any $Z_1 \subset Z_2$ integral closed subschemes of X, any two maximal chains of integral closed subschemes of

X containing Z_1 and contained in Z_2 have the same length. We say that X is **universally catenary** if X is Noetherian and every scheme Y of finite type over X is catenary.

It is easy to check the following:

Proposition 2.2. *Let X be catenary. Then:*

(i) *if $Z_1 \subset Z_2$ are irreducible closed subschemes of X, then* $\operatorname{codim}_X Z_2 + \operatorname{codim}_{Z_2} Z_1 = \operatorname{codim}_X Z_1$.
(ii) *if X is local, all irreducible closed subschemes Z of X satisfy (1.2.2);*
(iii) *if X satisfies (1.2.1), then all irreducible closed subschemes Z of X satisfy (1.2.2).*

We also have that universally catenary is a rather weak condition:

Theorem 2.3. *Let X be of finite type over a locally Cohen-Macaulay scheme. Then X is universally catenary.*

Proof. See [3, Thm. 17.9]. □

In particular, any scheme of finite type over a field or any other regular scheme is universally catenary.

3. Equidimensionality

If a scheme X is finite-dimensional, and further (1.2.2) holds for all closed subschemes of X, we say that X is **equidimensional**. As we have seen, this need not be the case for catenary schemes, nor even for regular, Noetherian schemes over a base field. However, we do have the following positive result:

Proposition 3.1. *Let X be an irreducible scheme of finite type over a field. Then X is equidimensional.*

Proof. See [5, Prop. 5.2.1] □

More generally, as remarked earlier, it is easy to check the following:

Proposition 3.2. *Let X be an irreducible catenary scheme satisfying (1.2.1). Then X is equidimensional.*

However, it is difficult to formulate more general conditions under which equidimensionality holds.

Example 3.3. We observe that it is not the case that an irreducible scheme of finite type over a DVR is equidimensional. Indeed, if we consider $X = \operatorname{Spec} R[x]$ for R a DVR with uniformizer t, and take any curve Z in X not having points in the closed fiber, such as $xt = 1$, that Z consists of a single closed point of codimension 1 in X.

We conclude with one case in which it is often useful to work with schemes which are not irreducible. While irreducible fibers of morphisms of finite type are necessarily equidimensional by the above, the main purpose of the following is to conclude equidimensionality for possibly reducible fibers in flat families with irreducible total space. As a bonus, we obtain a generalization of Proposition 3.1.

Theorem 3.4. *Let X, Y be irreducible schemes, and $f : X \to Y$ an open morphism of finite type. Then for all $y \in Y$, the fiber $f^{-1}y$ is equidimensional, of dimension equal to $\dim f^{-1}\eta$, where η is the generic point of Y.*

If further f is closed and Y is equidimensional and universally catenary, then X is equidimensional.

Proof. For the first assertion see [6, Cor. 14.2.2]. For the second part, since X will be catenary, we have to show that for any closed point $x \in X$, $\dim \mathscr{O}_{X,x} = \dim X$. First note that because f is closed, any closed point $x \in X$ necessarily lies in a closed fiber. Thus if $y = f(x)$, by equidimensionality of Y we have $\dim \mathscr{O}_{Y,y} = \dim Y$. Further, by our first assertion, and since x is closed, we have $\dim \mathscr{O}_{f^{-1}y,x} = \dim f^{-1}\eta$. Thus by [6, Thm. 14.2.1] or Theorem 4.1 below, $\dim \mathscr{O}_{X,x} = \dim Y + \dim f^{-1}\eta$, and since this is independent of x, we find $\dim \mathscr{O}_{X,x} = \dim X$, as desired. □

4. Basic operations

Finally, we review certain positive results which allow one to compute (co)dimensions in practice.

Theorem 4.1. *Let $f : X \to Y$ be a morphism and $x \in X$ any point. Then*

$$\dim \mathscr{O}_{X,x} \leqslant \dim \mathscr{O}_{Y,f(x)} + \dim \mathscr{O}_{f^{-1}(f(x)),x},$$

with equality holding if f is open.

Proof. See [1, Thm. 10.10] for the inequality, and equality in the case that f is flat; the more general statement is [6, Thm. 14.2.1]. □

Theorem 4.2. *Let $f : X \to Y$ be a morphism of irreducible schemes, with Y regular, and Z an irreducible closed subscheme of Y. Then any irreducible component Z' of $f^{-1}(Z)$ has codimension in X at most equal to the codimension of Z in Y.*

Proof. In fact, Hochster shows [2, Thm. 7.1] that this result follows from the case that $X \to Y$ is a closed immersion, which is the following (deep) theorem of Serre. □

Theorem 4.3. *Let Z_1, Z_2 be irreducible subschemes of an irreducible regular scheme X, and Z_3 any irreducible component of $Z_1 \cap Z_2$. Then $\operatorname{codim} Z_3 \leqslant \operatorname{codim} Z_1 + \operatorname{codim} Z_2$.*

Proof. See [4, Thm. V.3] for the statement in terms of local rings. □

Remark 4.4. The above results are much easier to prove in the case that the base scheme is, instead of regular, smooth over a field, by reducing to the diagonal and using that the diagonal is a local complete intersection.

Example 4.5. Note that the regularity is a vital hypothesis for the above to be true: if X is a cone over a quadric surface, and Z_1, Z_2 are cones corresponding to two distinct lines in one ruling of the surface, then each has codimension 1, but their intersection is only at the cone point, which has codimension 3.

Theorem 4.6. *Let $f : X \to Y$ be a morphism locally of finite type. For any $n \geqslant 0$, the locus $F_n(X) := \{x \in X : \dim_x(f^{-1}(f(x))) \geqslant n\}$ is closed in X, where $\dim_x$ denotes the dimension of any components meeting x.*

In particular, if also f is closed, the locus $F_n(Y) := \{y \in Y : \dim f^{-1}(y) \geqslant n\}$ is closed.

Proof. See [7, Thm. 13.1.3], [7, Cor. 13.1.5]. □

5. A sample argument

We conclude with a sample argument putting together these techniques to make a classical dimension-theoretic argument in the context of rather general base schemes. This argument is a key part of the machinery of Eisenbud and Harris' theory of limit linear series, and it is in two parts: we suppose we have a regular base B, and we construct a scheme X of finite type over B. Conceptually, we first want to show that every component of X has dimension at least $\rho+\dim B$ for some $\rho \geqslant 0$. We then want to conclude that if for some $b \in B$, the fiber of X over b has dimension exactly ρ, then X dominates B, and if further X is proper over B, every non-empty fiber of X over B has dimension ρ.

X is constructed (non-canonically) as the intersection of a number of closed subschemes in a scheme Y smooth over B of relative dimension d. These closed subschemes have codimension adding up to at most $d-\rho$. We can easily conclude a bound on the codimension of X in Y, but because X is canonical and of interest, and Y is non-canonical and used only to construct X, we would rather have a statement involving only X and B. For this, we turn to dimensions of local rings:

Theorem 5.1. *Let B be a regular scheme, and Y be smooth and finite type over B, of relative dimension d. Suppose that $Z_1, \dots, Z_n$ are irreducible closed subschemes of Y of codimension $d_1, \dots, d_n$, with $\sum_i d_i \leqslant d-\rho$ for some $\rho \geqslant 0$, and let X be $Z_1 \cap \cdots \cap Z_n$. Then for any $x \in X$ with image $b \in B$, such that x is closed in the fiber of X over b, we have*

$$\dim \mathscr{O}_{X,x} \geqslant \dim \mathscr{O}_{B,b} + \rho.$$

Proof. Since Y is smooth over a regular scheme, it is regular. By inductively applying Theorem 4.3, we find that any component of X has codimension at most $d-\rho$ in Y. Because Y is catenary, we conclude that $\dim \mathscr{O}_{Y,x} - \dim \mathscr{O}_{X,x} \leqslant d-\rho$, and by Theorem 4.1, we have $\dim \mathscr{O}_{Y,x} = \dim \mathscr{O}_{B,b} + d$, giving the desired result. □

Observe that as long as X is not supported only over non-closed points of B (which in particular cannot occur if B is of finite type over a field), we have as a consequence that $\dim X \geqslant \dim B + \rho$.

In addition, it immediately follows that every fiber of X over B has dimension at least ρ. We now wish to explore the consequences of a single fiber of X over B having dimension precisely ρ. We have:

Theorem 5.2. *Let B be a regular scheme, and X an irreducible scheme of finite type over B, such that for any $x \in X$ with image $b \in B$, with x closed in the fiber of X over b, we have*

$$\dim \mathscr{O}_{X,x} \geqslant \dim \mathscr{O}_{B,b} + \rho.$$

Suppose further that for some $b \in B$, we have $\dim f^{-1}(b) = \rho$. Then X dominates B.

If further X is proper over B, then every fiber of X over some open neighborhood of $b \in B$ has dimension ρ.

Proof. Let B' be the scheme-theoretic image of X inside B, and x a closed point of $f^{-1}(b)$. By Theorem 4.1, we have $\dim \mathscr{O}_{X,x} \leqslant \dim \mathscr{O}_{B',b} + \dim f^{-1}(b) = \dim \mathscr{O}_{B',b} +$

ρ. But by hypothesis, $\dim \mathscr{O}_{X,x} \geqslant \dim \mathscr{O}_{B,b} + \rho$, so we have $\dim \mathscr{O}_{B,b} = \dim \mathscr{O}_{B',b}$, and X dominates B.

By the same argument, X cannot have any fibers of dimension smaller than ρ, so by Theorem 4.6, we find that if X is proper over B, the fiber dimension is ρ in a neighborhood of b. □

References

1. David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag, 1995.
2. Melvin Hochster, *Big cohen-macaulay modules and algebras and embeddability in rings of witt vectors*, Queen's Papers on Pure and Applied Math **42** (1975), 106–195.
3. Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
4. J. P. Serre, *Algèbre locale. multiplicités*, Lecture Notes in Mathematics, no. 11, Springer-Verlag, 1965.
5. Alexander Grothendieck with Jean Dieudonné, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, seconde partie*, vol. 24, Publications mathématiques de l'I.H.É.S., no. 2, Institut des Hautes Études Scientifiques, 1965.
6. ———, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, troisiéme partie*, vol. 28, Publications mathématiques de l'I.H.É.S., no. 2, Institut des Hautes Études Scientifiques, 1966.
7. ———, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, quatriéme partie*, vol. 32, Publications mathématiques de l'I.H.É.S., no. 2, Institut des Hautes Études Scientifiques, 1967.