LIMIT LINEAR SERIES AND THE MAXIMAL RANK CONJECTURE

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Abstract. The maximal rank conjecture addresses the degrees of equations cutting out suitably general curves in projective spaces. We describe an approach to this conjecture involving degenerating to a chain of genus-1 curves, and using ideas from limit linear series. More speculatively, we will describe the prospects for applications to related problems such as the strong maximal rank conjecture. This is joint work with Fu Liu, Montserrat Teixidor i Bigas, and Naizhen Zhang.

1. The maximal rank conjecture

The classical Brill-Noether theorem states that if we are given $g, r, d \geqslant 0$, a general curve X of genus g carries a linear series $(\mathscr{L}, V)$ of rank r and degree d (equivalently, a nondegenerate morphism to $\mathbb{P}^r$ of degree d) if and only if the quantity

$$\rho := g - (r+1)(r+g-d)$$

is nonnegative. Eisenbud and Harris proved that (at least in characteristic 0) when $r \geqslant 3$, a general such linear series on X will define an imbedding of X in $\mathbb{P}^r$. One of the most basic questions one might then ask is: what are the degrees of the equations defining X? More precisely, we have the following question:

Question 1.1. In the above setting, for each $m \geqslant 2$, what is the dimension of the space of homogeneous polynomials of degree m vanishing on the image of X?

Stated this way, the question is about the kernel of the natural restriction map

$$\Gamma(\mathbb{P}^r, \mathscr{O}(m)) \to \Gamma(X, \mathscr{L}^{\otimes m}). \tag{1}$$

Equivalently, if the imbedding is given by a linear series $(\mathscr{L}, V)$ as above, then we have canonically

$$\Gamma(\mathbb{P}^r, \mathscr{O}(m)) = \operatorname{Sym}^m V,$$

so we can rewrite (1) as a multiplication map on global sections:

$$\operatorname{Sym}^m V \to \Gamma(X, \mathscr{L}^{\otimes m}). \tag{2}$$

Either way, the dimension of the source space is $\binom{r+m}{m}$, and it is a consequence of the Gieseker-Petri theorem that the dimension of the target space is $md + 1 - g$. The *Maximal Rank Conjecture* states that the kernel of this map is as small as possible, or equivalently, that the rank of this map is as large as possible.

Conjecture 1.2. *The rank of the restriction map* (1) *is*

$$\min\{\binom{r+m}{m}, md + 1 - g\},$$

so the answer to Question 1.1 is

$$\max\{0, \binom{r+m}{m} - (md+1-g)\}.$$

At least in part, this conjecture goes back to work of Noether in the late 1800's, and of Severi in the early 1900's, but it was stated explicitly by Harris in 1982, and has received considerable attention since then. It has been worked on by many people, including Voisin, Farkas, Teixidor i Bigas, Ballico and Ellia, and most recently Larson and Jensen and Payne, and partial results toward the conjecture have led to applications:

- to the surjectivity of the Wahl map;
- to the construction of families of effective divisors on the moduli space of curves;
- and to Brill-Noether theory for higher-rank vector bundles.

The conjecture is now known:

- when $r = 3$;
- when $d \geqslant r + g$;
- when $m = 2$;
- when d is sufficiently large relative to r and m;
- in various ranges of cases for $m = 3$.

An advantage of the perspective of (2) is that it makes sense for arbitrary linear series. In this context, Aprodu and Farkas posed a *Strong Maximal Rank Conjecture*, which we state vaguely as follows:

Conjecture 1.3. *The codimension of the locus of linear series on which the rank of the multiplication map* (2) *is submaximal is the given by the usual determinantal codimension (in particular, the locus is empty when the expected codimension is greater than ρ).*

This conjecture is wide open even for $m = 2$ and small ρ, and would have important consequences for construction of effective divisors on $\mathscr{M}_g$. Work of Farkas and Ortega also connects it to higher-rank Brill-Noether theory.

2. An approach via limit linear series

Nearly all of the known results on the maximal rank conjecture were proved by degenerations, but other than the tropical approach of Jensen and Payne, this has typically involved imbedded degenerations, which are less well suited to approaching the strong maximal rank conjecture.

Until now, no one had systematically approached the maximal rank conjecture using what is normally the most powerful tool for analyzing linear series via degeneration: the Eisenbud-Harris theory of limit linear series. The typical approach for using limit linear series to study (injectivity of) multiplication maps involves supposing one had an element of the kernel, and studying its specialization to a reducible curve X_0, which gives a tuple of sections, one on each component of X_0. One then derives a contradiction by looking at the orders of vanishing which must occur at the nodes. This was the approach taken by Eisenbud and Harris in their simple proof of the Gieseker-Petri theorem, and also by Teixidor i Bigas, Farkas and Popa in proving some special cases of the maximal rank conjecture in the $m = 2$ case. However, this approach does not appear to yield a systematic approach to the conjecture, even in the $m = 2$ case.

In joint work with Fu Liu, Montserrat Teixidor i Bigas, and Naizhen Zhang, we take a different approach to using limit linear series to study the maximal rank conjecture. Instead of working with hypothetical elements of the kernel and restrictions to components of X_0, we work with global sections of line bundles of different multidegrees on X_0, and maps from one multidegree into another (here multidegree is simply the collection of degrees of the restrictions to each component of X_0). The setup is as follows:

Situation 2.1. Suppose we have a one-parameter family of curves $\pi : X \to B$, in which all fibers are smooth, except for one fiber X_0 over $b_0 \in B$. Suppose further that X_0 is a chain of genus-1 curves, and the total space of X is regular. Set $U = X \smallsetminus X_0$.

In this situation, any line bundle on U (or even on base changes thereof) extends to a line bundle on all of X, and line bundles on X_0 are determined by their restrictions to the components. Now, if Z is a component of X_0, then it is a Cartier divisor on X. If we are given a line bundle $\mathscr{L}$ on X, we see that $\mathscr{L}(Z)$ agrees with $\mathscr{L}$ on U, but is different on X_0: in fact, if Y is the union of the other components of X_0, then $\mathscr{L}(Z)|_Y = \mathscr{L}|_Y(Z \cap Y)$, but $\mathscr{L}(Z)|_Z \cong \mathscr{L}|_Z(-Z \cap Y)$. In particular, twisting by Z changes the multidegree of $\mathscr{L}$ (i.e., the tuple of degrees of restrictions to the components of X_0).

It is not hard to verify that given a line bundle $\mathscr{L}_U$ on U of degree d, and a multidegree w on X_0 of total degree d, there is a unique extension of $\mathscr{L}_U$ to a line bundle $\widetilde{\mathscr{L}_w}$ on X such that $\mathscr{L}_w := \widetilde{\mathscr{L}_w}|_{X_0}$ has multidegree w. Moreover, any two such extensions are related by twisting by components of X_0 as above. Now, for any linear series $(\mathscr{L}_U, V_U)$ on U, and a choice of multidegree w on X_0, we get a unique extension $\widetilde{\mathscr{L}_w}$ of $\mathscr{L}_U$ to X, and then a unique extension

$$\widetilde{V}_w = V_U \cap \Gamma(\mathscr{L}_w) \subseteq \Gamma(\mathscr{L}_U)$$

of V_U to X. Restricting to X_0 thus gives a collection of linear series $(\mathscr{L}_w, V_w)$ on X_0, one for each multidegree, related by maps induced by twisting by components of X_0. Thus, these maps are always injections on some components and uniformly zero on others.

For our approach to the maximal rank conjecture, we have the following

Key Observation: apply the above construction to the image W_U of the m-fold multiplication map. Thus, if we fix w of total degree md, we obtain a limit W_w with dimension equal to the (generic) rank of W_U. Moreover, for any $w_1, \ldots, w_m$, if we take the composed map

$$V_{w_1} \otimes \cdots \otimes V_{w_m} \to \Gamma(\mathscr{L}^{\otimes m}_{\sum_i w_i}) \to \Gamma(\mathscr{L}^{\otimes m}_w),$$

the image will be contained in W_w.

Thus, we can prove that the generic rank of W_U is at least n by varying the w_i and using the above to construct a collection of vectors in W_w with an n-dimensional span.

In order to make this work, we need to have a good understanding of global sections of the spaces V_w occurring in different multidegrees w. We also need to incorporate the Eisenbud-Harris theory of limit linear series, which we briefly recall. This theory revolves around the observation that if we choose a component Z of X_0, and consider the multidegree w_Z which is d on Z and 0 on other components, then we can restrict further from X_0 to Z to obtain a linear series $(\mathscr{L}^Z, V^Z) := (\mathscr{L}_{w_Z}|_Z, V_{w_Z}|_Z)$. They then make the following definition:

Definition 2.2. A **limit linear series** of rank r and degree d on X_0 consists of a tuple of linear series $(\mathscr{L}^Z, V^Z)$ on each component Z of X_0 (of the same rank and degree), satisfying the following compatibility conditions at nodes: if Z, Z' are two components meeting at a node P, and $a_0^{Z,P}, \dots, a_r^{Z,P}$ and $a_0^{Z',P}, \dots, a_r^{Z',P}$ are the vanishing sequences at P of $(\mathscr{L}^Z, V^Z)$ and $(\mathscr{L}^{Z'}, V^{Z'})$ respectively, then

$$a_j^{Z,P} + a_{r-j}^{Z',P} \geqslant d$$

for $j = 0, \dots, r$.

Furthermore, a limit linear series as above is said to be **refined** if the above inequalities are all equalities.

3. An elementary criterion

We use the above ideas (together with an interesting analysis of the behavior of certain families of linear series on twice-marked genus-1 curves) to develop a completely elementary criterion for proving cases of the maximal rank conjecture, which we summarize (very vaguely) as follows.

Given (g, r, d, m), we describe a purely combinatorial process of looking at vanishing orders at nodes to associate a table of integers to a choice of limit linear series on X_0 and a choice of multidegree. We also describe a numerical process for iteratively erasing rows of the constructed table.

Theorem 3.1. *(Liu-O-Teixidor-Zhang) If there exists a choice of limit linear series and multidegree such that the aforementioned process erases all rows of the constructed table, then the maximal rank conjecture holds for the case (g, r, d, m).*

Under a mild additional hypothesis, we also conclude that the locus of chains of genus-1 curves in $\overline{\mathscr{M}}_g$ is not contained in the closure of the locus in $\mathscr{M}_g$ where the maximal rank condition fails for (g, r, d, m).

Using Theorem 3.1, we are then able to prove many cases of the maximal rank conjecture. Specifically, we have the following:

Theorem 3.2. *(Liu-O-Teixidor-Zhang) Given g, r, d with $r \geqslant 3$, $r + g > d$, and $\rho \geqslant 0$, the Maximal Rank Conjecture holds in the following circumstances:*

- (i) *when $m = 2$;*
- (ii) *when $m = 3$, and either $r = 3$ with $g \geqslant 7$, $r = 4$ with $g \geqslant 16$, or $r = 5$ with $g \geqslant 26$;*
- (iii) *when $g \geqslant (r + 1)\left((m + 1)^{r-1} - r\right)$;*
- (iv) *when $m \geqslant 3$, and either $r + g - d = 1$ with $2r - 3 \geqslant \rho + 1$, or $r + g - d = 2$ with $r \geqslant 4$ and $2r - 3 \geqslant \rho + 2$.*

Moreover, in each of the above cases, a general chain of g genus-1 curves is not in the closure of the locus in $\mathscr{M}_g$ for which the maximal rank condition fails.

Empirically, the cases which appear hardest are the ones where $\binom{r+m}{m}$ is equal to (or very close to) $md+1-g$. The first and second statements of the theorem are the most substantive, insofar as they treat cases of this type. The third and fourth statements are based on some rather simple observations which are intended more to illustrate the variety of ways one can apply Theorem 3.1 than to provide any potentially systematic approach to the conjecture.

Most of the above cases are already known, although we believe that some of the $m = 3$ cases are new. Examples for $m = 3$, $r = 6$ get large very quickly, but it is possible that to handle the hardest cases, we will need to make use of restrictions on the direction of approach to the chosen chain of genus-1 curves (this could be necessary if for instance in the given case every chain of genus-1 does in fact lie in the closure of the locus for which the maximal rank condition fails). Our approach does allow us to make use of such restrictions, in fact, in two different ways.

Currently, we are focused on adapting this approach to the strong maximal rank conjecture, and related statements on other multiplication maps. In principle, our general approach should apply. However, there is one crucial foundational difficulty, involving developing a good understanding about what we can say about global sections in different multidegrees if all we know is the Eisenbud-Harris limit linear series. Once this is addressed, the elementary arguments in order to apply (the appropriate analogue of) Theorem 3.1 are also much more complicated, but at least they appear promising in examples we have considered so far.