RELATIVE DIMENSION OF STACKS

Abstract. Infinitesimal deformation theory is a common tool for showing that objects can be deformed in families. However, in some cases, such as the theory of limit linear series and its higher-rank generalization, it works better to use more naive dimension-based arguments. For morphisms of varieties over a field, there is no subtlety in carrying this out, but for schemes or stacks, the pathologies of dimension theory demand a more careful approach. With this motivation, we describe a notion of a morphism having "at least" a given relative dimension. This condition is quite well behaved, works equally well for algebraic stacks, and arises naturally in many settings.

1. What we want

Suppose X/S is a relative moduli space, where S is positive-dimensional. If we have a point $s_0 \in S$ where we understand the fiber X_{s_0}, we then frequently want to draw conclusions on the other fibers X_s. One standard tool for doing this is infinitesimal deformation theory. However, another, more naive, approach was used very successfully by Eisenbud and Harris in their of limit linear series: they simply counted dimensions.

A piece of an argument. If X and S are both of finite type over a field (and assume for simplicity that S is irreducible), the idea is quite simple: if we know that every component of X has dimension at least $n + \dim S$, and if a fiber X_{s_0} is nonempty of dimension exactly n, then this should tell us quite a lot about what happens in a neighborhood of s_0. To start with, sufficiently small open neighborhoods U of X_{s_0} will map dominantly onto S, with all nonempty fibers pure of dimension n. In particular, every point of X_{s_0} is a limit of points of X in nearby fibers. If further X is proper over S, we can choose U to be the preimage of an open neighborhood of s_0, so that all fibers X_s are nonempty of dimension n for s sufficiently close to s_0.

Now, if X and S are not of finite type over a field, naive notions of dimensions become very problematic, and it is easy to make incorrect statements. In the scheme setting, options include making statements in terms of dimensions of local rings, or working with codimension instead. The latter is undesirable because it complicates statements and requires the use of (often non-canonical) ambient spaces, while the former doesn't work in the context of algebraic stacks. We'd thus like to develop a definition which will cleanly capture the above ideas, but which will generalize transparently to more general classes of schemes, and to algebraic stacks. This audience presumably doesn't need to be sold on the value of the latter, but for the sake of concrete motivation, what we will describe is applied in joint work with Teixidor i Bigas which uses generalized limit linear series to prove new theorems on existence of stable rank-2 vector bundles on smooth curves having large numbers of global sections.

The basic idea is that in the above situation, the statement that "every component of X has dimension at least $n + \dim S$" should be interpreted as saying that "X has relative dimension at least n over S." Accordingly, we want to define what this should mean more generally, in such a way that the key properties of the above situation generalize.

2. A definition

It turns out that we will be able to get away with rather mild hypotheses. Our definition is as follows:

Definition 2.1. Let $f : X \to Y$ be a morphism locally of finite type of universally catenary schemes. We say that f has **relative dimension at least** n if for every irreducible closed subscheme Y' of Y, and every irreducible component X' of $X|_{Y'}$, with generic point η, we have

$$\dim X'_{f(\eta)} - \operatorname{codim}_{Y'} \overline{f(X')} \geqslant n. \tag{2.1.1}$$

We say that f has **universal relative dimension at least** n if for all universally catenary Y-schemes S, the base change $X \times_Y S \to S$ has relative dimension at least n.

Notice that if X and Y happen to be of finite type over a field, then (2.1.1) is equivalent to requiring that $\dim X' \geqslant \dim Y' + n$. However, even in this case, the condition of having relative dimension at least n depends on the morphism f, due to the consideration of preimages of closed subsets of Y.

Example 2.2. If f is flat with every component of every fiber having dimension at least n, then f has universal relative dimension at least n.

Example 2.3. Suppose f is a closed immersion, and X has pure codimension c. Then f need not in general have relative dimension at least $-c$; in fact, this says that codimension can only drop when intersecting with other closed subschemes of Y. In this situation, universal relative dimension corresponds to Hochster's notion of "superheight."

Thus, if Y is regular, Serre's theorem implies that f has relative dimension at least $-c$, and Hochster showed that Serre's theorem generalizes to superheight, so in fact in this case, X has universal relative dimension at least $-c$.

On the other hand, if Y is not necessarily regular, but X is locally cut out by c equations, then again f has relative dimension at least $-c$. This generalizes further to the case that X is locally determinantal, or locally cut out by Schubert conditions in Y.

Example 2.4. The normalization of a nodal curve has relative dimension at least 0, but not universal relative dimension at least 0. One can see this by taking the base change to either the normalized curve, or to the product of the curve with its normalization.

3. Formal properties

It is straightforward to check the following.

Proposition 3.1. *Suppose that $f : X \to Y$ is smooth of relative dimension m, and $g : Y \to Z$ a morphism with Z universally catenary. We have that g has relative dimension at least n if and only if $g \circ f$ has relative dimension at least $n+m$. Additionally, the same statement holds for universal relative dimension.*

It takes a bit more work to prove that relative dimension behaves well under composition.

Lemma 3.2. *Suppose that $f : X \to Y$ has relative dimension at least m, and $g : Y \to Z$ has relative dimension at least n. Then $g \circ f$ has relative dimension at least $m + n$.*

The same holds for universal relative dimension.

Using the previous results, we can prove the following, which will allow us to apply the notion of universal relative dimension to algebraic stacks.

Corollary 3.3. *Suppose that $f : X \to Y$ is a morphism, with Y universally catenary. Let $g : Y' \to Y$ be a smooth cover of relative dimension m, and let $h : X'' \to X'$ be a smooth cover, where $X' = X \times_Y Y'$. Denote by f'' the induced morphism $X'' \to Y'$. Then f has universal relative dimension at least n if and only if f'' has universal relative dimension at least $n + m$.*

We also find:

Corollary 3.4. *Given morphisms $f : X \to Y$ and $g : Y \to Z$, suppose g is smooth of relative dimension n, with Z universally catenary. Then f has universal relative dimension at least m if and only if $g \circ f$ has universal relative dimension at least $m + n$.*

Note that the last two corollaries are false if we drop the universality condition: one can construct counterexamples to both using Example 2.4.

4. Consequences

Putting together previous examples and Lemma 3.2, we see that our definition is satisfied quite frequently. Specifically:

Corollary 4.1. *Let $f : X \to Y$ be a closed immersion, and let $g : Y \to Z$ be a smooth morphism of relative dimension n, with Z universally catenary. Suppose that either Z is regular and every component of X has codimension at most c in Y, or X may be expressed locally as an intersection of determinantal conditions with expected codimensions adding up to c. Then $g \circ f$ has universal relative dimension at least $n - c$.*

Alternatively, suppose that $f : X \to Y$ is a morphism of smooth S-schemes, with S universally catenary, and m and n the relative dimensions of X and Y over S, respectively. Then f has universal relative dimension at least $m - n$.

I also believe that when deformation theory gives a lower bound on dimension in terms of tangent and obstruction spaces, the corresponding morphism should have at least the same universal relative dimension, but I haven't checked the details of this yet.

Now, we see that relative dimension also has the sort of consequences we want in the context of smoothing arguments. We have:

Proposition 4.2. *Given $f : X \to Y$, suppose that f has relative dimension at least n, and for some $y \in Y$, the fiber X_y has dimension n. Then there exists a neighborhood U of X_y on which f has pure fiber dimension n, and on any such neighborhood f is open.*

If further f has universal relative dimension at least n, then f is universally open on U.

If further Y is reduced and the fiber X_y is geometrically reduced, then f is flat at every point of X_y.

Obviously, the above only applies in the case $n \geqslant 0$, but there is an analogous but slightly weaker statement if $n < 0$, where in place of a fiber, we consider the restriction to a closed subscheme of Y of dimension at least $-n$.

Put together, these results allow us to make very transparent smoothing arguments as in the situation for Eisenbud-Harris limit linear series. The constructions together with (a slight generalization of) Corollary 4.1 allow us to say that we have a suitable relative moduli space with relative dimension at least a certain n, so that then if the space of limit linear series on a given reducible curve has dimension exactly n, Proposition 4.2 allows us to conclude that every limit linear series arises as a limit of linear series on nearby smooth curves. However, in order to apply these ideas to higher-rank vector bundles, we need to generalize the theory to moduli spaces which are algebraic stacks.

5. The situation for stacks

Dimension theory for algebraic stacks is a bit subtle, due to the tendency of smooth (and even etale) covers to break irreducible schemes into reducible ones. For instance, it is due to this behavior that the property of being universally catenary is not etale local, and hence not well-defined even for Deligne-Mumford stacks. However, a natural substitute is to work with stacks which are locally of finite type over a universally catenary base scheme, and in this case, both usual notions of codimension and our definition of relative dimension of a morphism generalize well.

Indeed, both can be defined in the usual way in terms of smooth covers. In the case of relative dimension, Corollary 3.3 above implies that the definition is well defined. Moreover, one can show that our original definition of relative dimension can be restated directly in the context of stacks, and gives a definition equivalent to the one in terms of smooth covers. Our previous results on schemes generalize transparently to the stack setting, so that again we find that having a certain relative dimension is very common, and has the desired consequences from the point of view of smoothing arguments.