FOUNDATIONS OF LIMIT LINEAR SERIES: AN UPDATE

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Abstract. Linear series are fundamental to the study of algebraic curves, and the most powerful technique to date for studying linear series is the theory of limit linear series, a degeneration technique introduced by Eisenbud and Harris. After reviewing their original theory, we will discuss how an alternative point of view sheds new light both on the original definition, and on generalizations to higher-rank vector bundles and to curves not of compact type.

1. Linear series

Linear series arise naturally from the study of maps of varieties into projective space. Let $X$ be a smooth proper curve over a field $k$. Then we define:

Definition 1.1. A $g^r_d$ on $X$ is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle of degree $d$ on $X$, and $V$ is an $(r + 1)$-dimensional subspace of $H^0(X, \mathcal{L})$.

The idea of ramification will also be important:

Definition 1.2. Given a linear series $(\mathcal{L}, V)$ and a point $P \in X$, the vanishing sequence $a_0(P) < \cdots < a_r(P)$ of $(\mathcal{L}, V)$ at $P$ is defined to be the increasing sequence of orders of vanishing at $P$ of sections of $V$.

The non-decreasing ramification sequence $\alpha_0(P), \ldots, \alpha_r(P)$ is defined by $\alpha_i(P) = a_i - i$.

Example 1.3. A few basic cases:

1. The pair $(\mathcal{L}, V)$ corresponds to a map to $\mathbb{P}^r$ when $a_0(P) = 0$ for all $P$. In this case, we say $(\mathcal{L}, V)$ is basepoint-free.
2. If $r = 1$, and $(\mathcal{L}, V)$ is basepoint-free, then $a_1(P)$ (or $\alpha_1(P) := a_1(P) - 1$, depending on convention) corresponds to the usual ramification index at $P$ of the map of curves.
3. If $r = 2$ and $(\mathcal{L}, V)$ defines a birational map onto its image, then a point with $a_1(P) > 1$ corresponds to a cusp-type singularity in the plane curve image of $X$, while a point with $a_1(P) = 1$ but $a_2(P) > 2$ corresponds to an inflection point.

The following question is then quite natural:

Question 1.4. Given a curve $X$, and $r, d$, does $X$ have a $g^r_d$? If so, what is the dimension of the space of $g^r_d$'s? What about if we also impose ramification conditions?

This question is answered for general curves by the famous Brill-Noether theorem, and its generalization due to Eisenbud and Harris:

Theorem 1.5. Given $g, r, d, n$, and $n$ nondecreasing sequences $\alpha^j := \alpha^j_0 \leq \cdots \leq \alpha^j_r$ of nonnegative integers bounded by $d - r$, let

$$\rho = (r + 1)(d - r) - rg - \sum_{j=1}^n \sum_{i=0}^r \alpha^j_i.$$ 

Suppose that one of the following holds:

1. $\text{char } k = 0$ or $d < \text{char } k$;
2. $n \leq 2$.

If $X$ together with $P_1, \ldots, P_n$ is a general (smooth) proper $n$-marked curve of genus $g$, the space of $g^r_d$'s on $X$ with ramification sequence at least $\alpha^j$ at every $P_j$ has dimension exactly $\rho$ if it is nonempty.
2. Limit Linear Series

Theorem 1.5 was proved via degeneration techniques, using the Eisenbud-Harris theory of limit linear series. We describe their theory and briefly sketch this proof. As further motivation, we mention that Eisenbud and Harris gave a range of other applications, including proving the existence of a range of types of Weierstrass points, and proving that the moduli space of curves of genus at least 24 is of general type. For the latter argument, the key point is the explicit computation of the classes in the Picard group of \(\mathcal{M}_g\) of some effective divisors defined in terms of Brill-Noether theory.

As the name suggests, the theory of limit linear series attempts to understand what happens to a linear series as a smooth curve degenerates to a singular one. Eisenbud and Harris considered the case of curves of compact type, of which we will focus on the simplest case: a curve \(X_0\) consisting of two smooth components \(Y\) and \(Z\) glued at a single simple node \(Q\). The idea here is relatively simple: imagine a family \(\mathcal{X}\) of curves over the spectrum of a DVR, with regular total space, \(X_0\) as the special fiber, and smooth generic fiber \(X_\eta\); then we want to think about how a \(g^r_d\) on \(X_\eta\) extends to \(X_0\). First, any line bundle \(\mathcal{L}\) of degree \(d\) on \(X_\eta\) can be extended over \(X_0\), but not uniquely: \(Y\) and \(Z\) are divisors on \(\mathcal{X}\), and twisting by them doesn’t change anything away from \(X_\eta\). However, it turns out that this is the only ambiguity, and it follows that if we specify a pair of degrees \((i, d-i)\) for the restrictions to \(Y\) and \(Z\) there is a unique extension which we denote by \(\mathcal{L}_{(i,d-i)}\). If we have a linear series \((\mathcal{L}, V)\) on \(X_\eta\), then given a choice of the extension of \(\mathcal{L}\), the space \(V\) of global sections extends uniquely. Thus, we find that we have an infinite family of extensions \((\mathcal{L}_{(i,d-i)}, V_{(i,d-i)})\) of \((\mathcal{L}, V)\) to \(X_0\).

Eisenbud and Harris consider the extensions \((\mathcal{L}_{(d,0)}, V_{(d,0)})\) and \((\mathcal{L}_{(0,d)}, V_{(0,d)})\) and observe that in this case, no information is lost by restricting to \(Y\) and \(Z\) respectively. Thus, from a \(g^r_d\) on \(X_\eta\) defined away from \(X_0\), one obtains a pair \((\mathcal{L}^Y, V^Y)\) and \((\mathcal{L}^Z, V^Z)\) of \(g^r_d\)s on \(Y\) and \(Z\) respectively. One then wants to determine what additional condition best captures that they arose as limits of a \(g^r_d\) on \(X_\eta\). Their definition is as follows:

**Definition 2.1.** Let \((\mathcal{L}^Y, V^Y)\) and \((\mathcal{L}^Z, V^Z)\) be \(g^r_d\)s on \(Y\) and \(Z\) respectively. Then they constitute an (Eisenbud-Harris) limit \(g^r_d\) on \(X_0\) if they satisfy the following condition:

\[
(2.1.1) \quad a^Y_i + a^Z_{d-i} \geq d, \quad \forall i: 0 \leq i \leq r,
\]

where \(a^Y_i, a^Z_i\) denote the vanishing sequences at \(P\) of \((\mathcal{L}^Y, V^Y)\) and \((\mathcal{L}^Z, V^Z)\) respectively. If further (2.1.1) is an equality for all \(i\), we say the limit \(g^r_d\) is refined.

The same definition works for more complicated curves of compact type, working with a collection of \(g^r_d\)s for each component, with the condition (2.1.1) imposed independently at each node.

Eisenbud and Harris showed that if \((\mathcal{L}^Y, V^Y)\) and \((\mathcal{L}^Z, V^Z)\) are obtained as the limit of a \(g^r_d\) on \(X_\eta\), then they satisfy (2.1.1). They also showed that in characteristic 0, after base change and blowing up the special fiber (introducing new rational components to \(X_0\)) we can arrange so that the resulting limit \(g^r_d\) is refined. On the other hand, they showed that if the space of refined limit \(g^r_d\)s on \(X_0\) has the expected dimension \(\rho\), then every refined limit \(g^r_d\) arises as the limit of \(g^r_d\)s on \(X_\eta\). This last result is proved by constructing a single scheme parametrizing \(g^r_d\)s on \(X_\eta\) and refined limit \(g^r_d\)s on \(X_0\). One then carries out a dimension count to prove the desired statement.

Given these results, to study limits of linear series on \(X_\eta\), it more or less suffices to study linear series on \(Y\) and \(Z\) (which can be chosen to have smaller genus), at the price of considering also imposed ramification conditions. Thus, the context of Theorem 1.5 is the natural setting to apply the theory.

The first case of Theorem 1.5 can be proved via a relatively straightforward inductive argument using limit linear series. This proof encapsulates the power of the Eisenbud-Harris approach: limit linear series on the reducible curve can be described in terms of independent linear series on each component. However, it also points to the weakness of the approach: it is not at all obvious how to construct a moduli space in families as discussed above. Even the construction of Eisenbud and Harris, which only contains refined limit \(g^r_d\)s on \(X_0\), is difficult and technical, and it has remained open until now to produce a proper moduli space which contains not necessarily refined limit \(g^r_d\)s as well. The fact that the space constructed only contains refined limit \(g^r_d\)s means that we need an alternate construction to handle the second case of the theorem.
In my thesis, I introduced an alternate construction of limit linear series, which has recently been applied in joint work with Eduardo Esteves to fibers of Abel map for reducible curves, and in joint work with Montserrat Teixidor i Bigas to Brill-Noether theory for vector bundles of rank 2 with special determinant. Inspired by this construction, I have recently realized how to construct a proper space of (limit) linear series in families, so that fibers parametrize linear series for smooth curves, and (not necessarily refined) limit linear series for reducible curves. This construction also works for higher rank, although does not produce a proper space in this context since the moduli spaces of vector bundles are no longer proper. The first observation is that if we have a $g^r_d$ on the smooth generic fiber $X_\eta$ of $\mathcal{X}$ as above, and $(\mathcal{L}^Y, V^Y), (\mathcal{L}^Z, V^Z)$ is the resulting limit $g^r_d$, if we consider the $g^r_d$ on $X_0$ given by $(\mathcal{L}^{(i,d-i)}, V^{(i,d-i)})$ for some $i$ between 0 and $d$, the sections of $V^{(i,d-i)}$ are glued together from sections of $V^Y$ vanishing to order at least $d-i$ at the node $Q$, and sections of $V^Z$ vanishing to order at least $i$ at $Q$.

The next observation is that in fact the Eisenbud-Harris conditions (2.1.1) are equivalent to the condition that for every nonnegative degree pair $(i, d-i)$, it is possible to construct an $(r+1)$-dimensional space of global sections of $\mathcal{L}(i, d-i)$ by gluing together sections from $V^Y$ and $V^Z$. To make sense of this in smoothing families, we can also work equivalently with $V_{(d,0)}$ and $V_{(0,d)}$; the idea is that we are looking for the locus on which $\mathcal{L}^{(i,d-i)}$ has an $(r+1)$-dimensional space of global sections mapping into both $V_{(d,0)}$ and $V_{(0,d)}$, or equivalently, the locus on which the natural map

$$H^0(X_0, \mathcal{L}^{(i,d-i)}) \to H^0(X_0, \mathcal{L}^{(d,0)})/V_{(d,0)} \oplus H^0(X_0, \mathcal{L}^{(0,d)})/V_{(0,d)}$$

has kernel of dimension at least $r+1$. Now, this wants to be a determinantal locus, but it isn’t quite, due to the pushforwards involved and the fact that pushforward need not commute with base change.

The solution is to develop a generalized notion of determinantal locus involving pushforwards of coherent sheaves flat over a base, which turns out to be relatively straightforward, although it leads to what seems to be new understanding even of classical determinantal ideals. As motivated by the above, the perspective we wish to take is that a determinantal locus should be thought of not as the locus where the rank is at most a given value, but rather the locus on which the kernel is at least a certain size. We then have the following lemma (proof due largely to David Eisenbud):

**Lemma 3.1.** Let $X$ be a scheme, and for $i = 1, 2$ let $\mathcal{E}_i$ be locally free sheaves of rank $r_i$, $\mathcal{F}_i$ locally free sheaves of finite rank, and $f_i : \mathcal{E}_i \to \mathcal{F}_i$ morphisms. Suppose that on all closed subschemes $Z \subseteq X$, the kernel of $f_1|_Z$ is isomorphic to the kernel of $f_2|_Z$. Then for any $n$, the determinantal ideals of $(r_i - n)$ minors of $f_i$ are equal to one another.

The proof of the lemma proceeds in steps: first, we may mod out by one determinantal ideal to reduce to the statement that we cannot have one ideal equal to zero with the other nonzero. Then further modding out allows us to reduce to the case that $X$ is 0-dimensional and Gorenstein, where duality lets us invoke standard independence results on Fitting ideals.

We thus obtain a generalization of determinantal loci to pushforwards (note that the classical case is recovered as the special case that the morphism is the identity), and the construction generalizes rather trivially to allow modding out the target by subbundles on $S$, as we need to do for our above description of limit linear series. Thus, we have a functorial, proper construction of Eisenbud-Harris limit linear series which works in families. From this point of view, we see that one may think of Eisenbud-Harris limit linear series spaces as something of a hybrid between classical $G^r_d$ and $W^r_d$ spaces.

4. CURVES NOT OF COMPACT TYPE

Developing a robust theory of limit linear series for (reducible) curves not of compact type has been a long-standing open problem. One hopes that such a theory would help compute cohomology classes of effective cycles on $\overline{M}_g$ having codimension higher than 1. Also, the flexibility to degenerate to curves not of compact type may prove useful in dealing with remaining open questions on linear series on curves, such as the maximal rank conjecture. In any event, the first test of a robust theory would be to provide a new proof of the Brill-Noether via degeneration to a reducible curve not of compact type.

I have recently started to work with Eduardo Esteves on developing a theory of limit linear series for curves not of compact type. For concreteness, we will focus on the case that $X_0$ is a curve with two components
Y and Z glued at nodes $Q_1,\ldots,Q_m$. Complications include that line bundles are no longer determined by their restrictions to components, that the effect of twisting by Y or Z depends on the family $X$, and that the moduli space of line bundles is no longer proper. However, these issues appear surmountable, so we will instead focus on the question of what is the right generalization of the Eisenbud-Harris conditions for limit linear series. As in the case of a single node, given a family of curves $X$ degenerating to $X_0$, there are infinitely many extensions to $X_0$ of line bundles on $X_1$. Again, an extension is uniquely determined by its degrees on $Y$ and $Z$, but not every degree can be achieved: the parity is predetermined. Again for simplicity, let’s consider the case that $d$ is a multiple of $m$ and $(d,0)$ (equivalently) $(0,d)$ can be achieved as degrees on $Y$ and $Z$. Then as in the Eisenbud-Harris case, from a family of $g^r_d$s on smooth fibers we obtain a pair of $g^r_d$s $(\mathcal{L}^Y, V^Y)$ and $(\mathcal{L}^Z, V^Z)$ on $Y$ and $Z$ respectively. The natural question is what additional conditions this pair will satisfy, analogous to (2.1.1).

Here, the philosophy espoused above has been useful: the conditions should ensure that for any intermediate degree pair $(d - m, m)$, we should have an $(r+1)$-dimensional space of sections of the corresponding extended line bundle, obtained by gluing together sections of $V^Y$ and $V^Z$. Following through this idea, we are led to the following sequence giving orders of simultaneous vanishing at a collection of points:

**Definition 4.1.** Given points $P_1,\ldots,P_m$ on a smooth curve $X$, and $(\mathcal{L}, V)$ a $g^r_d$ on $X$, the multivanishing sequence $a_0 \leq a_1 \leq \ldots \leq a_r$ of $(\mathcal{L}, V)$ at the $P_i$ is the sequence in which the number of times $jm$ appears is equal to the dimension of

$$V(−j(P_1 + \cdots + P_m))/V(−(j + 1)(P_1 + \cdots + P_m)),$$

where for an effective divisor $D$, $V(−D)$ denotes $V \cap \Gamma(\mathcal{L}(−D))$.

Thus, the sequence is allowed to have up to $m$ repetitions of any given number. The $m=1$ case is simply the usual vanishing sequence at a point. It then turns out that the correct generalization of the Eisenbud-Harris condition (2.1.1) to the case of the banana curve is (still in the case that $d$ is even and $(d,0)$ can be achieved as an extension of $\mathcal{L}$) given by

$$(4.1.1) \quad a_i^Y + a_{r-i}^Z \geq d, \quad \forall i: 0 \leq i \leq r,$$

where the $a_i^Y$ and $a_i^Z$ are now the multivanishing sequences of $(\mathcal{L}^Y, V^Y)$ and $(\mathcal{L}^Z, V^Z)$ at $P_1$ and $P_2$. However, this condition is not enough: in indices where (4.1.1) achieves equality and a particular value shows up without repetition in the vanishing sequence, we also need to impose a gluing condition on the sections with this vanishing. Imposing both (4.1.1) and the gluing condition appears to produce the correct expected dimension, at least in the case that $Y$ and $Z$ are rational, so we expect to produce a new proof of the Brill-Noether theorem soon using these techniques.

Finally, we remark that just as this whole circle of ideas was originally developed in the higher-rank context, the above definitions end up being remarkably similar to the generalization to higher rank by Teixidor of Eisenbud-Harris limit series. The main new ingredient is the introduction of the new vanishing sequences, although we have now also gone further in dealing with torsion-free sheaves and compactification issues as well. For curves with more than two components, we expect the non-compact type case to become substantially more complicated, and we will start working on that as soon as we have finished the case of two components.