LIMIT LINEAR SERIES: CONSTRUCTIONS AND APPLICATIONS

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Abstract. We discuss the Eisenbud-Harris theory of limit linear series, and survey a number of applications, ranging from the geometry of moduli spaces of curves to connectedness of certain Hurwitz spaces. We also present a more recent construction of limit linear series, and discuss applications and potential applications both to classical linear series and to their higher-rank generalization.

In this series of lectures, we will survey the theory of linear series on curves, with a particular emphasis on degeneration techniques. We will also discuss the generalization to higher-rank vector bundles. The lecture plan is as follows:

(1) Review of the basics of linear series on smooth curves and their ramification; discusssion of the theory of limit linear series developed by Eisenbud and Harris.
(2) Discussion of several different applications of the Eisenbud-Harris theory.
(3) Presentation of a more recent approach to limit linear series, and examination of the relationship to the Eisenbud-Harris approach.
(4) Construction of linear series and limit linear series spaces, and the related notion of linked Grassmannians.
(5) An application of the new construction to fibers of generalized Abel maps; discussion of higher-rank Brill-Noether theory, and approaches to it via degeneration techniques.

We work throughout over an algebraically closed field F of characteristic 0, although we will periodically remark on the situation for positive characteristic, and some of the foundational results hold over much more general schemes.

1. Eisenbud-Harris limit linear series

In this lecture, we begin by recalling basic definitions relating to linear series on smooth curves and ramification sequences, and preview some of the applications of the theory. We then discuss the theory of limit linear series developed by Eisenbud and Harris, which describes how linear series specialize under degenerations from smooth curves to curves of compact type.

1.1. Linear series on curves.

Linear series arise naturally from the study of maps of varieties into projective space. Specifically, if X is a smooth proper curve over F, then F-linear maps from X to $\mathbb{P}^r$ of degree d correspond to pairs $(\mathscr{L}, \{v_0, \dots, v_r\})$ of line bundles $\mathscr{L}$ on X of degree d together with sections $\{v_0, \dots, v_r\} \in H^0(X, \mathscr{L})$ such that for every $P \in X$, at least one of the v_i is non-vanishing at P. The map is non-degenerate (that is, does not have image contained in a hyperplane) if and only if the v_i are linearly independent, and if we wish to work up to automorphism of $\mathbb{P}^r$, we replace the v_i by the $(r+1)$-dimensional vector space V which they span. Finally, it turns out we can obtain a natural compactification of this space simply by dropping the requirement that V be non-vanishing at every point of X. Thus motivated, we have:

Definition 1.1.1. A **linear series** of degree d and dimension r (also called a $\mathfrak{g}^r_d$) on X is a pair $(\mathscr{L}, V)$ where $\mathscr{L}$ is a line bundle of degree d on X, and V is an $(r+1)$-dimensional subspace of $H^0(X, \mathscr{L})$.

We also briefly mention the idea of ramification, which plays an important role later:

Definition 1.1.2. Given a linear series $(\mathscr{L}, V)$ and a point $P \in X$, the **vanishing sequence** $a_0(P), \dots, a_r(P)$ of $(\mathscr{L}, V)$ at P is defined to be the strictly increasing sequence of orders of vanishing at P of sections of V. The non-decreasing **ramification sequence** $\alpha_0(P), \dots, \alpha_r(P)$ is defined by $\alpha_i(P) = a_i(P) - i$. If $\alpha_i(P) = 0$ for all i, we say P is **unramified**, otherwise P is **ramified**, or a **ramification point**.

Example 1.1.3. A few basic cases:

(1) We see that $(\mathscr{L}, V)$ corresponds to a map to $\mathbb{P}^r$ when $a_0(P) = 0$ for all P. In this case, we say $(\mathscr{L}, V)$ is **basepoint-free**. If $a_0(P) > 0$ for some P, we say that P is a **base point** of $(\mathscr{L}, V)$. We see immediately that a linear series can have at most finitely many base points.

(2) If $r = 1$, and $(\mathscr{L}, V)$ is basepoint-free, then $a_1(P)$ (or $\alpha_1(P) := a_1(P) - 1$, depending on convention) corresponds to the usual ramification index at P of the corresponding map of curves $X \to \mathbb{P}^1$.

(3) If $r = 2$ and $(\mathscr{L}, V)$ is basepoint-free, suppose also that the induced map $X \to \mathbb{P}^2$ is birational onto its image. Then a point with $\alpha_1(P) > 0$ corresponds to a cusp-type singularity in the image of X, while a point with $\alpha_1(P) = 0$ but $\alpha_2(P) > 0$ corresponds to a flex point.

(4) If $r = g - 1$, and $d = 2g - 2$, the only $\mathfrak{g}^r_d$ on X is the **canonical series** $(\Omega^1_X, H^0(X, \Omega^1_X))$, and ramification points correspond precisely to the Weierstrass points of X.

A basic formula for the total ramification of a linear series is the following Plucker formula (see Proposition 1.1 of [4]):

Proposition 1.1.4. *Let $(\mathscr{L}, V)$ be a $\mathfrak{g}^r_d$ on a smooth projective curve X of genus g. Then*

$$\sum_{P \in X} \sum_{i=0}^{r} \alpha_i(P) = (r+1)d + \binom{r+1}{2}(2g-2).$$

In particular, $(\mathscr{L}, V)$ has only finitely many ramification points.

Note that this generalizes the formula for the total weight of the Weierstrass points on a curve, and also agrees with the Riemann-Hurwitz formula for branched covers of $\mathbb{P}^1$ when we set $r = 1$.

The study of linear series is a natural way to try to better understand X. For instance, the classical theorems that every curve of genus 1 can be imbedded as a cubic plane curve, and that every curve of genus 2 can be represented as a 2-to-1 cover of $\mathbb{P}^1$ branched over 6 points, are powerful tools for understanding such curves. One is thus led to Brill-Noether theory, which can be roughly summarized as the study of the following question:

Question 1.1.5. Given a curve X, and r, d, does X have a linear series of degree d and dimension r? If so, what is the dimension of the space of the $\mathfrak{g}^r_d$'s?

The proof of the following result is not difficult, and we will discuss it later:

Proposition 1.1.6. *Given g, r, d and a smooth projective curve X of genus g, there is a projective moduli space $G^r_d(X)$ of linear series on X of dimension r and degree d. Moreover, if we set*

$$\rho = (r+1)(d-r) - rg,$$

every component of $G^r_d(X)$ has dimension at least ρ.

Question 1.1.5 is answered more comprehensively by the famous Brill-Noether theorem, which was ultimately proved by Kempf, Kleiman, Laksov and Griffiths-Harris (based also work of Severi and Castelnuovo):

Theorem 1.1.7. *Given g, r, d, let*

$$\rho = (r+1)(d-r) - rg.$$

If $\rho \geqslant 0$, then for all smooth, projective curves X of genus g, the space $G^r_d(X)$ is non-empty with every component of dimension at least ρ. On a general curve X of genus g, the space $G^r_d(X)$ has dimension exactly ρ, and in particular is empty if $\rho < 0$.

Moreover, it was proved by Fulton and Lazarsfeld [7] that $G^r_d(X)$ is always connected when $\rho > 0$, and then by Gieseker [8] (based on work of Petri) that for X general the space $G^r_d(X)$ is in fact smooth. For our purposes, our interest is focused on the following generalization of Theorem 1.1.7, proved by Eisenbud and Harris:

Theorem 1.1.8. *Given g, r, d, n, and nondecreasing sequences $0 \leqslant \alpha^j_0, \dots, \alpha^j_r \leqslant d - r$ for each $j = 1, \dots, n$, let*

$$\rho = (r+1)(d-r) - rg - \sum_{i,j} \alpha^j_i.$$

Then for all smooth, projective curves X of genus g, and distinct marked points $P_1, \dots, P_n \in X$, there is a projective moduli space $G^r_d(X, P_1, \dots, P_n, \alpha^1, \dots, \alpha^n)$ of $\mathfrak{g}^r_d$'s on X with ramification at least α^j at each P_j, and it has every component of dimension at least ρ if it is non-empty. On a general curve X of genus g, the space of $\mathfrak{g}^r_d$'s has dimension exactly ρ, and in particular is empty if $\rho < 0$.

Note that in this case, the non-emptiness statement does not hold, although Eisenbud and Harris also give an explicit criterion in terms of Schubert calculus for when it does. The construction of the moduli space and dimensional lower bound proceeds in this case just as for Proposition 1.1.6.

Remark 1.1.9. We make a brief detour to observe that according to Proposition 1.1.4, if we attempt to specify all ramification of a linear series, we will have

$$\rho = -r(r+2)g.$$

Thus, for $g > 0$ and a fixed choice of curve, we cannot hope to specify all ramification and still get a linear series.

Another application of the theory of linear series is to the study of Weierstrass points. For instance, Eisenbud and Harris proved in [5]:

Theorem 1.1.10. *Fix $g > 1$, and let ω be a Weierstrass semigroup having weight at most $g/2$. Then there exist smooth curves X of genus g with a Weierstrass point having semigroup ω.*

In fact, there are far subtler ways to apply linear series. One such application is as follows: in the cases of genus 1 and genus 2 curves above, we have that any curve of genus 1 can be written as $y^2 = x^3 - ax + b$, and any curve of genus 2 can be written as $y^2 = x^6 + a_5x^5 + \cdots + a_1x + a_0$ (where we have to normalize to get rid of the singularity at infinity). In particular, in both cases we see that we can write down very explicit families of curves with rational parameters which contain the general curve of genus 1 and 2. It is natural to ask whether it is possible to do this more generally, or equivalently:

Question 1.1.11. Is the moduli space $\mathscr{M}_g$ of curves of genus g always unirational?

This question leads to the study of effective divisors on $\mathscr{M}_g$. One way of constructing effective divisors is to use Brill-Noether theory, by looking at loci of curves which have a $\mathfrak{g}^r_d$ where $\rho = -1$. It turns out that divisors described in this way, together with related variants, play a vital role in understanding the geometry of $\mathscr{M}_g$, and allowed Eisenbud and Harris [6] to prove:

Theorem 1.1.12. *$\mathscr{M}_g$ is not unirational for $g \geqslant 23$ (and is in fact of general type for $g \geqslant 24$).*

Using similar techniques, this theorem was generalized to $\mathscr{M}_{g,n}$ by Adam Logan in [12].

Given such a range of applications, Eisenbud and Harris were perhaps justified in asserting in [4] that "most problems of interest about curves are, or can be, formulated in terms of (families of) linear series."

Remark 1.1.13. If F has characteristic p, the Brill-Noether theorem Theorem 1.1.7 still holds as stated. The generalized version Theorem 1.1.8 does not hold, due to inseparable linear series. We say that a linear series is **inseparable** if every point is a ramification point; otherwise it is **separable**. We say a ramification point P is **tame** if the $a_i(P)$ are maximally distributed modulo p; otherwise we say it is **wild**. Then the usual argument for Proposition 1.1.4 shows that if $(\mathscr{L}, V)$ is separable, the same expression gives an upper bound for the total ramification, with equality if and only if $(\mathscr{L}, V)$ has no wild ramification. See also Proposition 2.4 of [19]. An alternate approach to generalizing the Plucker formula to positive characteristic is given by Stohr and Voloch in [23].

If we restrict to separable linear series, as far as I know it is an open question whether Theorem 1.1.8 holds, although it is known in the case $r = 1$; see [18]. On the other hand, in this case the space G^r_d is no longer proper, which causes serious difficulties for many degeneration arguments.

1.2. **Limit linear series.** The proofs of the Brill-Noether theorem and the theorems of Eisenbud and Harris have a key point in common: in order to give the desired analysis of linear series on smooth curves, they rely on analysis of degenerations to singular curves. The most powerful and general such technique is the theory of limit linear series, introduced by Eisenbud and Harris in [4] in order to prove the above-mentioned theorems. We briefly outline the issues involved, and the idea of their construction.

We first note that the definition of a linear series still makes perfect sense on a singular curve. However, let us suppose we have a (flat, projective) one-parameter family X/B of curves, smooth over the generic point $\eta \in B$, but degenerating to a singular curve over a point $b_0 \in B$. Suppose also that the total space of X is regular. We see that if we want to study the linear series on X_η in terms of those on $X_0 := X|_{b_0}$, we immediately run into problems. The subspaces V of $H^0(X, \mathscr{L})$ are not the issue: given a line bundle $\mathscr{L}$ on the whole family, and a $V_\eta \subset H^0(X_\eta, \mathscr{L}|_{X_\eta})$, there is a unique extension V to the whole family. The problem is rather with the line bundles $\mathscr{L}$ themselves. There are essentially two cases to consider, and in both we assume that the singular fiber X_0 remains smooth except for a single node:

(1) If the curve degenerates to an irreducible nodal curve, then a line bundle $\mathscr{L}_\eta$ on X_η may not extend to any line bundle on all of X: i.e., the relative Picard scheme of X/B is not universally closed.

(2) If the curve degenerates to a reducible curve, with two smooth components Y and Z glued at a node, then any $\mathscr{L}_\eta$ extends to X/B, but does not do so uniquely. Because the total space of X is regular, we may see this by noting that Y and Z are Cartier divisors on X, so given an extension $\mathscr{L}$ to all of X, we can also consider $\mathscr{L}(iY)$ for $i \in \mathbb{Z}$, which does not affect $\mathscr{L}_\eta$, but does change the line bundle over b_0. Thus, the relative Picard scheme of X/B is not separated.

It turns out that – at least thus far – the second case is more fruitful to study than the first, and Eisenbud and Harris focused on that case. It is well known that by also specifying the degree of $\mathscr{L}$ on Y and Z, one in fact obtains a separated, hence proper Picard scheme. Thus, given a linear series $(\mathscr{L}_\eta, V_\eta)$ for each i, we obtain a unique family of extensions $(\mathscr{L}^i, V_i)$ to the whole family, characterized by the condition that $\mathscr{L}^i$ has degrees $d - i$ and i when restricted to Y and Z, respectively.

The main insight of Eisenbud and Harris was that for many purposes, it is enough to consider the two linear series $(\mathscr{L}^0, V_0), (\mathscr{L}^d, V_d)$. Because $\mathscr{L}^0$ has degree 0 on Z, and $\mathscr{L}^d$ has degree 0 on Y, we don't lose any information by restricting to $(\mathscr{L}^Y, V^Y) := (\mathscr{L}^0|_Y, V_0|_Y)$ and $(\mathscr{L}^Z, V^Z) := (\mathscr{L}^d|_Z, V_d|_Z)$, giving us a pair of $\mathfrak{g}^r_d$'s on Y and Z. Thus, for every linear series $(\mathscr{L}_\eta, V_\eta)$, we obtain a pair $(\mathscr{L}^Y, V^Y), (\mathscr{L}^Z, V^Z)$. However, it is easy to see that there are far more pairs of $\mathfrak{g}^r_d$'s on Y and Z than there could be limits from X_η, so the question becomes which additional conditions describe the pairs arising in this way. Eisenbud and Harris showed:

Proposition 1.2.1. *If $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ arise as the limit of $(\mathscr{L}_\eta, V_\eta)$, a $\mathfrak{g}^r_d$ on X_η, and if we write $P := Y \cap Z$, and a^Y_i, a^Z_i for the vanishing sequences at P of $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ respectively, then we have for all $i = 0, \ldots, r$:*

$$a^Y_i + a^Z_{r-i} \geqslant d. \tag{1.1}$$

Eisenbud and Harris treat more generally the case of curves X_0 of **compact type**: these are projective nodal curves for which the dual graph is a tree, or equivalently, the Jacobian is compact. If X_0 is of compact type, its Jacobian is simply the product of the Jacobians of its irreducible components, each of which is necessarily nonsingular. That is, a line bundle on X_0 is uniquely determined by the restrictions to each component.

Now suppose we are in the same situation as above, but we allow X_0 to be any curve of compact type. Let $Y \subseteq X_0$ be an irreducible component. Then, as before, a line bundle $\mathscr{L}_\eta$ of degree d on X_η has a unique extension $\mathscr{L}^Y$ to X_0 having degree d on Y and degree 0 on all other components, and if $V_\eta \subseteq \Gamma(X_\eta, \mathscr{L}_\eta)$ determines a $\mathfrak{g}^r_d$ on X_η, we have a limit $V^Y \subseteq \Gamma(Y, \mathscr{L}^Y)$ giving a $\mathfrak{g}^r_d$ on Y. Thus, a $\mathfrak{g}^r_d$ $(\mathscr{L}_\eta, V_\eta)$ yields a collection of $\mathfrak{g}^r_d$'s $(\mathscr{L}^Y, V^Y)$ for each component Y of X_0. Eisenbud and Harris showed that we have the same compatibility conditions in this setting:

Proposition 1.2.2. *In the above setting, if $(\mathscr{L}^Y, V^Y)_{Y \subseteq X_0}$ arises as the limit of $(\mathscr{L}_\eta, V_\eta)$ a $\mathfrak{g}^r_d$ on X_η, suppose $P \in X_0$ is a node, and let Y, Z be the irreducible components of X_0 containing P. If we write a^Y_i, a^Z_i for the vanishing sequences at P of $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ respectively, then we have for all $0 \leqslant i \leqslant r$:*

$$a^Y_i + a^Z_{r-i} \geqslant d. \tag{1.2}$$

Eisenbud and Harris are thus motivated to define:

Definition 1.2.3. Given a curve X_0 of compact type, a tuple of $\mathfrak{g}^r_d$'s $(\mathscr{L}^Y, V^Y)_{Y \subseteq X_0}$ is called a **limit (linear) series** on X_0 if it satisfies (1.2).

If further (1.2) is an equality for all i, the tuple is called a **refined** limit series, and otherwise it is called a **crude** limit series.

Eisenbud and Harris refined Proposition 1.2.2 as follows:

Proposition 1.2.4. *In the situation of Proposition 1.2.2, the tuple $(\mathscr{L}^Y, V^Y)_{Y \subseteq X_0}$ gives a refined limit series on X_0 if and only if no ramification points of X_η specialize to nodes of X_0.*

Given $(\mathscr{L}_\eta, V_\eta)$ on X_η, after a finite base change and blowing up X_0 finitely many times at its nodes to resolve the resulting singularities of X, we may assume $(\mathscr{L}^Y, V^Y)_{Y \subseteq X_0}$ is a refined limit series.

The first statement follows from the Plucker formula Proposition 1.1.4, while the second follows from the first by choosing a base change such that all ramification points specializing to nodes become rational.

We have thus seen that any limit of a linear series on smooth curves is in fact a limit linear series. The question then becomes whether all limit series arise in this way. The answer is in general no, but Eisenbud and Harris were able to prove a remarkable smoothing theorem depending on having the expected dimension in the special fiber. They consider families X/B with all fibers of compact type (possibly including smooth fibers) and some additional technical hypotheses. They first give a complicated construction of a scheme parametrizing linear series and refined limit linear series for X/B and then carry out a relatively simple dimension-counting argument, to conclude the following:

Theorem 1.2.5. *Given X/B, smooth sections P_i, and integers r, d, α_i^j, there is a quasiprojective scheme $G_d^{r,\mathrm{EH}}$ parametrizing linear series on smooth fibers of X, and refined limit series on singular fibers of X, both of degree d and dimension r, and having ramification at least α_i^j at each P_j.*

The dimension of any component of $G_d^{r,\mathrm{EH}}$ is at least $\rho + \dim B$.

If either

$$\sum_{i,j} \alpha_i^j = (r+1)d + \binom{r+1}{2}(2g-2),$$

or no reducible fibers X_0 of X have crude $\mathfrak{g}_d^r$'s with the prescribed ramifications, then $G_d^{r,\mathrm{EH}}$ is proper over B.

Note that $G_d^{r,\mathrm{EH}}$ is not proper in general because the crude limit series are omitted. From the basic inequality on dimension and fiber dimension, Eisenbud and Harris can deduce from the theorem the following smoothing result:

Corollary 1.2.6. *In the situation of Theorem 1.2.5, if the special fiber $G_d^{r,\mathrm{EH}}(X_0)$ has dimension exactly ρ, then every refined limit series on X_0 can be smoothed to linear series on nearby fibers.*

This result is unusual in that it gives a criterion for deforming objects (in this case, (limit) linear series) which does not rely at all on infinitesimal deformation theory and cohomological conditions, but rather using only elementary dimension-counting.

Remark 1.2.7. In fact, Eisenbud and Harris called anything satisfying (1.2) a crude limit series, and frequently use the term “limit series” to refer to refined limit series. The reason for this was that their key smoothing theorem was restricted to refined limit series. However, in our construction of limit linear series, discussed below, the smoothing theorem is generalized to hold for all limit linear series, so we see no need to exclude crude limit series from the default terminology.

Remark 1.2.8. Although Eisenbud and Harris stated their results only in characteristic 0, their definitions face no obstacles in positive characteristic, and it appears that the arguments for most of their foundational results do not face any problems in positive characteristic. However, Proposition 1.2.4 uses the Plucker formula in a key way, and consequently fails in positive characteristic.

2. A sampling of applications

We have already mentioned several applications of degeneration techniques. We now revisit these to sketch the ideas behind the proofs, and we also discuss some other applications of limit linear series theory.

2.1. **Eisenbud-Harris applications.** We begin with the results proved by Eisenbud and Harris using limit linear series.

The generalized Brill-Noether theorem. Although the Brill-Noether theorem was first proved by degeneration to irreducible nodal curves, Eisenbud and Harris gave a new and simpler proof using limit linear series ideas, and in the process generalized it to allow prescribed ramification at marked points, as described in Theorem 1.1.8. The key point is the following additivity observation, which is straightforward to verify and absolutely crucial to getting anything useful out of the general theory.

Proposition 2.1.1. *Let X_0 be a curve of compact type of genus g. For every irreducible component Y of X_0, and each node P of X_0, choose any increasing sequence $a^{Y,P}$, subject to the condition that for every such P and Y, if Z is the other component of X_0 containing P, then we have for each $i = 0, \dots, r$*

$$a_i^{Y,P} + a_{r-i}^{Z,P} = d.$$

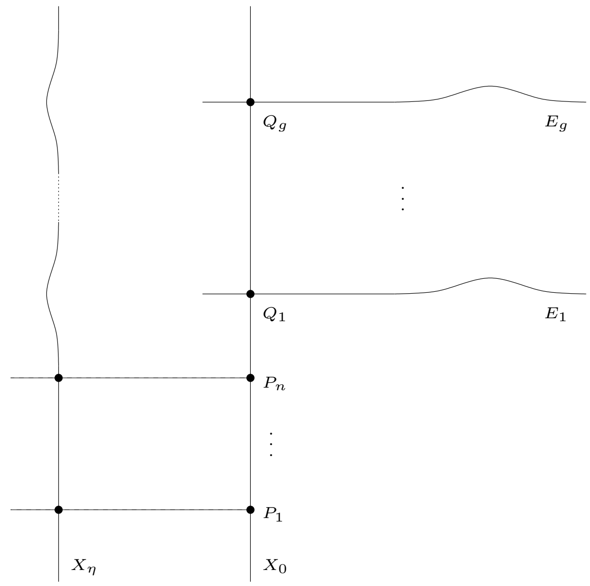

Figure 1

Then, given r, d and any ramification conditions imposed at smooth points $Q_1, \dots, Q_n$ of X_0, let ρ be the corresponding expected dimension of $\mathfrak{g}^r_d$'s on a curve of genus g with the imposed ramification at the Q_i. For each irreducible component Y of X_0, let ρ_Y be the expected dimension of $\mathfrak{g}^r_d$'s on Y with the imposed ramification at all of the Q_i lying on Y, and also the ramification imposed by the vanishing sequences $a^{Y,P}$ for each node P lying on Y. Then we have

$$\sum_{Y \subseteq X_0} \rho_Y = \rho.$$

Given this statement, the argument for the Brill-Noether theorem can be summarized as follows: we construct a one-parameter family of curves with smooth generic fiber X_η of genus g, and special fiber X_0 a "general n-marked comb curve" obtained by gluing g elliptic curves to a copy of $\mathbb{P}^1$ at general points, and then adding n general marked points on the $\mathbb{P}^1$. Then, if we prove the theorem for elliptic curves with ramification prescribed at a single point, and linear series on $\mathbb{P}^1$ with ramification prescribed at general points, by additivity of ρ we obtain the desired statement (for refined limit series) on X_0. Both of these cases can be proved by direct argument. By Theorem 1.2.5, we can smooth from X_0 to X_η, and this implies the nonemptiness statement of Brill-Noether. It would also imply the expected dimension statement if the Eisenbud-Harris space $G^{r,\text{EH}}_d$ of limit linear series were proper. However, it is not difficult to get around that it is not: given any point of $G^r_d(X_\eta)$, by Proposition 1.2.4 after base change and blowing up nodes of X_0 we have that the point extends to a refined limit series on X_0. Blowing up will only introduce new rational components, so arguing as before we still find that Brill-Noether holds for X_0, and by semicontinuity of fiber dimension we have that our original point of $G^r_d(X_\eta)$ must have had dimension ρ as well.

Existence of Weierstrass points. Moving on to Theorem 1.1.10 on the existence of Weierstrass points, the basic idea is as follows: since a Weierstrass point is simply a ramification point of the canonical linear series, and the canonical linear series is the unique $\mathfrak{g}^{g-1}_{2g-2}$ on a smooth curve of genus g, it is enough to produce a (refined) limit $\mathfrak{g}^{g-1}_{2g-2}$ with the desired ramification at a marked point, which occurs in the expected dimension; we can then apply Theorem 1.2.5. The only problem is that in this case, the expected dimension is negative! Eisenbud and Harris dealt with this by working with a higher-dimensional family of curves, so that even the degenerate curves move in a family. In this case, even though $\rho < 0$, as long as the limit series in question occur over a locus in the base of codimension $-\rho$, they have the expected dimension, and the argument goes through.

The geometry of moduli spaces of curves. Finally, we discuss the argument for Theorem 1.1.12, on the Kodaira dimension of moduli spaces of curves. The Picard group of $\overline{\mathscr{M}}_g$ had already been computed by Harer, and the class of the canonical divisor was likewise known. Following earlier work of Harris and Mumford, to prove that the space is of general type it was necessary to produce effective divisors on it, to compute their class in the Picard group of the moduli space, and to check that the coefficients satisfied certain inequalities. Eisenbud and Harris studied two families of divisors in order to obtain their results, of which we will discuss the first. We suppose that $g+1$ is composite, and write $g+1 = (r+1)(r+g-d)$ for some $r, d \geqslant 1$. We then have

$$\rho = (r+1)(d-r) - rg = (r+1)(d-r-g) + g = -1,$$

so if we consider the universal space of $\mathfrak{g}^r_d$'s over $\mathscr{M}_g$, we expect (and indeed Eisenbud and Harris showed) that it is supported over an effective divisor $\mathcal{D}$ on $\mathscr{M}_g$ (together with possibly higher-codimension components). They further computed the class of $\mathcal{D}$, and found that for $g \geqslant 24$, it satisfies the necessary inequalities to prove that $\mathscr{M}_g$ is of general type. In computing the class of $\mathcal{D}$, they computed how it intersected two loci of reducible curves. They first observed that $\mathcal{D}$ is in fact disjoint from the locus of general comb curves used in the proof of the Brill-Noether theorem, and deduced strong restrictions the class of the divisor from that observation. They then considered curves obtained by gluing a fixed general curve of genus $g-2$ to a curve of genus 2 at a single point, and showed that $\mathcal{D}$ meets this locus when the genus-2 curve is glued to the other component at a Weierstrass point. Using these observations, they were able to compute the class of $\mathcal{D}$ (up to positive rational scalar multiple), completing the proof for the case that $g+1$ is composite. They then separately handled the case that g is even and at least 28, via a similar but more difficult argument involving a different divisor on $\mathscr{M}_g$. Considering both divisors in the case $g = 23$ let them prove that $\mathscr{M}_{23}$ is not unirational, and they were thus able to conclude Theorem 1.1.12.

2.2. **Enumeration of linear series.** Using degeneration techniques, Castelnuovo had already studied the question of the number of $\mathfrak{g}^r_d$'s on a general curve of genus g when g, r, d are chosen such that $\rho = 0$, and in fact it was these techniques which motivated the original proof of the Brill-Noether theorem. However, one may naturally generalize the question to allow the imposition of ramification as well:

Question 2.2.1. Fix g, r, d and n sequences $\alpha^j = \alpha^j_0, \dots, \alpha^j_r$ such that

$$\rho := (r+1)(d-r) - rg - \sum_{j=1}^{n} \sum_{i=0}^{r} \alpha^j_i = 0.$$

Then what is the number of $\mathfrak{g}^r_d$'s on a general n-marked curve of genus g, having ramification sequences at least α^j at the marked points?

We remark that in the case $g = 0$, this asks what is the number of points in the intersection of certain Schubert cycles (specifically, those associated to the osculating flags at n general points of a rational normal curve in $\mathbb{P}^d$). Thus, a very special case of Question 2.2.1 is the following:

Question 2.2.2. Fix r, d and 3 sequences $a^j = a^j_0 < \cdots < a^j_r$ such that

$$(r+1)(d-r) = \sum_{j=1}^{3} \sum_{i=0}^{r} (a^j_i - i).$$

For $\ell = 0, \dots, d$ write

$$b^j_\ell = \#\{i : a^j_i \geqslant d - \ell\}.$$

Let P_1, P_2, P_3 be three distinct points of the rational normal curve $C_d \subseteq \mathbb{P}^d$, and let $F^j_0 \subseteq \cdots \subseteq F^j_d$ be the osculating flag to C_d at P_j for $j = 1, 2, 3$. Finally, for $j = 1, 2, 3$ let $\Sigma_j \subseteq G(r+1, d+1)$ be the Schubert cycle of r-planes $W \subseteq \mathbb{P}^d$ such that for each $\ell = 0, \dots, d$ we have

$$\dim(W \cap F^j_\ell) \geqslant b^j_\ell - 1.$$

Then, do $\Sigma_1, \Sigma_2, \Sigma_3$ intersect transversely?

The following theorem is from [17]:

Theorem 2.2.3. *For a given r, suppose that Question 2.2.2 has a positive answer. Then Question 2.2.1 has an explicit solution in terms of Schubert calculus, and the relevant spaces of $\mathfrak{g}^r_d$'s all consist of reduced points.*

We will sketch the use of limit linear series to reduce Question 2.2.1 to the case $g = 0$; a similar argument can then reduce to the case $n = 3$, proving the theorem. We mention that in the case $r = 1$, it is easy to see that Question 2.2.2 is answered positively, so we get a complete answer in that case. More recently, Mukhin, Tarasov and Varchenko proved in [14] that Question 2.2.2 has a positive answer in general (and indeed they proved the far sharper statement that for any number of distinct real points of the rational normal curve, the corresponding Schubert cycle intersection is transverse and contains only real points). Thus, we now have a complete solution to Question 2.2.1.

Sketch of proof of Theorem 2.2.3. Choose a one-parameter family of n-marked curves with smooth generic fiber, nonsingular total space, and special fiber X_0 a general comb curve with n marked points $Q_1, \dots, Q_n$, all on the spine. We note that because of the generalized Brill-Noether theorem Theorem 1.1.8, and because X_0 is constructed from general marked curves, if the expected dimension of linear series (including ramification conditions at the nodes) is negative on any component, there will be no linear series on that component, and hence no corresponding limit series on X_0. Now, since $\rho = 0$ by hypothesis, and because ρ is additive on components (Proposition 2.1.1), we see that in order to prevent the expected dimension on any component from becoming negative, we must have ramification at nodes so that the expected dimension on each component is precisely 0. The space of limit linear series on X_0 will thus be the product of G^r_d spaces on each component, with this additional ramification imposed at the node.

In particular, on each elliptic component E_i we must have enough ramification at Q_i alone to push the expected dimension to 0, and this will allow us to show that the G^r_d space on each E_i is simply a single reduced point. We start by observing that the maximal vanishing sequence is *a priori* $d-r, \dots, d-1, d$, which imposes $(r+1)(d-r)$ conditions on the $(r+1)(d-r)-rg = (r+1)(d-r)-r$ dimensions of the space of g^r_d's on E_i. Thus, we can reduce the total ramification weight only by r from this maximal sequence if we wish the expected dimension to be 0. In particular, the last vanishing order must remain d, meaning that we have a section vanishing to order d at our point, which automatically restricts our line bundle to $\mathscr{O}(dQ_i)$. Now, we note that since the genus of E_i is positive, we cannot have a section of $\mathscr{O}(dQ_i)$ vanishing to order exactly $d-1$ at Q_i, since that would correspond to a rational function with a simple pole at Q_i, and regular elsewhere. Thus, the actual maximal vanishing sequence is $d-r-1, \dots, d-2, d$, which imposes $(r+1)(d-r)-r$ conditions, and therefore has expected dimension 0, and must be the unique vanishing sequence which can have a non-empty G^r_d space of expected dimension 0 for E_i. Thus, the G^r_d space in question is now simply a Schubert cycle inside our Grassmannian, so if it is non-empty and of dimension 0, it must be simply a single reduced point. Lastly, it must be non-empty and of dimension 0 simply by the Riemann-Roch theorem. We conclude that on X_0, the ramification conditions at each Q_i are uniquely determined, giving ramification indices of $0, 1, \ldots, 1$ at the Q_i on the rational component. Furthermore, since the G^r_d spaces on each E_i are single reduced points, the $G^{r,\mathrm{EH}}_d$ space for X_0 is simply the G^r_d space for $\mathbb{P}^1$ with the given ramification at the P_i and Q_i.

Since we noted above that the expected dimension imposed by ramification conditions at the nodes would have to be precisely 0 on each component, because this can be achieved only for ramification conditions corresponding to refined limit series, we see that we have no crude limit series on X_0. By Theorem 1.2.5, this implies that the space $G^{r,\mathrm{EH}}_d$ for the entire smoothing family is proper. Since the P_i and Q_i are general, we have by the $g = 0$ case of Theorem 1.1.8 that $G^{r,\mathrm{EH}}_d$ has the expected dimension on X_0. By Corollary 1.2.6, each limit $\mathfrak{g}^r_d$ on X_0 must smooth to the family, and by properness any $\mathfrak{g}^r_d$ on the generic fiber X_η limits to a limit $\mathfrak{g}^r_d$ on X_0, so we find that if our G^r_d space for X_0 is made up of reduced points, then the G^r_d space for X_η is made up of the same number of reduced points. We have thus shown that given the $g = 0$ case of the theorem, we have produced at least one smooth curve with the right G^r_d space. Standard techniques then show that the statement holds for a general curve of genus g, as desired.

This gives the reduction to the case $g = 0$, and the reduction to the case $n = 3$ is accomplished by a similar argument, but this time using genus-0 degenerations with 2 marked points on one component and $n - 2$ on the other to inductively reduce the number of marked points. □

2.3. **Pure-cycle Hurwitz spaces.** When Eisenbud and Harris introduced limit linear series, they took the point of view that the $r = 1$ case already known, in the form of the theory of admissible covers. Our point of view is that these are distinct and complementary theories, and we mention one final application to underline this point. We consider Hurwitz spaces parametrizing branched covers of given degree of the projective line with prescribed branch type. Explicitly, given d, n and $T_1, \ldots, T_n$ partitions of d we have the Hurwitz space $\mathscr{H}(d; T_1, \ldots, T_n)$ parametrizing branched covers of $\mathbb{P}^1$ of degree d together with (distinct) branch points $Q_1, \ldots, Q_n \in \mathbb{P}^1$ such that the ramification indices of the points lying over each Q_i are given by the partition T_i, and there are no other branch points. Note that the genus of every cover in the Hurwitz space is determined via the Riemann-Hurwitz formula from d and the T_i, so we say that a Hurwitz space is **genus** g if the covers parametrized by that space have genus g. We also say that a Hurwitz space is **pure-cycle** if each T_i is of the form $(e_i, 1, \ldots, 1)$ for some $e_i > 1$, or equivalently if the corresponding covers all have exactly one ramification point over each branch point. In [11], Liu and I prove:

Theorem 2.3.1. *Every genus-*0 *pure-cycle Hurwitz space is irreducible.*

Because the problem is stated in terms of branched covers, it seems like a natural candidate for study via the techniques of admissible covers. However, one quickly runs into a serious problem: if we degenerate the base and look at the corresponding degeneration of the cover, there is no reason for the pure-cycle condition to be preserved, so we are not able to give an inductive argument. In a group theory setting, this corresponds to the fact that frequently, the product of two cycles is no longer a cycle. If instead we shift our perspective and consider the Hurwitz space to be parametrizing $\mathfrak{g}^1_d$'s with prescribed ramification on $\mathbb{P}^1$ (and with the ramification points mapping to distinct points, which gives an open subset of the total space of $\mathfrak{g}^1_d$'s), we can instead degenerate via limit linear series, which by allowing us to control the degeneration of the covering curve, sets up an inductive argument reducing to the case of four branch points. This case is then handled by direct analysis in the group-theoretic setting to complete the proof of Theorem 2.3.1.

Remark 2.3.2. The distinction between the situation of linear series and branched covers is somewhat evident already for smooth curves in characteristic 0: for instance, spaces of branched covers always have the expected dimension for any fixed distinct branch points, while we expect families of linear series with fixed ramification to have larger dimension for "special" configurations of branch points. This distinction is highlighted further in characteristic p. For branched covers, it is well known that pathologies (such as infinite families of covers with fixed branching) occur only in the case of wild ramification. In contrast, from the linear series point of view the distinction between wild and tame is largely irrelevant for the dimensions of the relevant spaces [18], and what pathologies do occur seem more related to the size of ramification.

We discuss this further in the case of maps $\mathbb{P}^1 \to \mathbb{P}^1$. In characteristic 0, it is known that for arbitrary distinct ramification points, there are only finitely many such maps (up to automorphism of the image) and more generally this is true for $\mathfrak{g}^r_d$'s on $\mathbb{P}^1$ via a simple inductive argument; see Theorem 2.3 of [3]. However, as shown by the example

$$x^{p+2} + tx^p + x,$$

this can fail in positive characteristic, if some ramification indices are greater than p. In contrast, if all ramification indices are less than p, this does not occur; see [22] for the case all ramification indices are odd, but the general case should follow along similar lines. Remarkably, the proof exploits a relationship between such ramified maps and Mochizuki's dormant torally indigenous bundles. No elementary proof is known.

Finally, we remark that the enumerative formulas of Theorem 2.2.3 no longer hold in positive characteristic, and determining the correct formulas is far more difficult. This is carried out in the special case $r = 1$, $g = 0$, and all ramification indices less than p in [21].

3. The new approach

There are two substantive drawbacks to Eisenbud and Harris' construction. The first is that the construction itself is rather *ad hoc*, and does not obviously represent any natural functor. The second is that it does not include the crude limit series, so the space constructed is not proper, and although we've seen that Eisenbud and Harris have tools for dealing with this, certain arguments become more awkward than they should be.

3.1. **The moduli functor.** In [19], it is observed that the way around these problems is to remember all extensions of the generic linear series to the reducible fibers, and not only those in extremal degrees. We work in the special case of curves with at most two components:

Situation 3.1.1. Suppose $\pi : X \to B$ is a one-parameter family with smooth generic fiber X_η and special fiber X_0 a union of two smooth components Y and Z intersecting at a single node P. Suppose also that X has regular total space.

In this situation, given $(\mathscr{L}_\eta, V_\eta)$ a $\mathfrak{g}^r_d$ on X_η, we have seen that for any i there is a unique extension $(\mathscr{L}^i, V_i)$ to X such that $\mathscr{L}^i$ has degree $d - i$ on Y and degree i on Z. The $\mathscr{L}^i$ are related by the identity $\mathscr{L}^{i+1} = \mathscr{L}^i(Y)$. We thus have maps $\mathscr{L}^i \to \mathscr{L}^{i+1}$ for each i. On the other hand, if s is a uniformizer on B, then

$$\mathscr{O}(Y + Z) = \mathscr{O}(X_0) = \mathscr{O}(\pi^* s) \cong \mathscr{O},$$

so if we fix a choice of such an isomorphism, we obtain maps

$$\mathscr{L}^{i+1} = \mathscr{L}^i(Y) \cong \mathscr{L}^i(-Z) \to \mathscr{L}^i,$$

with the property that the composed maps $\mathscr{L}^i \to \mathscr{L}^{i+1} \to \mathscr{L}^i$ and $\mathscr{L}^{i+1} \to \mathscr{L}^i \to \mathscr{L}^{i+1}$ are equal to multiplication by s. For the spaces V_i constructed as above, it is easy to see that we will have V_i mapping to V_{i+1} under the induced map $\pi_* \mathscr{L}^i \to \pi^* \mathscr{L}^{i+1}$, and vice versa.

We can thus define a moduli functor $\mathcal{G}^r_d$ parametrizing choices of a line bundle $\mathscr{L}^0$ of degree d on Y and degree 0 on Z, together with $(r + 1)$-dimensional spaces of global sections V_i of $\mathscr{L}^i := \mathscr{L}^0(iY)$, with the property that the V_i map into one another under the above maps. We can impose ramification along smooth sections just as in the Eisenbud-Harris situation (where we impose it either on V_0 or V_d according to whether the section specializes to Y or Z). Also as in the

Eisenbud-Harris situation, we can make the definition work for families with $\dim B > 1$. We then prove:

Theorem 3.1.2. *Given X/B, smooth sections P_i, and integers $r, d, \alpha_i(P_j)$, the functor $\mathcal{G}^r_d$ of limit series on X/B having ramification at least $\alpha_i(P_j)$ at each P_j is compatible with base change, and representable by a scheme G^r_d projective over B. Every component of G^r_d has dimension at least $\rho + \dim B$.*

If the dimension of a fiber of G^r_d is exactly ρ, then every limit series in that fiber can be smoothed to (limit) linear series on nearby fibers.

We remark that the most difficult part of the theorem is the dimension count, which is subtler than in the Eisenbud-Harris construction. This is addressed by the theory of linked Grassmannians, which we discuss below. From now on, we will use the term "limit series" for this new definition, and will refer to those defined by Eisenbud and Harris as "Eisenbud-Harris limit series."

We briefly consider what the space G^r_d parametrizes on fibers. Suppose X_b is a smooth fiber of X/B. Then the maps $\mathscr{L}^i|_{X_b} \to \mathscr{L}^{i+1}|_{X_b}$ are all isomorphisms, so all V_i are uniquely determined by V_0, and we see that we recover the usual functor of $\mathfrak{g}^r_d$'s on X_b.

On the other hand, if X_b is reducible, recall that line bundles on X_b are uniquely determined by their restrictions to components. We see that $\mathscr{L}^i|_Z = \mathscr{L}^0(iY)|_Z = \mathscr{L}^0|_Z(iP)$. Similarly, $\mathscr{L}^i|_Y \cong \mathscr{L}^0(-iZ)|_Y = \mathscr{L}^0|_Y(-iP)$. The maps $\mathscr{L}^i|_{X_b} \to \mathscr{L}^{i+1}_{X_b}$ are the canonical inclusion on Z, and the zero map on Y, and vice versa for the maps $\mathscr{L}^{i+1}|_{X_b} \to \mathscr{L}^i_{X_b}$. Thus the condition that V_i maps into V_{i+1} means simply that the spaces $V_i|_Z$ may be considered as an increasing filtration of $V_Z := V^d|_Z$, and similarly we have that the $V_i|_Y$ are a decreasing filtration of V_Y. Note however that V_i includes in general strictly more information than $V_i|_Y$ and $V_i|_Z$, as there are choices about how sections on Y and Z can be glued if they both vanish at P. This additional information will mean in particular that our space is not the same as the Eisenbud-Harris space.

Returning to the properties of the relative G^r_d space, in work with David Helm [9] we showed:

Theorem 3.1.3. *If G^r_d has the expected dimension $\rho + \dim B$, then it is flat and Cohen-Macaulay.*

3.2. **Comparison to the Eisenbud-Harris case.** We now restrict our attention to X_0 as in Situation 3.1.1. As discussed above, on reducible fibers the $\mathcal{G}^r_d$ functor parametrizes $(r+1)$-tuples of spaces of sections on X_0, and the gluing at nodes matters. Thus, a limit linear series on X_0 cannot be expressed solely in terms of objects on Y and Z, and the description does not immediately lend itself to induction. It is thus helpful to compare to the Eisenbud-Harris construction.

The first observation we make is that the conditions (1.2) give us a construction of a scheme

$$G^{r,\mathrm{EH}}_d(X_0) \subseteq G^r_d(Y) \times G^r_d(Z)$$

which includes crude limit series as well as refined ones. Namely, for each pair $a^{Y,P}, a^{Z,P}$ of increasing sequences satisfying (1.2), we have a closed subscheme of $G^r_d(Y) \times G^r_d(Z)$ consisting of pairs of $\mathfrak{g}^r_d$'s with at least the imposed vanishing at the nodes, and we can then define $G^{r,\mathrm{EH}}_d(X_0)$ to be the union of these subschemes. Note however that because we take unions, it is not obvious how to give a functorial description of $G^{r,\mathrm{EH}}_d(X_0)$. Regardless, in this discussion we assume we are working with this compactified version of $G^{r,\mathrm{EH}}_d$.

Next, it is clear that by taking $(\mathscr{L}^0|_Y, V_0|_Y)$ and $(\mathscr{L}^d|_Z, V_d|_Z)$ we have a forgetful morphism

$$G^r_d(X_0) \to G^r_d(Y) \times G^r_d(Z), \tag{3.1}$$

Our main results are then the following:

Theorem 3.2.1. *The morphism* (3.1) *induces a set-theoretic surjection $G^r_d \to G^{r,\mathrm{EH}}_d$.*

We can thus define a point of G^r_d to be refined (respectively, crude) if it maps to a refined (respectively, crude) point of $G^{r,\mathrm{EH}}_d$. Note that (at least in our situation of curves with two components) Theorem 3.1.2 together with Theorem 3.2.1 recover both the specialization and smoothing results (Proposition 1.2.2 and Theorem 1.2.5) of Eisenbud and Harris, with the specialization statement following simply from the properness of the G^r_d space.

Theorem 3.2.2. *The map $G^r_d \to G^{r,\mathrm{EH}}_d$ induces an isomorphism on the open subschemes corresponding to refined limit series.*

We describe how to understand these results, first sketching the proof that G^r_d maps into $G^{r,\mathrm{EH}}_d$. For each i, denote by $V^Y(-iP) \subseteq V^Y$ and $V^Z(-iP) \subseteq V^Z$ the subspaces vanishing to order i at P. The main point is that for each i, by definition $V_i|_Y \subseteq V^Y(-iP)$, and $V_i|_Z \subseteq V^Z(-(d-i)P)$. But because V_i is glued from sections in these two spaces, their dimension must add up to at least $r+1$. In fact, if either $V_i|_Y$ or $V_i|_Z$ has a section which vanishes to the minimal order at P, there is a nontrivial gluing condition at P, and the dimensions must add up to at least $r+2$. After some combinatorial manipulation, one precisely recovers the Eisenbud-Harris inequalities (1.2).

The essential content of Theorem 3.2.2 is then that given $(\mathscr{L}^0, V_0)$ and $(\mathscr{L}^d, V_d)$ determining a refined limit series, there is a unique way to fill in the intermediate V_i. In this case, the dimensions work out so that the spaces $V^Y(-iP)$ and $V^Z(-iP)$ have precisely the minimum allowed dimension, so V_i must consist of all possible gluings of sections from these two spaces, and is consequently uniquely determined.

However, this uniqueness is certainly not true in general, and the map $G^r_d \to D^{r,\mathrm{EH}}_d$ has higher-dimensional fibers in general. This poses the problem that, unlike in the Eisenbud-Harris setting, it is more difficult to understand crude limit series in terms of $\mathfrak{g}^r_d$'s on Y and Z, and this in turn complicates inductive arguments in cases where the crude series are included.

As an example of this, observe that Theorem 3.2.2 together with Eisenbud and Harris' generalized Brill-Noether theorem imply that the refined locus of G^r_d has pure dimension ρ when Y and Z are general (marked) curves. Loci of crude Eisenbud-Harris limit series will (subject to the generality hypothesis) only have smaller dimension. But since the fibers of the map $G^r_d \to G^{r,\mathrm{EH}}_d$ can have positive dimension, *a priori* we might have that G^r_d has dimension strictly bigger than ρ on the crude locus, even for a general curve. However, a careful analysis of the situation in [20] allows us to prove:

Theorem 3.2.3. *The dimension of the space of limit series corresponding to a given Eisenbud-Harris limit series $((\mathscr{L}^Y, V^Y), (\mathscr{L}^Z, V^Z))$ is bounded above by*

$$\sum_{i=0}^{r}(a^Y_i + a^Z_{r-i} - d).$$

This is exactly the bound needed to see that in fact all components of G^r_d on a general curve have dimension exactly ρ. It also allows us to make inductive statements on dimensions of spaces of linear series, and together with Theorem 3.1.2, yields a new proof of the generalized Brill-Noether theorem (Theorem 1.1.8) by inductively breaking off elliptic tails to reduce the genus, without any need for point-by-point argument requiring base change and blowing up.

In particular, we also conclude:

Corollary 3.2.4. *On a general curve of compact type with two components, every limit series, and not only refined limit series, may be smoothed to nearby smooth curves.*

We thus see that the compactification afforded by G^r_d on a general curve doesn't include any extraneous components, in the sense that every point is a limit on $\mathfrak{g}^r_d$'s on smooth curves. However, it is also natural to ask whether (again on a general curve) G^r_d may have components consisting entirely of crude limit series. It turns out that this is possible, and even typical, and in [10] Fu Liu precisely classified when this occurs, by reproving Theorem 3.2.3 in a way that allowed a criterion for exactly when the inequality is strict.

The original proof of Theorem 3.2.3 involved producing an explicit numerical formula for the dimension of the fibers of the forgetful map (3.1), but rather than using the actual formula, concluded the desired bounds from an indirect geometric argument. Liu's work analyzed the numerical fiber dimension formulas directly, simplifying them and allowing her to produce her sharper results.

We conclude with a discussion of a useful condition weaker than that of being refined:

Definition 3.2.5. A limit linear series $(\mathscr{L}^0, V_0, \dots, V_d)$ on X_0 is **exact** if for each $i < d$, every section in V_i vanishing on Z is in the image of V_{i+1}, and every section in V_{i+1} vanishing on Y is in the image of V_i.

It is easy to see by our discussion above that every refined limit series is exact, but the converse does not hold. Using her analysis, Liu was also able to show that although refined limit series are not dense in $G^r_d(X_0)$, we have:

Theorem 3.2.6. *If $X_0 = Y \cup Z$ has Y and Z general, the exact points are dense in $G^r_d(X_0)$.*

We conclude with an example of the difference in structure of G^r_d and $G^{r,\mathrm{EH}}_d$.

Example 3.2.7. Consider the case $g = 0$, $d = 2$, $r = 1$. Then Y and Z are each copies of $\mathbb{P}^1$, and we will assume they are glued at 0. We first consider the Eisenbud-Harris space. The line bundles $\mathscr{L}^Y$ and $\mathscr{L}^Z$ are necessarily isomorphic to $\mathscr{O}(2)$, so we can identify them both as $\mathscr{O}(2\infty)$ and write their sections as polynomials of degree at most 2. Thus, $G^{r,\mathrm{EH}}_d$ will be a closed subscheme of

$$G(2,3) \times G(2,3) \cong \mathbb{P}^2 \times \mathbb{P}^2.$$

The possibilities for vanishing sequences at 0 satisfying (1.2) with equality are $(0,1)$ on Y and $(1,2)$ on Z, $(0,2)$ on Y and $(0,2)$ on Z, and $(1,2)$ on Y and $(0,1)$ on Z. The possibilities with at least one strict inequality are then obtained by increasing one or both sequences from these possibilities. The space of pairs of $\mathfrak{g}^1_2$'s with vanishing at least $(0,1)$ on Y and $(1,2)$ on Z is described as follows: for V^Y, we can choose any 2-dimensional space of polynomials, while V^Z is uniquely determined as $\langle z, z^2 \rangle$. Thus we get a component $W_1 \cong \mathbb{P}^2 \times \mathbb{P}^0$. We similarly get $W_3 \cong \mathbb{P}^0 \times \mathbb{P}^2$ for the case of vanishing at least $(1,2)$ on Y and at least $(0,1)$ on Z. Finally, for the case $(0,2)$ on Y and $(0,2)$ on Z, we must have spaces of the form $\langle a_0 + a_1 y, y^2 \rangle$ on Y and $\langle b_0 + b_1 z, z^2 \rangle$, which each gives us a $\mathbb{P}^1$ of possibilities, so we get a component $W_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. We see that $W_2 \cap W_1 \cong \mathbb{P}^1 \times \mathbb{P}^0$, $W_2 \cap W_3 \cong \mathbb{P}^0 \times \mathbb{P}^1$, and $W_1 \cap W_3 \cong \mathbb{P}^0 \times \mathbb{P}^0$. These intersections correspond precisely to crude limit series.

We now consider the case of G^r_d. Here it matters (at least for notational purposes) precisely what gluings we choose along the nodes, so if we have a polynomial vanishing to order at least i on Y, and at least $d-i$ on Z, to glue them to obtain a section of V_i we will require simply that the ith coefficient of the former agrees with the $(d-i)$th of the latter. We know that G^r_d will be isomorphic to $G^{r,\mathrm{EH}}_d$ along refined limit series, and this is easy to verify: for instance, if we have the Eisenbud-Harris series $(\langle 1 + a_1 y, y^2 \rangle, \langle 1 + b_1 z, z^2 \rangle)$, we get $V_0 = \langle (1 + a_1 y, z^2), (y^2, 0) \rangle$ and $V_2 = \langle (y^2, 1 + b_1 z), (0, z^2) \rangle$, noting that the z^2 is required in V_0 in order to glue to $1 + a_1 y$, and similarly for y^2 in V_2. Now, since we must have vanishing order at least 1 on both Y and Z for V_1, we see that we must have $V_1 = \langle (y^2, 0), (0, z^2) \rangle$.

Moving on to the crude locus, we see that V_1 is still uniquely determined for crude limit series in codimension 1 (in fact, this is a general phenomenon). For instance, if we have $(\langle 1 + a_1 y, y^2 \rangle, \langle z, z^2 \rangle)$, we get $V_0 = \langle (1 + a_1 y, z^2), (y^2, 0) \rangle$ and $V_2 = \langle (0, z), (0, z^2) \rangle$, and then we are again forced to take $V_1 = \langle (y^2, 0), (0, z^2) \rangle$. Note here that although we have z vanishing to order 1 on Z, which could therefore in principle appear in V_1, it has a nontrivial gluing requirement, and since we have no sections vanishing to order 1 on Y, it cannot appear.

Thus, the only interesting case is the point $(\langle y, y^2\rangle, \langle z, z^2\rangle)$. Here, we get $V_0 = \langle (y,0),(y^2,0)\rangle$ and $V_2 = \langle (0,z),(0,z^2)\rangle$, but for V_1 we can take any 2-dimensional subspace of the 3-dimensional space of pairs of the form $(a_0 y + a_1 y^2, a_0 z + a_2 z^2)$, so we find that the single point of $G^{r,\mathrm{EH}}_d$ is replaced by a new component $W_4 \cong \mathbb{P}^2$ in G^r_d.

This new component intersects W_1 and W_3 in distinct lines: for instance, for $W_1 \cap W_4$, on the refined locus of W_1 we had $V_0 = \langle (1+a_0y^2, z^2), (y+a_1y^2, 0)\rangle$, $V_1 = \langle (y+a_1y^2, z), (0,z^2)\rangle$ and $V_2 = \langle (0,z),(0,z^2)\rangle$ some for a_1, a_2. If a point of W_4 is in the limit of spaces of this form, we see that we have V_1 of the above form, which gives a line of possibilities. Note that by rescaling, the point with $V_1 = \langle (y^2,0),(0,z^2)\rangle$ is also in the limit; in fact, this point is the intersection of all four W_i. However, we see that $W_2 \cap W_4$ is just a single point, since V_1 is uniquely determined as $\langle (y^2,0),(0,z^2)\rangle$ for any point of W_2. Thus, the overall picture is that W_1 and W_3 have been blown up at a point and intersect W_4 along the exceptional divisor, and we now have four components, each intersecting two others along distinct lines, and all four intersecting at a single point.

Note that in this example, both $G^{r,\mathrm{EH}}_d$ and G^r_d arise as flat degenerations of $\mathbb{P}^2$, which is the space of $\mathfrak{g}^r_d$'s on a smooth curve of genus 0.

Remark 3.2.8. The constructions discussed here are designed to work in any characteristic, and in mixed characteristic, without any restriction. Of course, it is still true that the space is not proper if one restricts to the open locus of separable linear series, and this still poses difficulties for certain degeneration arguments. In particular, the generalized Brill-Noether theorem and results such as Corollary 3.2.4 and Theorem 3.2.6 which rely on it are not known in positive characteristic.

4. Constructing the spaces

We now discuss how to construct limit linear series spaces. We start by recalling the situation for linear series on smooth curves, then discuss a related construction, called the linked Grassmannian, which for limit linear series spaces will play the role that the classical Grassmannian plays in the case of smooth curves. Finally, we discuss the construction.

4.1. The classical space.

We now describe a construction of spaces of linear series on smooth curves, and see where the number ρ comes from. Let X be a smooth projective curves of genus g. Given, r, d, we want to construct a space $G^r_d(X)$ which is a moduli space for linear series of dimension r and degree d on X.

As a warmup, we consider the case (uninteresting from the point of view of Brill-Noether theory) that $d > 2g-2$. In this case, every line bundle of degree d has a $(d+1-g)$-dimensional space of global sections, so assuming $r+1 \leqslant d+1-g$ we have a g-dimensional space of choices of the line bundle underlying our $\mathfrak{g}^r_d$, and an $(r+1)(d+1-g-(r+1))$-dimensional Grassmannian of choices for the $(r+1)$-dimensional space of global sections. This gives us a total dimension of

$$g + (r+1)(d-g-r) = (r+1)(d-r) - rg = \rho.$$

In this case, we can formalize the construction as follows: let $\mathrm{Pic}^d(X)$ be the Picard variety of degree-d line bundles on X, which is smooth and projective (indeed, an abelian variety) of dimension g. Let $\widetilde{\mathscr{L}}$ be the Poincare line bundle on $\mathrm{Pic}^d(X) \times X$ (considered as an invertible sheaf). Then because d is large enough, $p_{1*}\widetilde{\mathscr{L}}$ is locally free of rank $d+1-g$ on $\mathrm{Pic}^d(X)$. Moreover, we claim that the relative Grassmannian $G(r+1, p_{1*}\widetilde{\mathscr{L}})$ precisely parametrizes linear series on X. Given the claim, the Grassmannian is smooth and projective of relative dimension $(r+1)(d+1-g-(r+1))$ over $\mathrm{Pic}^d(X)$, so we conclude the desired dimension statement, and also see that $G^r_d(X)$ is smooth and projective in this case.

Now we consider the claim. The space $G^r_d(X)$ is supposed to parametrize a choice of a line bundle $\mathscr{L}$ together with an $(r+1)$-dimensional space of global sections. We can rephrase this as saying that it parametrizes a point $P \in \mathrm{Pic}^d(\mathscr{L})$ together with an $(r+1)$-dimensional subspace of

$$\Gamma(X, \widetilde{\mathscr{L}_P}) = \pi_*(\widetilde{\mathscr{L}_P}),$$

where $\pi : X \to \operatorname{Spec} F$ is the structure map, and $\widetilde{\mathscr{L}_P}$ is the line bundle corresponding to P, which we can think of as the fiber of $\widetilde{\mathscr{L}}$ over the point P. On the other hand, a point of the Grassmannian $G(r+1, p_{1*}\widetilde{\mathscr{L}})$ corresponds to a point P of $\mathrm{Pic}^d(X)$ together with an $(r+1)$-dimensional subspace of the fiber of $p_{1*}\widetilde{\mathscr{L}}$ at P. Thus, we see that for G^r_d we are taking the pushforward of a fiber, while for the Grassmannian we are taking the fiber of a pushforward. These are not the same in general (for instance, if a general choice of $\mathscr{L}$ has no nonzero global sections, then $p_{1*}\widetilde{\mathscr{L}} = 0$, but some special line bundles may still have nonzero global sections). However, in our situation they agree by the theory of cohomology and base change, because $d > 2g-2$ and hence the higher derived pushforwards necessarily vanish.

We now move on to the general construction. We still start with $\mathrm{Pic}^d(X)$ and the line bundle $\widetilde{\mathscr{L}}$. However, for the reasons discussed above, we cannot work directly with $p_{1*}\widetilde{\mathscr{L}}$. Instead, we let D be an effective divisor on X such that $d + \deg D > 2g-2$. Let $D' = p_2^* D$ on $\mathrm{Pic}^d(X) \times X$. Then we consider the relative Grassmannian $G(r+1, p_{1*}\widetilde{\mathscr{L}}(D'))$, which is smooth and projective, this time of (total) dimension

$$g + (r+1)(d + \deg D + 1 - g - (r+1)).$$

As above, by cohomology and base change and our hypothesis that $d + \deg D > 2g-2$, we see that this parametrizes a choice of line bundle $\mathscr{L}$ of degree d together with an $(r+1)$-dimensional space of global sections not of $\mathscr{L}$, but of $\mathscr{L}(D)$. We will describe $G^r_d(X)$ as a closed subscheme of this Grassmannian, as follows. Let $\mathscr{F} \subseteq q^* p_{1*}\widetilde{\mathscr{L}}(D')$ be the universal subbundle on $G(r+1, p_{1*}\widetilde{\mathscr{L}}(D'))$, where q is the structure morphism to $\mathrm{Pic}^d(X)$. We are interested in the fibers of $\mathscr{F}$ which lie inside $\widetilde{\mathscr{L}}$. Using the exact sequence

$$0 \to \widetilde{\mathscr{L}} \to \widetilde{\mathscr{L}}(D') \to \widetilde{\mathscr{L}}(D')|_{D'} \to 0,$$

we see that $G^r_d(X)$ is precisely given as the vanishing locus of the composed map

$$\mathscr{F} \to \widetilde{\mathscr{L}}(D')|_{D'}.$$

Since $\mathscr{F}$ is by definition locally free of rank $r+1$, and $\widetilde{\mathscr{L}}(D')|_{D'}$ is locally free of rank $\deg D'$, since it is the pushforward of a sheaf with discrete support along D', we see that the codimension of (each component of) $G^r_d(X)$ in the Grassmannian is at most $(r+1)(\deg D)$. We finally conclude that $G^r_d(X)$ is a projective scheme, with every component having dimension at least

$$g + (r+1)(d + \deg D + 1 - g - (r+1)) - (r+1)(\deg D) = \rho,$$

as desired. Note that $G^r_d(X)$ is not smooth in general, but we do at least see that if it has dimension ρ, it is a locally complete intersection.

4.2. **Linked Grassmannians and linked Hom spaces.** In order to generalize the construction of linear series to limit linear series, there are two points to address. The more substantive is that we need to replace the Grassmannian with a suitable substitute, the linked Grassmannian. It is also not immediately obvious how to impose vanishing along the divisor D without imposing too many conditions. In [21] this was addressed in a slightly *ad hoc* manner following Eisenbud and Harris, but we will give a more transparent (and more general) construction following [16].

Linked Grassmannians play the role of Grassmannians, and their theory is indispensible to carrying out the dimension count of Theorem 3.1.2, which in turn is precisely what is required for smoothing arguments. In defining linked Grassmannians, instead of looking at subbundles of a single vector bundle, we will consider a chain of vector bundles linked by maps between them, and look at the space parametrizing chains of linked subbundles.

The basic situation we will consider is the following:

Definition 4.2.1. Let S be an integral Cohen-Macaulay scheme, and $n, d > 1$. Let $\mathscr{E}_\bullet$ consist of vector bundles $\mathscr{E}_1, \dots, \mathscr{E}_n$ on S of rank d, with morphisms $f_i : \mathscr{E}_i \to \mathscr{E}_{i+1}$ and $f^i : \mathscr{E}_{i+1} \to \mathscr{E}_i$. Given $s \in \Gamma(S, \mathscr{O}_S)$, we say that $\mathscr{E}_\bullet$ is **s-linked** if the following conditions are satisfied:

(I) For each i,

$$f_i \circ f^i = s\,\mathrm{id}, \quad f^i \circ f_i = s\,\mathrm{id}\,.$$

(II) On the fibers of the $\mathscr{E}_i$ at any point with $s = 0$, we have that for each i,

$$\ker f^i = \mathrm{im}\, f_i, \quad \ker f_i = \mathrm{im}\, f^i.$$

(III) On the fibers of the $\mathscr{E}_i$ at any point with $s = 0$, we have that for each i,

$$\mathrm{im}\, f_i \cap \ker f_{i+1} = (0), \quad \mathrm{im}\, f^{i+1} \cap \ker f^i = (0).$$

We say that $\mathscr{E}_\bullet$ is **weakly s-linked** if conditions (I) and (III) are satisfied.

Note that we do not have $\mathscr{E}_\bullet$ a complex; in fact, condition (III) imposes quite the opposite. Note also that the condition of being weakly s-linked is preserved by passing to tuple of subbundles (as long as they all map into one another), but the condition of being s-linked is not necessarily preserved. We then define the linked Grassmannian as follows:

Definition 4.2.2. Given S and $\mathscr{E}_\bullet$ as in Definition 4.2.1, with $\mathscr{E}_\bullet$ s-linked, and $r < d$, the **linked Grassmannian** $LG(r, \mathscr{E}_\bullet)$ is the S-scheme representing the functor which associates to T/S the set of tuples of subbundles

$$\mathscr{F}_i \subseteq \mathscr{E}_i|_T$$

of rank r such that

$$f_i \mathscr{F}_i \subseteq \mathscr{F}_{i+1}, \quad f^i \mathscr{F}_{i+1} \subseteq \mathscr{F}_i$$

for each i.

It is clear that $LG(r, \mathscr{E}_\bullet)$ exists, as it can be cut out (even without the s-linking condition) as a closed subscheme of the Grassmannian product

$$G(r, \mathscr{E}_1) \times_S \cdots \times_S G(r, \mathscr{E}_n).$$

The importance of the s-linking condition is that it imposes very special behavior on the linked Grassmannian. Note that at fibers where $s \neq 0$, the f_i and f^i are isomorphisms, and any V_i uniquely determines the rest, so the linked Grassmannian is isomorphic to a standard Grassmannian. In general, we have from [19]:

Theorem 4.2.3. *A linked Grassmannian has pure relative dimension $r(d - r)$ over S, and its smooth points are dense in every fiber. In particular, every component dominates S.*

Theorem 3.1.3 is an easy consequence of the following further work with David Helm [9]:

Theorem 4.2.4. *If S is regular, any linked Grassmannian is flat over S, reduced and Cohen-Macaulay.*

Thus, linked Grassmannians provide flat degenerations of Grassmannians, and may therefore be interesting from the point of view of studying Grassmannians via degenerations. In addition, they are related to local models of certain Shimura varieties, which are described similarly except that there is a ring of vector bundles with maps going in only one direction. The two cases overlap when there are only two vector bundles.

We briefly discuss the proofs of the theorems. For the first, the key idea is the following definition:

Definition 4.2.5. A T-valued point $\mathscr{F}_\bullet$ of $LG(r, \mathscr{E}_\bullet)$ is an **exact point** if at every point of T with $s|_T = 0$, we have on the fibers of the $\mathscr{F}_i$ for each i that

$$\ker f^i = \operatorname{im} f_i, \quad \ker f_i = \operatorname{im} f^i.$$

Observe that this is equivalent to asking that $\mathscr{F}_\bullet$ be s-linked, since $\mathscr{F}_\bullet$ is always weakly s-linked. It is easy to see that the exact points form an open subscheme of $LG(r, \mathscr{E}_\bullet)$. The key results are then the following:

Proposition 4.2.6. *The exact points are smooth points of $LG(r, \mathscr{E}_\bullet)$ over S, and the relative tangent space dimension at an exact point is $r(d-r)$. Furthermore, the exact points are dense in every fiber of $LG(r, \mathscr{E}_\bullet)$.*

This immediately proves Theorem 4.2.3. Similar arguments show that in fact the non-exact points always lie in the intersection of components of the corresponding fiber of $LG(r, \mathscr{E}_\bullet)$, so we conclude that the components are smooth away from their intersection. However, examples show that even the individual components need not be smooth where they intersect other components.

For the proof of Theorem 4.2.4, the key point turns out to be to prove that fibers are Cohen-Macaulay. An induction arguments reduces to the case $n = 2$, and then explicit study of the local behavior of the linked Grassmannian gives a description in terms of matrix spaces which had already been proved by Zhaohui Zhang [24] to be Cohen-Macaulay, using Hodge algebra techniques introduced by de Concini, Eisenbud, and Procesi.

Returning to the linear series construction, in order to impose vanishing along the divisor D we develop linked Hom spaces, which are in spirit a Hom analogue of linked Grassmannians. We make the following definition:

Definition 4.2.7. Suppose that $\mathscr{E}_\bullet$ is s-linked. We say that $\mathscr{E}_\bullet$ is **rigidly** s-linked if on the fibers of the $\mathscr{E}_i$ at any point with $s = 0$, we have that for each $i = 2, \dots, n-1$,

$$\operatorname{im} f_i \oplus \ker f_{i+1} = \mathscr{E}_{i+1} = \operatorname{im} f^{i+1} \oplus \ker f^i.$$

Note that condition (III) of s-linkage is equivalent to requiring that $\operatorname{im} f_i \oplus \ker f_{i+1} \hookrightarrow \mathscr{E}_{i+1}$ and similarly for $\operatorname{im} f^{i+1} \oplus \ker f^i$, so the rigid linkage condition is a stronger version of this condition. We remark that this condition implies that $\operatorname{rk} f_i$ is constant for all i, which motivates the terminology.

Definition 4.2.8. Suppose that $\mathscr{F}_\bullet$ has maps f_i, f^i satisfying condition (I) of s-linkage, and $\mathscr{G}_\bullet$ is rigidly s-linked with maps g_i, g^i. The **linked Hom space** $LH(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$ is the scheme parametrized tuples of morphisms $\varphi_i : \mathscr{F}_i \to \mathscr{G}_i$ which commute with the f_i, g_i, f^i, g^i.

Note that it is easy to see that the linked Hom space exists, and without the linkage conditions. However, the main point from our perspective is the following theorem, which is proved rather easily:

Theorem 4.2.9. *Suppose that $\mathscr{F}_\bullet$ has rank r and satisfies condition (I) of s-linkage, and $\mathscr{G}_\bullet$ is rigidly s-linked of rank m. Then $LH(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$ is a vector bundle on S of rank rm, and the forgetful map realizes it as a subbundle of $\mathscr{H}om(\mathscr{F}_1, \mathcal{G}_1) \oplus \mathscr{H}om(\mathscr{F}_n, \mathcal{G}_n)$.*

4.3. **Constructing limit linear series spaces.** We can now describe the limit linear series construction. We remark that the construction of $G^r_d(X)$ discussed above works equally well for families of curves, and this point is crucial to degeneration arguments. The basic idea for the limit case is to replace the relative Grassmannian with a linked Grassmannian. First, we replace Pic^d with the relative $\operatorname{Pic}^{d,0}$ consisting of line bundles of degree d, having degree d on restriction to Y and degree 0 on restriction to Z. Note that the twistings by $\mathscr{O}_X(Y)$ and $\mathscr{O}_X(Z)$ induce canonical isomorphisms between the Picard schemes of different multidegrees, so if we let $\mathscr{L}^i$ be the universal line bundle of multidegree $(i, d-i)$, we can consider $p_{1*}\mathscr{L}^i(D')$ to be on $\operatorname{Pic}^{d,0}$ for all i. We thus take these as our chain of vector bundles, with the maps being the canonical ones induced by twisting, as in the definition of the $\mathcal{G}^r_d$ functor.

Because Theorem 4.2.3 implies that the dimension behavior of LG is the same as that of the classical Grassmannian, the construction almost goes through exactly as before. The one additional complication is that to impose vanishing along D', *a priori* we have to impose it separately for each subbundle in the chain, which gives far too many conditions. In [4] and [19], this is dealt with in a simple but somewhat *ad hoc* manner, by assuming we can write $D = D_Y + D_Z$ where D_Y and D_Z specialize only to Y and Z, respectively. Such a D is constructed trivially if we assume we have sections specializing to Y and to Z, but otherwise its existence is not at all obvious. We will instead use the linked Hom space construction to see that any D can be used. The observation is that if $\mathscr{F}_\bullet$ is the universal subbundle on LG, then our G^r_d space is cut out precisely by the condition that for all i, the composed map

$$\mathscr{F}_i \to p_{1*}\mathscr{L}^i(D') \to p_{1*}(\mathscr{L}^i(D')|_{D'})$$

be equal to zero. Moreover, the $p_{1*}(\mathscr{L}^i(D')|_{D'})$ form a rigidly s-linked chain of bundles, of rank equal to $\deg D'$. Thus, our composed map gives a morphism from LG to a linked Hom space, and the space G^r_d is precisely the preimage of the zero section. By Theorem 4.2.9, this is cut out by $(r+1)(\deg D)$ conditions, giving the desired dimension bound and completing the construction.

As compared to the construction in [19], we are thus able to weaken our hypotheses on the families of curves for which we make our construction, as we do not have to assume that they carry a divisor D of the special form $D_Y + D_Z$ as above.

5. Fibers of generalized Abel maps

In the classical situation of smooth projective curves, there are two aspects of the theory of linear series which are extremely important, but which have not played a role in our discussion of limit linear series: complete linear series, and families of effective divisors associated to linear series. Given a line bundle $\mathscr{L}$ on a smooth projective curve X, as long as $\mathscr{L}$ has nonzero global sections we always have the associated complete linear series $(\mathscr{L}, \Gamma(X, \mathscr{L}))$ which takes all global sections of $\mathscr{L}$. On the other hand, given a $\mathfrak{g}^r_d$ $(\mathscr{L}, V)$ on X, we have the associated family of effective divisors obtained by considering $\operatorname{div} s$ for all nonzero $s \in V$; these form an r-dimensional projective space, since s is uniquely determined by $\operatorname{div} s$ up to scaling. For the same reason, we can recover V uniquely from the associated family of divisors. Both of these ideas are related to fibers of the Abel map.

5.1. Abel maps and their fibers.

These two concepts are related by means of the fibers of the **Abel map**

$$A_d : S^d(X) \to \operatorname{Pic}^d(X)$$

constructed by sending a point $(P_1, \dots, P_d)$ to $\mathscr{O}(P_1 + \cdots + P_d)$. For historical reasons, this map is often constructed to the Jacobian via the choice of a base point, but we will find it convenient to use the above canonical form.

Note that we can consider the points of $S^d(X)$ to correspond to effective divisors on X of degree d. Then the complete linear series associated to a line bundle $\mathscr{L}$ of degree d arises from the family of divisors appearing in the fiber of A_d over $\mathscr{L}$; in particular, the fibers of A_d are all projective spaces (of varying dimension). On the other hand, given $(\mathscr{L}, V)$, we can think of the above description of the associated family of effective divisors as giving us an associated subset of $S^d(X)$, which is necessarily a linear subspace of the fiber $A_d^{-1}\mathscr{L}$. This construction behaves well in families, so that if we have a family of $\mathfrak{g}^r_d$'s on a smooth curve (or a family of smooth curves), we obtain a flat family of closed subschemes of the fibers of the associated Abel maps.

In [2], Coelho and Pacini construct Abel maps

$$A_d : S^d(X) \to \mathrm{Pic}^d(X)$$

where X is a curve of compact type, and the image is contained inside a particular multidegree of Pic^d. We will consider again the case that $X = Y \cup Z$, with $Y \cap Z = \{P\}$ the only node. From our point of view, instead of working in a particular multidegree it will be convenient to work as above, identifying all the different $\mathrm{Pic}^{i,d-i}(X)$ via twisting up and down at P. This makes it very clear how to define A_d in our case: given a point of $S^d(X)$, write it as $D_Y + D_Z$ for some effective divisors on Y and Z with degree summing to d, and choose the associated line bundle to be $\mathscr{O}_Y(D_Y)$ on Y and $\mathscr{O}_Z(D_Z)$ on Z. The only ambiguity is if D_Y or D_Z contain P (possibly with multiplicity), it may be moved between Y and Z, changing the multidegree. But this twisting is precisely what we have modded out by, so does not change the result.

The fibers of A_d are easy to describe: a point of our reduced Pic^d scheme has a unique representative $\mathscr{L}$ having degree d on Y and degree 0 on Z; then write $\mathscr{L}^Y := \mathscr{L}|_Y$, and $\mathscr{L}^Z := \mathscr{L}|_Z(dP)$. Also set $\Gamma_i^Y = \Gamma(Y, \mathscr{L}^Y(-iP))$ and $\Gamma_i^Z = \Gamma(Z, \mathscr{L}^Z(-iP))$. The corresponding fiber of A_d is then simply

$$\left(\mathbb{P}\Gamma_0^Y \times \mathbb{P}\Gamma_d^Z\right) \cup \left(\mathbb{P}\Gamma_1^Y \times \mathbb{P}\Gamma_{d-1}^Z\right) \cup \cdots \cup \left(\mathbb{P}\Gamma_d^Y \times \mathbb{P}\Gamma_0^Z\right) \subseteq \mathbb{P}\Gamma_0^Y \times \mathbb{P}\Gamma_0^Z.$$

However, despite being easy to describe, the fibers are not especially well behaved. For instance, they frequently have components of different dimensions, and do not look at all like flat limit of fibers of Abel maps on smooth curves.

5.2. **Limit linear series and fibers of Abel maps.** Eduardo Esteves and I have been investigating the relationship between limit linear series and fibers of Abel maps in this context. In analogy with the situation for smooth curves, we expect that a limit linear series should yield a subscheme of a fiber of A_d. Given an Eisenbud-Harris limit series $(\mathscr{L}^Y, V^Y), (\mathscr{L}^Z, V^Z)$, it is not obvious how to define such an associated subscheme. The only reasonable approach appears to be as follows: for each i, and each $s_Y \in V^Y$ vanishing to order at least i at P, and $s_Z \in V^Z$ vanishing to order at least $d - i$ at P, then take the point

$$\operatorname{div} s_Y - iP + \operatorname{div} s_Z - (d - i)P.$$

One might also require that either s_Y and s_Z both vanish to order exactly i and $d-i$ respectively at P, or both vanish to strictly higher order. This is precisely the condition implying that some nonzero scalar multiples of them can be glued together into a section of $(\mathscr{L}^Y(-iP), \mathscr{L}^Z(-(d-i)P)$. In the latter case, the behavior will be good for refined limit series, but not for crude ones, where we may again get non-equidimensional schemes.

On the other hand, if $(\mathscr{L}^0, V_0, \ldots, V_d)$ is an exact limit series on X, we can define an associated subscheme of the associated fiber of A_d as follows:

Definition 5.2.1. Suppose that $(\mathscr{L}^0, V_0, \ldots, V_d)$ is an exact limit series on X of dimension r and degree d. Denote by $V_i^{Y,0}$ the kernel of $V_i \to V_i|_Y$, and similarly for $V_i^{Z,0}$. Then let $\mathbb{P}((\mathscr{L}^i, V_i)_i)$ be the Zariski closure of the subset of $A_d^{-1}(\mathscr{L}^0)$ consisting of points of the form

$$\operatorname{div}(s|_Y) + \operatorname{div}(s|_Z) \in S^d(X),$$

where $s \in V_i \smallsetminus V_i^{Y,0} cup V_i^{Z,0}$ for some i.

One does not need exactness to make this definition, but it turns out that it is precisely the exact limit series for which it is well behaved. Specifically, we have the following result:

Theorem 5.2.2. *If $(\mathscr{L}^0, V_0, \ldots, V_d)$ is an exact limit series on X of dimension r and degree d, then $\mathbb{P}(\mathscr{L}^0, V_0, \ldots, V_d)$ has pure dimension r in $A_d^{-1}(\mathscr{L}^0)$. If we give it the induced reduced* *subscheme structure, it is Cohen-Macaulay, with the same Hilbert polynomial as the projective space $\mathbb{P}^r$. Furthermore, it is a flat degeneration of $\mathbb{P}^r$.*

We also show that this construction produces flat limits of linear series inside fibers of Abel maps:

Theorem 5.2.3. *If we have a one-parameter family X/B, with smooth generic fiber X_η, and limit X_0 reducible as above, and if $(\mathscr{L}^0, V_0, \ldots, V_d)$ on X_0 is the limit of a linear series $(\mathscr{L}_\eta, V_\eta)$ on X_η, then the union of $\mathbb{P}(V_\eta)$ in the generic fiber and $\mathbb{P}((\mathscr{L}^0, V_0, \ldots, V_d))$ in the closed fiber gives a closed subscheme of $S^d(X/B)$ which is flat over B.*

Note that the last assertion of Theorem 5.2.2 doesn't follow from Theorem 5.2.3 because not every limit linear series on an arbitrary curve is a limit of linear series on smooth curves.

Finally, we see that fibers of Abel maps can tell us something about limit linear series. Indeed, a simple numerical argument shows:

Proposition 5.2.4. *Given $\mathscr{L}^0$ of multidegree $(d, 0)$ on $X = Y \cup Z$, suppose that $A_d^{-1}(\mathscr{L}^0)$ has some irreducible component with dimension strictly less than r. Then there is no limit $\mathfrak{g}^r_d$ on X with underlying line bundle $\mathscr{L}^0$.*

6. Higher-rank Brill-Noether theory

As Brill-Noether theory studies pairs consisting of a line bundle with a vector space of global sections, higher-rank Brill-Noether theory studies pairs of a vector bundle together with a space of global sections. The line bundle case is closely related to maps to projective space, while the vector bundle case relates more generally to maps to Grassmannians.

6.1. **Expected dimension and its discontents.** As in the classical case, we can construct moduli spaces of such pairs. In the higher-rank case, it is convenient to work with the moduli stack $\mathscr{M}_{r,d}(X)$ of vector bundles of rank r and degree d on X, but in what follows one may restrict to stable bundles and work instead with coarse moduli spaces. Applying the same construction as in the classical case, and shifting our notation, we have:

Proposition 6.1.1. *Given r, d, k and a smooth projective curve of genus g, there is a moduli stack $\mathcal{G}^k_{r,d}(X)$ of pairs $(\mathscr{E}, V)$, where $\mathscr{E}$ is a vector bundle of rank r and degree d on X, and $V \subseteq \Gamma(X, \mathscr{E})$ is a k-dimensional space of global sections. Every component of $\mathcal{G}^k_{r,d}(X)$ has dimension at least*

$$\rho - 1 := r^2(g-1) - k(k - d + r(g-1)).$$

Note that the -1 comes from taking stack-theoretic dimension; in the context of coarse moduli spaces we instead get ρ. One can then ask all the same questions as in classical Brill-Noether theory: is the space $\mathcal{G}^k_{r,d}(X)$ nonempty when $\rho \geqslant 0$, and does it have dimension $\rho - 1$ when X is general? The answer to both questions turns out to be no in general. We emphasize that it is not the case that ρ is simply wrong: there are many cases when $\mathcal{G}^k_{r,d}(X)$ is known to have components of dimension $\rho - 1$. However, the behavior of $\mathcal{G}^k_{r,d}(X)$ is simply more complicated than in the classical case: it can be empty when $\rho \geqslant 0$, and it can have multiple components of different dimensions. Despite many papers on the subject giving partial results, we do not have a comprehensive conjecture even for the case $r = 2$. We will therefore focus our attention on this case.

6.2. **Rank 2 and special determinants.** The two phenomena which are known to create components of $\mathcal{G}^k_{2,d}(X)$ of dimension greater than $\rho - 1$ even when X is general are very unstable bundles, and bundles with special determinants. Since the first situation can be easily avoided by imposing suitable stability conditions, we focus our attention on the second.

The following is an alternate construction of $\mathcal{G}^k_{r,d}(X)$ which turns out to be important in understanding the case of special determinants. Let $\tilde{\mathscr{E}}$ be the universal bundle on $\mathscr{M}_{r,d}(X) \times X$. Let D be a sufficiently ample divisor on X, and let $D' = p_2^* D$ be the pullback divisor on $\mathscr{M}_{r,d}(X) \times C$. As in the earlier construction, we have that $p_{1*}\tilde{\mathscr{E}}(D')$ is a vector bundle of rank $d + r \deg D + r(1-g)$. We start with the Grassmannian $G(k, p_{1*}\tilde{\mathscr{E}}(D'))$ over $\mathscr{M}_{r,d}(X)$, which has dimension $r^2(g-1) + k(d + r\deg D + r(1-g) - k)$, since $\mathscr{M}_{r,d}(X)$ has dimension $r^2(g-1)$.

We can cut out the space of coherent systems in this Grassmannians by imposing that our subbundles in fact lie in $p_{1*}\tilde{\mathscr{E}}$. We do this differently from before, as follows: observe that $p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D'))$ and $p_{1*}(\tilde{\mathscr{E}}/\tilde{\mathscr{E}}(-D'))$ are vector bundles on $\mathscr{M}_{r,d}(X)$ of rank $2r\deg D$ and $r \deg D$ respectively, and the latter is a subbundle of the former. Moreover, because D is sufficiently ample, $p_{1*}\tilde{\mathscr{E}}(D')$ is a subbundle of $p_{1*}\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D')$. We see that a subbundle of $p_{1*}\tilde{\mathscr{E}}(D')$ is contained in $p_{1*}\tilde{\mathscr{E}}$ if and only its image in $p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D'))$ is contained in $p_{1*}(\tilde{\mathscr{E}}/\tilde{\mathscr{E}}(-D'))$.

Now, the universal subbundle induces a map to the Grassmannian $G(k, p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D')))$, while the pullback of $p_{1*}(\tilde{\mathscr{E}}/\tilde{\mathscr{E}}(-D'))$ induces a map to the Grassmannian $G(r\deg D, p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D')))$. We thus have a map

$$G(k, p_{1*}\tilde{\mathscr{E}}(D')) \to G(k, p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D'))) \times_{\mathscr{M}_{r,d}(X)} G(r\deg D, p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D'))),$$

and we can rephrase the above discussion by saying that the space $\mathcal{G}^k_{r,d}(X)$ is precisely the preimage of the incidence correspondence, which has codimension $kr\deg D$. We thus conclude (an alternate proof of) Proposition 6.1.1.

Given $\mathscr{L} \in \operatorname{Pic}^d(X)$, we can take instead the stack $\mathcal{G}^k_{r,\mathscr{L}}(X)$ of pairs $(\mathscr{E}, V)$ as before, except that $\mathscr{E}$ is required to have fixed determinant $\mathscr{L}$ (technically, a choice of this isomorphism is part of the data of the pair). The same construction works in this context, and gives lower bound on dimension of $\rho - g$.

But now suppose that we are in the case $r = 2$, and we consider the case $\mathscr{L} = \omega$, the canonical line bundle on X. Bertram, Feinberg and Mukai saw in [1] and [13] that in this case, we actually have a larger expected dimension, which they explained as follows: given an isomorphism

$$\varphi : \bigwedge^2 \tilde{\mathscr{E}} \xrightarrow{\sim} \omega,$$

we can place a symplectic form on $p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D'))$ by choosing local sections, applying φ, and summing over residues at each point of D. Moreover, $p_{1*}(\tilde{\mathscr{E}}/\tilde{\mathscr{E}}(-D'))$ is an isotropic subbundle because the resulting local forms will not have poles, and $p_{1*}(\tilde{\mathscr{E}}(D'))$ is an isotropic subbundle because in this case the forms are global, so we can apply the residue theorem. Thus, instead of mapping $G(k, p_{1*}\tilde{\mathscr{E}}(D'))$ to a product of Grassmannians, we can map it to a product of symplectic Grassmannians (i.e., spaces which parametrize isotropic subbundles for the given symplectic form), and the relevant incidence correspondence then has the smaller codimension $kr\deg D - \binom{k}{2}$. We conclude that in this case, the space $\mathcal{G}^k_{2,\omega}(X)$ actually has dimension at least $\rho - g + \binom{k}{2}$.

These ideas are generalized to other determinants $\mathscr{L}$ in [15], for instance for $\mathscr{L}$ with $h^1(X, \mathscr{L}) \geqslant 2$. However, more general, and possibly even comprehensive, results should be possible along these lines.

6.3. **Symplectic linked Grassmannians.** In order to apply degeneration arguments to prove existence of vector bundles of rank 2 with canonical determinant and k-dimensional spaces of sections, it becomes necessary to figure out how to combine the limit linear series dimension arguments with the modified expected dimensions discussed above. Montserrat Teixidor i Bigas has generalized Eisenbud-Harris limit linear series to the higher-rank case, but it is very unclear how to obtain the appropriate dimension counts for special determinants in this context. Teixidor and I have been discussing how to apply the ideas of the limit linear series construction of [19] to surmount this obstacle, and while our work is still ongoing, the main ideas are described below.

We wish to investigate the analogue of a symplectic Grassmannian in the context of linked Grassmannians. One could of course simply put symplectic forms on each vector bundle in the chain and require each subspace to be isotropic for the corresponding form, but it turns out that this does not give good behavior – the dimension of fibers with $s = 0$ will in general be too large. Put another way, if S is the spectrum of a DVR and s a uniformizer, then over the generic fiber the linked Grassmannian is isomorphic to the classical Grassmannian, and any sensible definition of symplectic linked Grassmannian will recover the classical symplectic Grassmannian. Then the condition that each subbundle be isotropic is not strong enough to isolate the tuples of subspaces in the special fiber coming from the usual symplectic Grassmannian in the generic fiber.

It looks like the correct idea is to consider not only forms on each individual space, but pairings $\langle , \rangle_{i,j}$ between each pair $\mathscr{E}_i, \mathscr{E}_j$, satisfying certain natural compatibility conditions. Then the subbundles $\mathscr{F}_i$ are required not only to be isotropic, but also orthogonal to one another under the pairings. This cuts out more conditions, and appears to give the desired behavior, at least on the smooth locus of the linked Grassmannian, which is enough for our applications.

We begin by defining the analogue of an alternating form:

Definition 6.3.1. Given $\mathscr{E}_\bullet$ s-linked of rank k, with maps $f_\bullet, f^\bullet$, and $m \in \frac{1}{2}\mathbb{Z}$ between 1 and n, a **linked alternating form** $\langle , \rangle_\bullet$ on $\mathscr{E}_\bullet$ of index m consists of a bilinear pairing

$$\langle , \rangle_{i,j} : \mathscr{E}_i \times \mathscr{E}_j \to \mathscr{O}_S$$

for each i, j satisfying the following conditions:

(I) For i, j between 2 and n, we have

$$\langle , \rangle_{i,j} \circ (f_{i-1} \times \mathrm{id}) = s^{\epsilon_{i,j}} \langle , \rangle_{i-1,j}\,, \text{ and}$$
$$\langle , \rangle_{i,j} \circ (\mathrm{id} \times f_{j-1}) = s^{\epsilon_{i,j}} \langle , \rangle_{i,j-1}\,,$$

where $\epsilon_{i,j} = 1$ if $i+j > 2m$, and $\epsilon_f = 0$ otherwise, and for i, j between 1 and $n-1$, we have

$$\langle , \rangle_{i,j} \circ (f^i \times \mathrm{id}) = s^{\epsilon^{i,j}} \langle , \rangle_{i+1,j}\,, \text{ and}$$
$$\langle , \rangle_{i,j} \circ (\mathrm{id} \times f^j) = s^{\epsilon^{i,j}} \langle , \rangle_{i,j+1}\,,$$

where $\epsilon^{i,j} = 1$ if $i + j < 2m$, and $\epsilon^{i,j} = 0$ otherwise;

(II) for each i, the form $\langle , \rangle_{i,i}$ is alternating, and for each $i \neq j$, we have

$$\langle , \rangle_{i,j} \circ \mathrm{sw}_{j,i} = - \langle , \rangle_{j,i}\,,$$

where $\mathrm{sw}_{j,i} : \mathscr{E}_j \times \mathscr{E}_i \to \mathscr{E}_i \times \mathscr{E}_j$ is the canonical switching map.

Note that the data of the linked alternating form is equivalent to an alternating form on $\bigoplus_i \mathscr{E}_i$, but it is difficult to state condition (I) in this context. Note also that for condition (II), if we are not in characteristic 2 we do not need to separate impose that $\langle , \rangle_{i,i}$ be alternating.

We then prove:

Theorem 6.3.2. *The space of linked alternating forms of index m on $\mathscr{E}_\bullet$ is a vector bundle on S of rank $\binom{k}{2}$.*

If we have a linked alternating form on $\mathscr{E}_\bullet$, then restriction of the form to the universal subbundle on a linked Grassmannian $LG(k, \mathscr{E}_\bullet)$ induces a morphism from LG to the space of linked alternating forms. If we then define a linked alternating Grassmannian as parametrizing tuples of subbundles which are pairwise orthogonal for a given linked alternating form, it will be cut out inside the linked Grassmannian precisely as the preimage of the zero section under the above map. Thus, according to the theorem it has codimension at most $\binom{k}{2}$.

The remaining issue is to come up with a suitable concept of nondegeneracy to define linked symplectic forms, in such a way that the linked symplectic Grassmannian is smooth of codimension exactly $\binom{k}{2}$ (at least away from the singularities of LG itself).

Definition 6.3.3. In the situation of Definition 6.3.1, the linked alternating form $\langle,\rangle_\bullet$ is **symplectic** if:

(I) for all i, j between 1 and n with $i + j = 2m$, we have $\langle,\rangle_{i,j}$ nondegenerate;
(II) if $2m < n + 1$, then on all fibers where $s = 0$, and for all i with $2m - 1 < i \leqslant n$, the degeneracy of $\langle,\rangle_{i,1}$ is equal to ker f^{i-1};
(III) if $2m > n + 1$, then on all fibers where $s = 0$, and for all i with $1 \leqslant i < 2m - n$, the degeneracy of $\langle,\rangle_{i,n}$ is equal to ker f_i.

Using explicit descriptions of the tangent spaces of LG at exact points, we then prove:

Theorem 6.3.4. *If $LSG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet) \subseteq LG(r, \mathscr{E}_\bullet)$ is a linked symplectic Grassmannian, and S is regular, and $z \in LSG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$ is a smooth point of $LG(r, \mathscr{E}_\bullet)$, then z is also smooth point of $LSG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$, and the latter has codimension $\binom{r}{2}$ in $LG(r, \mathscr{E}_\bullet)$ at z.*

These ideas are enough to conclude that that limit linear series techniques can be applied to prove existence results in rank-2 Brill-Noether theory in the case of canonical determinant, at least to curves with at most two irreducible components.

However, because higher-dimensional limit linear series do not have an inductive description, for our application it is crucial to generalize this to arbitrary curves of compact type, which of course depends on generalizing the original linked Grassmannian construction to arbitrary curves of compact type. This turns out to be more difficult than initially expected, as the arguments used in the original case do not work in the necessary generality, but I have made progress on this, and hope to complete it soon.

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