Pure-cycle Hurwitz factorizations and multi-noded rooted trees

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CMS winter meeting, Vancouver, British Columbia
December 4, 2010

This is joint work with Rosena R.X. Du.
PART I:

Definitions and Backgrounds
Hurwitz’s problem

Given integers $d$ and $r$, and $r$ partitions $\lambda^1, \ldots, \lambda^r \vdash d$, a Hurwitz factorization of type $(d, r, (\lambda^1, \ldots, \lambda^r))$ is an $r$-tuple $(\sigma_1, \ldots, \sigma_r)$ satisfying the following conditions:

1. $\sigma_i \in S_d$ has cycle type (or is in the conjugacy class) $\lambda^i$, for every $i$;
2. $\sigma_1 \cdot \cdots \cdot \sigma_r = 1$;
3. $M := \langle \sigma_1, \ldots, \sigma_r \rangle$ is a transitive subgroup of $S_d$. 
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**Question:** What is the Hurwitz number $h(d, r, (\lambda^1, \ldots, \lambda^r))$?

This question originally arises from geometry: Hurwitz number counts the number of degree-$d$ covers of the projective line with $r$ branch points where the monodromy over the $i$th branch point has cycle type $\lambda^i$. 