THE NUMBER OF LINEAR SERIES ON CURVES WITH GIVEN RAMIFICATION

BRIAN OSSERMAN

Abstract. We use Eisenbud and Harris' theory of limit linear series (1986) to show that for a general smooth curve of genus g in characteristic 0, with general points P_i and indices e_i such that $\sum_i (e_i - 1) = 2d - 2 - g$, $G^1_d(C, \{(P_i, e_i)\}_i)$ is made up of reduced points. We give a formula for the number of points, showing that it agrees with various known special cases. We also conjecture a corresponding reducedness result and formula for g^r_ds of any dimension, and reduce this to the case of three points on $\mathbb{P}^1$, where one need no longer consider moduli or generality.

1. Introduction

We work throughout over an algebraically closed field of characteristic 0.

We state the basic problem which will concern us herein:

Question 1.1. Given a general smooth, genus-g curve C, n points P_i on C and integers $e_i \geq 2$, with $\sum_i (e_i - 1) = 2d - 2 - g$, and $e_i \leq d$ for all i, modulo automorphism of the image, how many maps are there from C to $\mathbb{P}^1$ of degree d which ramify to order e_i at the P_i?

In the case of $C = \mathbb{P}^1$, the answer, as a reduction to easy Schubert calculus, has long been known, and was first proved by Eisenbud and Harris; see [3, Theorem 9.1]. L. Goldberg gave a closed-form combinatorial formula when all e_i are equal to 2 in [5], and I. Scherbak did likewise in the general case in [11]. On the other hand, the question is largely unanswered either when $\mathbb{P}^1$ is replaced by a higher-genus curve, or when the linear series being counted have dimension higher than 1. Beyond their instrinsic interest, in various known and not yet known cases these questions have applications to understanding the cohomology rings of the G^r_d spaces of higher-genus curves, the study of the solutions of an A_N Bethe equation of XXX type (Mukhin and Varchenko in [9]), the computation of the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$ (Logan in [8]), and, in characteristic p, the study of connections with vanishing p-curvature on rank 2 vector bundles on curves ([10]).

We begin by reviewing the reduction of the case of $\mathbb{P}^1$ to Schubert calculus, and give a new argument for reducedness which, save for a pesky base case, goes through unmodified in the generality of arbitrary r-dimensional linear series on $\mathbb{P}^1$ with specified ramification: that is, $\alpha^i = \{\alpha^i_j\}_j$ arbitrary ramification sequences for each P_i which satisfy the Plücker formula $\sum_{i,j} \alpha^i_j = (r+1)d + \binom{r+1}{2}(2g-2) = (r+1)(d-r)$ for $g = 0$. We then continue on to analyze the case of higher-genus curves, making a similar degeneration argument to reduce it down to the case of genus 0. Finally, we apply some combinatorial analysis of our formulas to rederive known special cases due to Castelnuovo, Goldberg, Scherbak, and Logan.

2. Notation And Results

We start with the following two notational caveats:

Warning 2.1. Without further comment, we assume $\sum_{i,j}(\alpha^i_j) = (r+1)(d-r)-rg$ throughout the remainder of this paper, so that the expected dimension of linear series is always 0.

Warning 2.2. When we restrict to the special case of $r = 1$, we will abuse notation by assuming our linear series to be base-point free, and merely writing the ramification indices e_i rather than a full ramification sequence at each point. To further add insult to injury, we use the convention of e_i as the order of vanishing of the function itself, and not of its differential.

The moduli space of linear series with prescribed ramification on a curve C is written $G^r_d(C, \{(P_i, \alpha^i)\}_i)$. Until we specify otherwise, we consider the case $C = \mathbb{P}^1$, and here the G^r_d space is naturally a closed subscheme of the Grassmannian $\mathbb{G}(r, d)$: one simply views such linear series on $\mathbb{P}^1$ as the projectivization of $(r + 1)$-dimensional vector spaces of polynomials of degree d, and the ramification conditions correspond to requiring the existence of bases with higher-order vanishing at the P_i, so we see that our G^r_d scheme is cut out by (equivalently, each ramification condition is described by) Schubert cycles of the form $\Sigma_{\alpha^i}(V_{P_i})$ for appropriate flags V_{P_i}. By abuse of notation, we will write these cycles as $\Sigma_{\alpha^i}(P_i)$, so that we have

$$G^r_d(\mathbb{P}^1, \{(P_i, \alpha^i)\}_i) = \bigcap_i \Sigma_{\alpha^i}(P_i). \tag{2.1}$$

Unfortunately, a general choice of ramification points P_i does not correspond to general translates of the corresponding Schubert cycles, so a priori we know very little about their intersection. This is highlighted by the characteristic p case, where $G^1_d(\mathbb{P}^1, \{(P_i, e_i)\}_i)$ only rarely has the expected dimension, thanks to a locus of inseparable maps with base points, and can even have the wrong dimension when restricted to the separable locus, even in the tamely ramified case. However, the characteristic p case is ultimately a far more difficult and interesting question, and is discussed in [10].

On the other hand, in characteristic 0, this intersection is well-known to be 0-dimensional (see Lemma 3.1), and we denote the number of points contained in it by $N(\{(P_i, \alpha^i)\}_i)$. We denote by $N(\{\alpha^i\}_i)$ the intersection class of the corresponding Schubert cycle classes σ_{α^i}. The equality of these two numbers is equivalent to the reducedness of the corresponding G^r_d scheme. We will use Schubert calculus to derive a recursive formula for $N(\{\alpha^i\}_i)$, and then apply Eisenbud and Harris' theory of limit linear series to get a corresponding inductive lower bound for $N(\{(P_i, \alpha^i)\}_i)$. The case of three points, where there is no longer moduli or generality to worry about, will act as a base case, so that we prove the following theorem.

Theorem 2.3. *In characteristic* 0*, if* $G^r_d(\mathbb{P}^1, (0, \alpha^1), (1, \alpha^2), (\infty, \alpha^3))$ *consists of* $N(\{\alpha^i\}_i)$ *(necessarily reduced) points for any fixed* r *and arbitrary* α^i*, then for any number of general points* P_i*, and ramification sequences* α^i*,* $G^r_d(\mathbb{P}^1, \{(P_i, \alpha^i)\}_i)$ *is made up of* $N(\{\alpha^i\}_i)$ *reduced points.*

Unfortunately, there does not appear to be any simple way of getting at the general three-point case, but when $r = 1$, the intersection product is always 1 in the three point case, so the reducedness there is tautological, and we conclude the following theorem.

Theorem 2.4. *In characteristic* 0*, for general points* P_i*, and* $\sum_i(e_i-1)=2d-2$*,* $G^1_d(\mathbb{P}^1,\{(P_i,e_i)\}_i)$ *is made up of* $N(\{e_i\}_i)$ *reduced points.*

During the course of the argument, we also derive a recursive formula:

$$N(\{\alpha^i\}_i)=\sum_{\alpha'}N(\alpha',\alpha^{n-1},\alpha^n)N(\{\alpha^i\}_{i<n-1},\hat{\alpha}') \tag{2.2}$$

where α' ranges over ramification sequences of the appropriate weight, and $\hat{\alpha'}_j := d-\alpha'_{r-j}$.

Moving on to the higher-genus case, we are able to prove the following theorem analogously:

Theorem 2.5. *In characteristic* 0*, if* $G^r_d(\mathbb{P}^1,(0,\alpha^1),(1,\alpha^2),(\infty,\alpha^3))$ *consists of* $N(\{\alpha^i\}_i)$ *(necessarily reduced) points for any fixed* r *and arbitrary* α^i*, then for any number of general points* P_i*, any general smooth, proper curve* C *of genus* g*, and ramification sequences* α^i*,* $G^r_d(C,\{(P_i,\alpha^i)\}_i)$ *is made up of* $N(\{\alpha^i\}_i,b^1,\dots,b^g)$ *reduced points, where each* b^i *is simply the ramification sequence* $0,1,\dots,1$*.*

As before, in the case $r=1$ we get the following theorem.

Theorem 2.6. *In characteristic* 0*, for general points* P_i*, a general smooth proper curve* C *of genus* g*, and* $\sum_i(e_i-1)=2d-2-g$*,* $G^1_d(C,\{(P_i,e_i)\}_i)$ *is made up of* $N(\{e_i\}_i,\underbrace{2,\dots,2}_{g})$ *reduced points.*

Remark 2.7. The argument given here underscores the fact that limit linear series in the case $r=1$ are not, as is commonly thought, simply a rephrasing of the theory of admissable covers, but provide new tools which are not available in the latter theory. The fundamental difference in perspective between linear series and covers is highlighted even more dramatically in characteristic p: when working with covers, wild ramification is always the pathological scenario, but when working with linear series, as mentioned earlier, tame ramification is no guarantee of good behavior, whereas there is no evidence that wild ramification is any more pathological than tame.

Remark 2.8. Our Schubert cycle description of the case of genus 0 can also readily be described dually in terms of $d-(r+1)$-planes in $\mathbb{P}^d$ with prescribed intersection dimension with osculating planes at the P_i of the rational normal curve in $\mathbb{P}^d$, where our maps are given by projection from the $d-(r+1)$ planes. Thus, an equivalent formulation of our main result will be a reducedness theorem for intersections of certain Schubert cycles associated to osculating flags at points of the rational normal curve. However, we will not make use of this description in our analysis.

3. The Case Of Genus 0

Throughout this section, our curve $C=\mathbb{P}^1$. The following lemma is well known:

Lemma 3.1. *Suppose* $\sum_{i,j}\alpha^i_j=(r+1)(d-r)-c$ *for some* $c\geq 0$*; then as long as the* P_i *are distinct,* $\cap_i\Sigma_{\alpha^i}(P_i)$ *has expected dimension* c*.*

Proof. The idea is simply to use the Plücker formula to conclude the statement when c is negative (that is, when the intersection should be empty), and then iteratively cut out hyperplane sections to reduce to this case; for the details, see [3, Theorem 2.3]. □

The case $c = 0$ is simply the full specification of all ramification, so we obtain the following corollary.

Corollary 3.2. *There are finitely many linear series on $\mathbb{P}^1$ with any given fully specified ramification.*

Now that we know our intersection is a finite set, we can derive an upper bound for its size by computing the cycle class of the intersection. We find the following formula.

Proposition 3.3.

$$N(\{\alpha^i\}_i) = \sum_{\alpha'} N(\alpha', \alpha^{n-1}, \alpha^n) N(\{\alpha^i\}_{i<n-1}, \hat{\alpha}') \tag{3.1}$$

where α' runs over all ramification sequences of the appropriate weight, and $\hat{\alpha}'_j := d - \alpha'_{r-j}$.

Proof. Write out the intersection product of the first $n-2$ of the σ_{α^i} as $\sum_{\alpha'} c_{\alpha'}\sigma_{\alpha'}$ where the α' range over all ramification sequences of the appropriate weight/codimension. Using the complementary dimensional intersection formula (see, e.g., [6, p. 198]), we can read off any $c_{\alpha'}$ simply by intersecting this product with $\sigma_{\hat{\alpha}'}$, that is to say that $c_{\alpha'} = N(\{\alpha^i\}_{i<n-1}, \hat{\alpha}')$. On the other hand, it is clear that $N(\{\alpha^i\}_i)$ will be given simply as the sum of the $c_{\alpha'}$ times the intersection product of each $\sigma_{\alpha'}$ with $\sigma_{\alpha^{n-1}}$ and σ_{α^n}, and this latter is simply $N(\alpha', \alpha^{n-1}, \alpha^n)$ by definition, yielding the desired result. □

If we restrict to $r = 1$, it is well-known that any three-cycle intersection product is either 0 or 1: one way to see this is to 'factor out' the base points, leaving only special cycles, and then apply Pieri's formula (see [6, p. 203]) once, followed by the complementary dimensional formula. This even gives explicitly when the intersection is 0 and when it is 1: in the base-point free case, if we think in terms of maps from $\mathbb{P}^1$ to $\mathbb{P}^1$, one will exist if and only if each ramification index is less than or equal to the degree. If we think of our linear series as maps (after removing base points), we can sum over d', the possible degrees of the maps in the recursion, and we find that we can write out the preceding proposition in this situation as follows.

Corollary 3.4. *For $n \geq 3$,*

$$N(\{e_i\}_i) = \sum_{\substack{d-e_{n-1}+1 \leq d' \leq d \\ d-e_n+1 \leq d' \leq d}} N(\{e_i\}_{i \leq n-2}, e), \text{ with } e = 2d' - 2d + e_n + e_{n-1} - 1 \tag{3.2}$$

For $n = 3$,

$$N(e_1, e_2, e_3) = \begin{cases} 1 & e_1, e_2, e_3 \leq d \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

Remark 3.5. A priori, we must also require in our inequalities that our three-cycle intersection happens in degree d, which after removing base points is equivalent to asking that the corresponding map has degree less than or equal to d. The

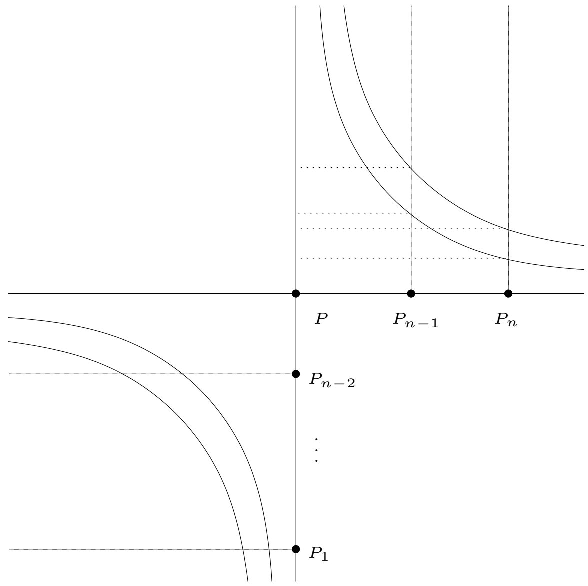

FIGURE 3.1

degree of this map is $\frac{e+e_{n-1}+e_n-1}{2}$, so this gives the additional requirement $e \leq 2d - e_{n-1} - e_n + 1$, which can be written equivalently as $d' \leq 2d - e_{n-1} - e_n + 1$. But the righthand side of this is simply $2d' + e$, so this inequality is equivalent to $e \leq d'$, and if it is violated, then $N(\{e_i\}_{i \leq n-2}, e) = 0$, and we get no contribution to the sum.

We now make use of Eisenbud and Harris' theory of limit linear series to provide a lower bound for $N(\{(P_i, \alpha^i)\}_i)$ for general P_i which agrees with the upper bound just derived:

Proposition 3.6. *For P_{n-1}, P_n general, and any P distinct from the P_i,*

$$N(\{(P_i, \alpha^i)\}_i) \geq \sum_{\alpha'} N((P, \alpha'), (P_{n-1}, \alpha^{n-1}), (P_n, \alpha^n)) N(\{(P_i, \alpha^i)\}_{i<n-1}, (P, \hat{\alpha}')) \tag{3.4}$$

where α' runs over all ramification sequences of the appropriate weight, and $\hat{\alpha'}_j := d - \alpha'_{r-j}$.

Proof. Inside of $\mathbb{P}^1 \times \mathbb{P}^1$, take the family of hyperbolas given in affine coordinates by $xy = t$, degenerating at $t = 0$ to the union of the x-axis and y-axis. For each $t \neq 0$, we get a smooth $\mathbb{P}^1$, and fix the isomorphism between them given by projection to the y-axis. Choose an isomorphism between our abstract $\mathbb{P}^1$ and the y-axis sending P to the node; we can now speak of $P_1, \dots, P_{n-2}$ as well as P as fixed points on the y-axis and simultaneously on all the $\mathbb{P}^1$s in our family; they are (constant) sections of our family. Now, choose any two points P_{n-1}^0 and P_n^0 on the x-axis away from 0, and define sections P_{n-1}^t and P_n^t similarly via projection from our family to the x-axis rather than the y-axis. Under our fixed trivialization of the smooth fibers of the family, both these sections tend towards the section defined by P as t goes to 0 (see Figure 3.1).

Now, for any α' of the required codimension/weight, and any pair of linear series in $G^r_d(\mathbb{P}^1, \{(P_i, \alpha^i)\}_{i\leq n-2}, (P, \alpha'))$ and $G^r_d(\mathbb{P}^1, (P, \alpha'), (P^0_{n-1}, \alpha^{n-1}), (P^0_n, \alpha^n))$, we get, by definition, a (refined) limit linear series on the union of the x and y-axes in the sense of [4]. It follows easily from Lemma 3.1 that any scheme of limit linear series with prescribed ramification on a curve with only rational components must have the expected dimension, so in particular we have the expected dimension for our special fiber, and by [4, Corollary 3.5], we can smooth any limit g^r_d to a family of linear series on our entire family, having ramification sequences α^i along our sections P_i. But given any two distinct linear series on the y-axis to start with, such a smoothing must give us distinct linear series on our family for all but finitely many values of t. Thus, the desired inequality holds for our chosen P^t_{n-1} and P^t_n for all but finitely many t. On the other hand, since we are working with intersections of families of Schubert cycles, the number of intersection points is constant for a general choice of P_{n-1} and P_n and can only drop under specialization, so if we establish a lower bound for any choice of P_{n-1} and P_n, it applies to a general choice, as desired. □

By choosing all our points P_i general, if we knew that the numbers agreed in the three-point case, we could inductively apply the two proceeding propositions to conclude that they agree in general; that is to say, we have proven Theorem 2.3. As we have already noted, for $r = 1$ the three-point case is tautological, so we also conclude Theorem 2.4.

4. Curves of Higher Genus

In this section, we reduce the case of general genus g curves to the case of $\mathbb{P}^1$ via a similar degeneration argument.

Definition 4.1. Denote by $N^g(\{\alpha^i\}_i)$ the number of linear series of degree d on a general smooth proper curve of genus g, with prescribed ramification sequences α^i at general points P_i, where $\sum_{i,j}(\alpha^i_j) = (r+1)(d-r) - rg$, so that the expected number is finite.

We immediately jump to the following proof.

Proof of Theorem 2.5. We proceed similarly to our argument in the case of genus 0, but this time we will work entirely within the framework of limit linear series, without any upper bound shortcuts involving intersection theory. We choose the special fiber X_0 of our family to be a general comb curve with n marked points: that is, we choose g general elliptic curves $E_1, \dots, E_g$, glue them to a copy of $\mathbb{P}^1$ at g general points $Q_1, \dots, Q_g$, and then we choose an additional n marked points which are required to lie on the $\mathbb{P}^1$ but are otherwise general. Winters shows in [12] that we can construct a one-parameter family of curves with the desired special fiber, with smooth generic fiber, and with the total space and base both non-singular. By [1, Proposition 2.3.8 (b)], after étale base change we can also produce sections $P_1, \dots, P_n$ of our family which specialize to our marked points at X_0. We thus obtain our smoothing family as in Figure4.1.

We note that because of [4, Theorem 4.5], and because all our curves and points are general in X_0, if the expected dimension of linear series (including ramification conditions at the nodes) is negative, there will be no linear series on that component, and hence no corresponding limit series on X_0. Now, since the total expected

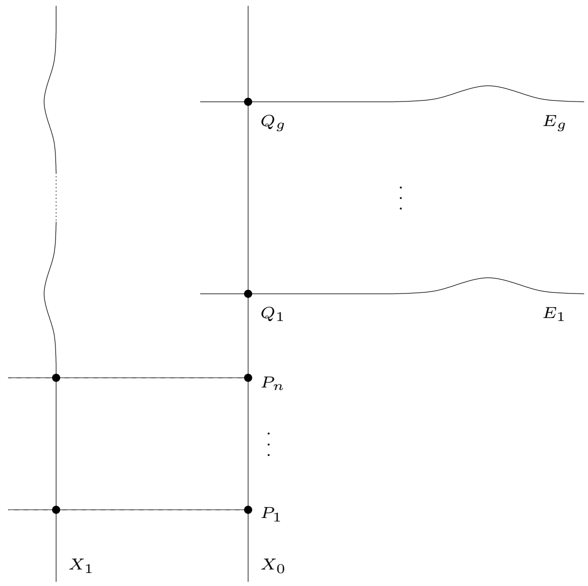

FIGURE 4.1

dimension of the G^r_d space is 0 by hypothesis, the agreement conditions at the nodes of the special fiber X_0 mandate that in order to prevent the expected dimension of the space of linear series on any component from becoming negative, on each component we must impose ramification conditions at the nodes which lower the total expected dimension to precisely 0. Our G^r_d space on X_0 will thus be the product of G^r_d spaces on each component, with this additional ramification imposed at the node. In particular, on each E_i we must have enough ramification at Q_i alone to push the expected dimension to 0, and this will allow us to show that the G^r_d space on each E_i is simply a single reduced point. We start by observing that the maximal vanishing sequence is a priori $d-r, \dots, d-1, d$, which imposes $(r+1)(d-r)$ conditions on the $(r+1)(d-r)-rg = (r+1)(d-r)-r$ dimensions of the space of g^r_d's on E_i. Thus, we can reduce the total ramification weight only by r from this maximal sequence if we wish the expected dimension to be 0. In particular, the last vanishing order must remain d, meaning that we have a section vanishing to order d at our point, which automatically restricts our line bundle to $\mathcal{O}(dQ_i)$. This remains true over an arbitrary base, so our G^r_d space is inside the Grassmannian of $(r+1)$-dimensional subspaces of $H^0(E_i, \mathcal{O}(dQ_i))$, even scheme-theoretically. Now, we note that since the genus of E_i is positive, we cannot have a section of $\mathcal{O}(dQ_i)$ vanishing to order exactly $d-1$ at Q_i, since that would correspond to a rational function with a simple pole at Q_i, and regular elsewhere. Thus, the actual maximal vanishing sequence is $d-r-1, \dots, d-2, d$, which imposes $(r+1)(d-r)-r$ conditions, and therefore has expected dimension 0, and must be the unique vanishing sequence which can have a non-empty G^r_d space of expected dimension 0 for E_i. Thus, the G^r_d space in question is now simply a Schubert cycle inside our Grassmannian, so if it is non-empty and of dimension 0, it must be simply a single reduced point. Lastly, it must be non-empty and of dimension 0 simply by the Riemann-Roch theorem.

Since we noted above that the expected dimension imposed by ramification conditions at the nodes would have to be precisely 0 on each component, because this can be achieved only for ramification conditions corresponding to refined limit series, we see that we have no crude limit series on X_0. As stated in the proof of [4, Theorem 3.3], the non-existence of crude limit series on X_0 implies that the G^r_d space for the entire smoothing family is proper. On X_0, we see that the ramification conditions at each Q_i are uniquely determined, giving ramification indices of $0, 1, \dots, 1$ at the Q_i on the rational component. Since the G^r_d spaces on each E_i are single reduced points, the G^r_d space for X_0 is simply the G^r_d space for $\mathbb{P}^1$ with the given ramification at the P_i and Q_i. By [4, Corollary 3.5] each limit g^r_d on X_0 must smooth to the family, and by properness any g^r_d on the generic fiber X_1 limits to a limit g^r_d on X_0, so we find that if our G^r_d space for X_0 is made up of reduced points, then the G^r_d space for X_1 is made up of the same number of reduced points. By Theorem 2.3, the hypotheses of our theorem imply that we do in fact have $N(\{\alpha^i\}_i, b^1, \dots, b^g)$ reduced points for X_0, whence we get the same number for X_1, and conclude that the general curve in our family has the expected G^r_d space; in particular, we have produced at least one smooth curve with the right G^r_d space.

Since $M_{g,n}$ is irreducible (see, e.g., [2]), it remains only to show that the G^r_d space having a given number of reduced points is an open condition on the space of smooth curves. However, this involves only very standard techniques: we choose a smooth integral base scheme B and a family of curves $\mathcal{C}$ over B which contains all curves in $M_{g,n}$; for instance, we may take B to be $M_{g,n}$ with sufficiently high level structure (see [2, 5.14] for representability, and [2, Theorem 5.2, Proposition 5.8, and proof of Theorem 5.15] for smoothness). We claim that $\mathcal{C}$ is relatively projective over B: if n is at least 1, this is automatic, since we have a B-valued point of $\mathcal{C}$ to use as an ample divisor. On the other hand, if $g \neq 1$, we have projectivity simply by taking the relative canonical or anticanonical bundle. But $n = 0$, $g = 1$ can't happen, since then the expected dimension is $(r+1)(d-r) - r$; as long as $r < d$, this is positive, but we cannot have $r = d$ if the genus is positive (and in particular if $g = 1$). Thus, our family is relatively projective, and we can construct its relative Picard scheme (see, for instance, [1, Theorem 8.2.1]), and also its relative G^r_d scheme with prescribed ramification (for instance, by twisting sufficiently many times with a relatively ample divisor and then taking the appropriate closed subscheme of a Grassmannian bundle, as in the proof of [4, Theorem 3.3]). Now, by construction G^r_d will be proper over B, and every component will have dimension at least $\dim B$, so if we restrict B to the quasifinite locus, we get a finite morphism, with every component of G^r_d mapping dominantly onto B. In this situation, if one fiber is made up of reduced points, the map is not everywhere ramified, so the étale locus is non-empty, and since every point in the chosen fiber must lie on a component mapping dominantly to B, the fiber must be in the étale locus, and the degree of the map is given by the number of points of our fiber. This implies that to show that a general smooth curve with marked points has a G^r_d space made up of a certain number of reduced points, it suffices to produce one such curve, which completes the proof of the theorem. □

As with genus 0, in the case $r = 1$ we no longer need any assumptions, and can conclude Theorem 2.6.

5. Combinatorial Analysis and Previously Known Cases

Finally, we simplify some of our formulas in the case $r = 1$, and show that they agree with four known formulas in special cases due to Castelnuovo, Goldberg, Logan, and Scherbak. We begin with a couple of combinatorial lemmas.

Definition 5.1. A sequence of integers $a_1, \dots, a_n$ is said to be *admissible* if $a_1 = 1$, $a_i = a_{i-1} \pm 1$ for all $i > 1$, and all the a_i are strictly positive. When $U \geq L \geq 1$, and $L+n$ and $U+n$ are both odd, we denote by $N_{\mathrm{ad}}(n, L, U)$ the number of admissible sequences of length n such that $L \leq a_n \leq U$; if $E \geq 1$ with $E + n$ odd, we write $N_{\mathrm{ad}}(n, E)$ for the number of admissible sequences of length n with $a_n = E$.

Lemma 5.2. $N_{\mathrm{ad}}(n, L, U) = \binom{n}{(n+L-1)/2} - \binom{n}{(n+U+1)/2}$

Proof. We proceed by induction on n. We begin with $n = 0$; here, $N_{\mathrm{ad}}(n, L, U) = 1$ if and only if $L = 1$, and 0 otherwise. But the same is clearly true of the asserted formula. For the induction step, we simply use that every admissible sequence is determined by an admissible sequence of length one less, together with the final term, for which there are either one or two possibilities. We have two cases to consider: $L > 1$ and $L = 1$. In the first case, it is evident that $N_{\mathrm{ad}}(n, L, U) = N_{\mathrm{ad}}(n-1, L-1, U-1) + N_{\mathrm{ad}}(n-1, L+1, U+1)$, and one checks easily that our asserted formula satisfies the same recursion, using the induction hypothesis and the fact that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$. In case $L = 1$, because the final term of the sequence is not allowed to drop to zero, we get $N_{\mathrm{ad}}(n, L, U) = N_{\mathrm{ad}}(n-1, L+1, U-1) + N_{\mathrm{ad}}(n-1, L+1, U+1)$. But because $L = 1$, $\binom{n}{(n+(L+1)-1)/2} = \binom{n}{(n+1)/2} = \binom{n}{(n-1)/2} = \binom{n}{(n+(L-1)-1)/2}$, and we are reduced to the same identity as before. $\square$

The next corollary easily follows.

Corollary 5.3. $N_{\mathrm{ad}}(n, E) = \frac{2E}{n+E+1}\binom{n}{(n+E-1)/2}$

Remark 5.4. These numbers are actually generalizations of Catalan numbers, which correspond to $n = 2d$ even, and $E = 1$, where we get $\frac{1}{d+1}\binom{2d}{d}$. This description of Catalan numbers is known, see for instance [5, Theorem 1.6]. I do not know whether or not this particular generalization is also known.

We now apply these to get the following theorem:

Theorem 5.5.

$$N^g(\{e_i\}_i) = N(\{e_i\}_i, \underbrace{2, \dots, 2}_{g}) = \sum_{\substack{1 \leq j \leq g+1 \\ j+g+1 \text{ even}}} N_{\mathrm{ad}}(g, j) N(\{e_i\}_i, j). \tag{5.1}$$

In particular,

$$N^g(\{e_i\}_i) = N(\{e_i\}_i, \underbrace{2, \dots, 2}_{g}) = \sum_{\substack{1 \leq j \leq g+1 \\ j+g+1 \text{ even}}} \frac{2j}{g+j+1}\binom{g}{(g+j-1)/2} N(\{e_i\}_i, j). \tag{5.2}$$

Proof. We simply make use of our recursive formula, applied in the case when $e_{n-1} = 2$. We had

$$N(\{e_i\}_i) = \sum_{\substack{d - e_{n-1} + 1 \leq d' \leq d \\ d - e_n + 1 \leq d' \leq d}} N(\{e_i\}_{i \leq n-2}, e), \text{ with } e = 2d' - 2d + e_n + e_{n-1} - 1 \tag{5.3}$$

Now, if $e_{n-1} = 2$, we see that as long as $e_n \geq 2$, we simply have $d-1 \leq d' \leq d$, which in turn gives us the possibilities $e = e_n+1$ or $e = e_n-1$. If $e_n = 1$, we get only the former possibility for e. We therefore add a dummy ramification index, initially set to 1, to the end of our list of indices; since a ramification index of 1 imposes no condition, this doesn't change $N(\{e_i\}_i)$. We can then use the recursive formula to remove one index of 2 at a time, each time either raising or lowering our dummy index by 1, never letting it drop to 0, and summing over all possible choices of such sequences. But such a choice corresponds precisely to an admissible sequence for the dummy index, and the number of choices that will terminate with $N(\{e_i\}_i, j)$ for any particular j is then $N_{\mathrm{ad}}(g, j)$ by definition, giving the first assertion of the theorem. The second follows immediately as an application of Corollary 5.3. □

We apply this theorem to recover the formulas of Logan in [8, Theorem 3.1], Castelnuovo in [7, p. 238] and Goldberg in [5], respectively. Logan's formula is for two ramification points on a curve of arbitrary genus. Castelnuovo's is the case of arbitrary genus and no ramification. However, while the formula was due to Castelnuovo, the first rigorous proof is apparently due to Griffiths and Harris in the above citation. Finally, Goldberg's formula is for genus 0, and minimal ramification indices.

Corollary 5.6. *In the special case of Theorem 5.5 with only two* e_i*,*

$$N^g(\{e_i\}_i) = N(\{e_i\}_i, \underbrace{2, \dots, 2}_{g}) = \binom{g}{d-e_2} - \binom{g}{d}. \tag{5.4}$$

Similarly,

$$N^{2d-2}(\{\}) = N(\underbrace{2, \dots, 2}_{2d-2}) = \frac{1}{d}\binom{2d-2}{d-1}, \tag{5.5}$$

where $N^{2d-2}(\{\})$ *is the number of linear series of degree* d *and dimension* 1 *on a general curve of genus* $2d-2$*, with no ramification specified.*

Proof. For the first part, we simply use Theorem 5.5 together with the fact that $N(e_1, e_2, j)$ will be 1 if and only if j satisfies $e_1-e_2+1 \leq j \leq e_1+e_2-1, e_2-e_1+1 \leq j \leq e_1+e_2-1$, and 0 otherwise; if $e_1 \geq e_2$, this simply says that the answer will be given by $N_{\mathrm{ad}}(g, e_1-e_2+1, e_1+e_2-1)$, which is $\binom{g}{(g+e_1-e_2)/2} - \binom{g}{(g+e_1+e_2)/2}$ by Lemma 5.2. Now, to get the expected dimension to be 0, we had $e_1+e_2 = 2d-g$, so we simply get the asserted formula $\binom{g}{d-e_2} - \binom{g}{d}$. On the other hand, we note that since $g-(d-e_2) = d-e_1$, the asserted formula is symmetric under exchanging e_1 and e_2, as it should be. Hence, the assumption that $e_1 \geq e_2$ in fact caused no loss of generality, so we have proven the desired formulas.

The second part is even simpler, once we note that our recursion formula, and hence our formulas from the previous theorem, are valid all the way down to the case with a single ramification index for the terms in the sum, which corresponds in our degeneration argument to ramification being imposed only at the node on the relevant branch. In this case, we have $e \leq d$, $e-1 = 2d-2$, so $d \leq 1$, and the only possible case for $N(j)$ to be nonzero is when $j = 1$, when $N(1) = 1$. Hence the number of maps is the number of admissible sequences of length $2d-2$ ending at 1, which is precisely the aforementioned special case of the Catalan numbers, and by the formula of Corollary 5.3, we get the desired expression $\frac{1}{d}\binom{2d-2}{d-1}$. □

For our final formula, we will use the following standard combinatorial identity, easily obtained inductively by repeated applications of the identity $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$.

Lemma 5.7. $\sum_{i=0}^{k-1} \binom{n+i}{r-1} = \binom{n+k}{r} - \binom{n}{r}$

Finally, we recover Scherbak's formula for the number of maps in genus 0 with arbitrary ramification, in [11]:

Corollary 5.8.

$$N^0(\{e_i\}_i) = \sum_{j=1}^{n} (-1)^{n-j} \sum_{1 \leq i_1 < \cdots < i_j \leq n} \binom{\sum_m e_{i_m} - d - 1}{n-2}. \tag{5.6}$$

Proof. One may verify directly that this formula agrees with Corollary 3.4 in the case $n = 3$. We therefore need only show that the asserted formula is compatible with the recursion formula of the same corollary, which we repeat here:

$$N(\{e_i\}_i) = \sum_{\substack{d - e_{n-1} + 1 \leq d' \leq d \\ d - e_n + 1 \leq d' \leq d}} N(\{e_i\}_{i \leq n-2}, e), \text{ with } e = 2d' - 2d + e_n + e_{n-1} - 1 \tag{5.7}$$

Note that for any given e, replacing e_{n-1} and e_n with e will change the degree from d to $d - \frac{e_{n-1}+e_n-1-e}{2}$. For each term on the right, Scherbak's formula gives, after separating out terms with and without the e,

$$\sum_{j=1}^{n-2} (-1)^{n-1-j} \sum_{1 \leq i_1 < \cdots < i_j \leq n-2} \left(\binom{S_{\bar{i}} - d - 1}{n-3} - \binom{S_{\bar{i}} + e - d - 1}{n-3} \right) \tag{5.8}$$

with $S_{\bar{i}} = \sum_m e_{i_m} + \frac{e_{n-1}+e_n-1-e}{2}$.

In order to prove the recursion, it will suffice, if we fix a j and a j-tuple $1 \leq i_1 < \cdots < i_j \leq n-2$, and factor out a $(-1)^{n-j}$, to show that as e is allowed to vary over its full range, the resulting sum

$$\sum_e \left(\binom{\sum_m e_{i_m} + \frac{e_{n-1}+e_n-1+e}{2} - d - 1}{n-3} - \binom{\sum_m e_{i_m} + \frac{e_{n-1}+e_n-1-e}{2} - d - 1}{n-3} \right) \tag{5.9}$$

will give precisely the four terms of Scherbak's formula corresponding to that j-tuple, namely:

$$\binom{\sum_m e_{i_m} - d - 1}{n-2} - \binom{\sum_m e_{i_m} + e_{n-1} - d - 1}{n-2} - \binom{\sum_m e_{i_m} + e_n - d - 1}{n-2} + \binom{\sum_m e_{i_m} + e_{n-1} + e_n - d - 1}{n-2} \tag{5.10}$$

Both of the formulas we wish to compare are visibly symmetric in e_{n-1}, e_n, so without loss of generality we may assume that $e_{n-1} \leq e_n$. In this case, e will range from $e_n - e_{n-1} + 1$ to $e_n + e_{n-1} - 1$, always increasing by 2 to keep the degree an integer. The lefthand part of the resulting sum is $\sum_{k=0}^{e_{n-1}-1} \binom{\sum_m e_{i_m} + e_n - d - 1 + k}{n-3}$, which by Lemma 5.7 is none other than $\binom{\sum_m e_{i_m}+e_n+e_{n-1}-d-1}{n-2} - \binom{\sum_m e_{i_m}+e_n-d-1}{n-2}$. Similarly, the righthand part is $-\sum_{k=0}^{e_{n-1}-1}\binom{\sum_m e_{i_m}-d-1+k}{n-3} = -\binom{\sum_m e_{i_m}+e_{n-1}-d-1}{n-2} + \binom{\sum_m e_{i_m}-d-1}{n-2}$. Putting these together yields the desired formula. □

Remark 5.9. We remark finally that both the recursive formula of this paper and the combinatorial formula of Scherbak leave a certain amount to be desired in terms of simplicity. For instance, while Scherbak's formula has the advantage of being visibly symmetric in the e_i, even in the case of three points, where the formula always gives 0 or 1, it has several terms. In the case of four points, it is easy to verify from our recursive formula that one may write the answer simply as $\min\{e_i, d+1-e_i\}$ (setting it to zero whenever this is negative), and this seems a far more satisfying form than one would get directly from either of the above formulas. This suggests that there may be a better closed form for the answer to our problem than is achieved either here or by Scherbak.

Acknowledgements

This paper was partially supported by a fellowship from the National Science Foundation (NSF). I would like to thank A. J. de Jong for his guidance, as well as Joe Harris, David Eisenbud, Frank Sottile, and Jason Starr for their helpful conversations during the course of preparing this paper.

References

1. Siegfried Bosch, Werner Lutkebohmert, and Michel Raynaud, *Neron models*, Springer-Verlag, 1991.
2. P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Institut Des Hautes Etudes Scientifiques Publications Mathematiques (1969), no. 36, 75–109.
3. David Eisenbud and Joe Harris, *Divisors on general curves and cuspidal rational curves*, Inventiones Mathematicae **74** (1983), 371–418.
4. ———, *Limit linear series: Basic theory*, Inventiones Mathematicae **85** (1986), 337–371.
5. Lisa R. Goldberg, *Catalan numbers and branched coverings by the Riemann sphere*, Advances in Mathematics **85** (1991), 129–144.
6. Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience, 1978.
7. ———, *On the variety of special linear systems on a general algebraic curve*, Duke Mathematical Journal **47** (1980), 233–272.
8. A. Logan, *The Kodaira dimension of moduli spaces of curves with marked points*, American Journal of Mathematics **125** (2003), no. 1, 105–138.
9. E. Mukhin and A. Varchenko, *Solutions to the XXX type Bethe ansatz equations and flag varieties*, preprint.
10. B. Osserman, *Limit linear series in positive characteristic and Frobenius-unstable vector bundles on curves*, PhD thesis, MIT.
11. I. Scherbak, *Rational functions with prescribed critical points*, Geometric And Functional Analysis **12** (2002), no. 6, 1365–1380.
12. Gayn B. Winters, *On the existence of certain families of curves*, American Journal Of Mathematics **96** (1974), no. 2, 215–228.