Limit linear series

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Preface

I hope that this book is simultaneously accessible on two different levels: first, most of the ideas should be comprehensible to the reader who has studied varieties but not schemes; and second, formal definitions and proofs are provided throughout for the reader who is familiar with schemes. On a third level, because it is not my intention to attempt to treat the moduli space of curves, for applications of limit linear series involving the moduli space I have taken the approach of giving precise statements and proofs for the portions of the arguments which can be stated without reference to the moduli space, and then sketching the remainder of the argument. Part of the motivation for the multilayered approach is that although limit linear series theory involves working with moduli spaces and proving deformation results, the arguments involved are based to a remarkable extent on naive dimension counts which can be carried out without reference to schemes.

In order to maximize compatibility with first-year algebraic geometry courses out of books such as Hartshorne [Har77] and Shafarevich [Sha94a] and [Sha94b], I have included appendices on supplemental topics such as representable functors and dimension theory for schemes. At the same time, while there is inevitably redundancy with the books of Arborello, Cornalba, Griffiths and Harris [ACGH85] and of Harris and Morrison [HM98], I have attempted to keep this to a minimum, in part by treating all results algebraically and in arbitrary characteristic wherever possible, and in part by focusing much more narrowly on limit linear series and their applications.

This first draft is the result of a one-quarter topics course in the Winter of 2013. As such, it is rough, and very incomplete. Most glaringly, it is missing pictures, appendices filling in important technical background (some of which is original), and several of the most striking applications of limit linear series (including the proof that moduli spaces of curves are of general type). It is also very short on examples and exercises. Finally, I expect to add a chapter on limit linear series for curves not of compact type once the theory is sufficiently mature.
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CHAPTER 1

Introduction

Our basic object of study will be (connected) nonsingular projective curves over an algebraically closed field \( k \). For us, a curve will always be connected, but not necessarily irreducible – as we will see, even if we are primarily interested in nonsingular curves, we can learn a lot about their geometry by studying reducible curves. A nonsingular projective curve \( C \) has a genus \( g \), which can be defined as the dimension over \( k \) of the vector space of global (algebraic) differential forms. In general, we have only the Zariski topology on \( C \), which carries exceedingly little information: the closed sets are precisely the finite sets, so over a given field \( k \), any two curves are homeomorphic. However, when \( k \) is the field of complex numbers, the curve \( C \) also inherits the richer topology of the complex numbers, and in this topology it takes the form of an orientable surface of genus \( g \).

1.1. Some questions on curves

There are of course many different questions one could ask about curves, but we break them down into three different categories:

(1) Intrinsic geometry of the curve.
(2) Extrinsic geometry of the curve.
(3) Classification of curves.

These categories overlap to some extent, but we illustrate with some sample questions. One example of intrinsic geometry is given by Weierstrass points: if \( k \) has characteristic 0, then for all but finitely many points \( P \) of the curve, if we consider rational functions having a pole only at \( P \), there are no functions with a pole of order less than or equal to \( g \), while every pole order greater than or equal to \( g + 1 \) occurs. However, for \( g \geq 2 \) there is always a finite set of points where the possible pole orders are different. Such points are called Weierstrass points. One could ask how many Weierstrass points a given curve has, and with appropriate weighting, the answer is classical, and turns out to be \((g - 1)g(g + 1)\). On the other hand, one could ask:

**Question 1.1.1.** What sorts of Weierstrass points (in terms of their possible pole orders) can exist for some curve of genus \( g \)?

Although there are some good partial results which we will discuss, this remains an open question.

For extrinsic geometry, we consider a curve together with some additional information: an imbedding in projective space, a morphism to the projective line, or something of that nature. We then ask what we can say about the combined geometry, typically with a focus on relating the intrinsic and extrinsic properties. For instance, if a curve of genus \( g \) is imbedded in the projective plane as the zero
set of a polynomial of degree $d$, the **degree-genus formula** states that

$$g = \frac{(d - 1)(d - 2)}{2}.$$  

Another formula states that for $d > 1$, the number of inflection points, counted with appropriate weights, is given by

$$3d + 6(g - 1) = 3d^2 - 6d.$$ 

As before, we can shift our attention from a fixed curve to a varying curve and ask:

**Question 1.1.2.** For which triples $(g, r, d)$ can some curve of genus $g$ be realized inside $\mathbb{P}^r_k$ as a nondegenerate curve of degree $d$?

Or alternatively:

**Question 1.1.3.** For which triples $(g, r, d)$ does every curve of genus $g$ have some nondegenerate map to $\mathbb{P}^r_k$ of degree less than or equal to $d$?

Here, nondegenerate simply means that the image is not contained inside any hyperplane, so that we have not increased $r$ artificially. Despite some partial results, the first question remains open, even for $\mathbb{P}^3_k$. In contrast, the second question is the subject of the famous “Brill-Noether theorem,” which we will discuss in detail below. In essence, the reason that the second question is more approachable is that it deals with the situation for general curves rather than special curves, so that it is well-suited to techniques involving specializing the curves in families.

Finally, for classification of curves, we begin with some simple examples. Any curve of genus 0 is isomorphic to $\mathbb{P}^1_k$, the projective line. Any curve of genus 1 can be realized as a plane cubic, and conversely every nonsingular plane cubic is a curve of genus 1. The genus-2 curves can all be written in the form $y^2 = f(x)$ where $f(x)$ is a polynomial of degree 5, without repeated roots.\(^1\) Curves of genus 3 break into two types: some are **hyperelliptic**, meaning that they admit a 2-to-1 map to $\mathbb{P}^1_k$, while the rest can be written as a curve of degree 4 in $\mathbb{P}^2_k$. Finally, curves of genus 4 have a similar dichotomy: some are hyperelliptic, and the rest can be written as the intersection of the zero sets of two polynomials in $\mathbb{P}^3_k$, one of degree 2 and one of degree 3. Conversely, if a curve can be written as such an intersection, it has genus 4. The common thread among these examples is that we can write down all such curves explicitly. In fact, in each case, we can write down a single family of one or more polynomials with freely varying coefficients whose zero sets give all (or almost all, in a suitable sense) curves of the given genus. This can be expressed as saying that “the moduli space $\mathcal{M}_g$ parametrizing curves of genus $g$ is unirational,” a statement which we now explain.

Recall that a variety $X$ is **unirational** if there exists a map from an open subset of $\mathbb{A}^n_k$ to $X$ which has dense image. We will not be able to properly develop the theory of $\mathcal{M}_g$, but for now we only need to know a couple of its properties: first, it is a variety (or something of that nature) whose points correspond to curves of genus $g$, and second, given a family of curves of genus $g$, parametrizing by the points of a variety $X$, we obtain a morphism $X \to \mathcal{M}_g$ induced by, for each point of $X$, taking the corresponding curve in the family. Now, if we have a family of

\(^1\)More precisely, the curve is the nonsingular projective curve compactifying the given affine curve; one can write this down explicitly by gluing the given curve to the curve $u^2 = v^5 f(1/v)$, with gluing map given by $(x, y) \mapsto (1/x, y/x^3)$.  

polynomials with freely varying coefficients, we can think of their zero sets as a family parametrized by $\mathbb{A}^n_k$, where $n$ is the number of coefficients. In the cases above, we'll have to throw away some choices of coefficients if we want the zero sets to give nonsingular curves, but the ‘bad’ choices will be a closed subset, so we still obtain a family of curves parametrized by an open subset of $\mathbb{A}^n_k$, and as a result, we get a map from an open subset of $\mathbb{A}^k_n$ to $\mathcal{M}_g$. As we will see, the non-hyperelliptic curves always constitute an open (hence dense) subset of $\mathcal{M}_g$ for $g \geq 3$, so we get that our map has dense image in $\mathcal{M}_g$, and conclude that $\mathcal{M}_g$ is unirational for $g \leq 4$.

It is then natural to ask:

**Question 1.1.4.** Is $\mathcal{M}_g$ unirational for all $g$?

Famously, Severi erroneously claimed an affirmative answer to this question, but work of Harris, Mumford and Eisenbud proved that in fact, $\mathcal{M}_g$ is not unirational for $g \geq 23$. In practice, this means that it is not possible to explicitly write down a “general” curve of large genus.

### 1.2. Linear series

What all of the above questions have in common is that they can be analyzed (with varying degrees of directness) in terms of the theory of linear series on curves. Recall that a **linear series** of degree $d$ and dimension $r$ (often called a $g^r_d$) on a curve $C$ is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle of degree $d$ and $V \subseteq \Gamma(C, \mathcal{L})$ is an $(r + 1)$-dimensional vector space of global sections of $\mathcal{L}$.

Linear series are closely related to maps from $C$ to projective space: recall (Theorem II.7.1 of [Har77]) that maps $C \to \mathbb{P}^r_k$ of degree $d$ are in bijection with tuples $(\mathcal{L}, s_0, \ldots, s_r)$ where $\mathcal{L}$ is a line bundle on $C$ of degree $d$, and the $s_i$ are global sections of $\mathcal{L}$ such that for all $P \in C$, at least one of the $s_i$ is nonvanishing at $P$. The map is nondegenerate if the $s_i$ are linearly independent. Thus, if we have a nondegenerate map $C \to \mathbb{P}^r_k$ of degree $d$ we obtain a $g^r_d$ on $C$ by letting $V$ be the vector space spanned by the $s_i$. Two maps yield the same linear series if and only if they differ by a linear change of coordinates on (equivalently, an automorphism of; Example II.7.1.1 of [Har77]) $\mathbb{P}^r_k$. Thus motivated, we define a **base point** of a linear series $(\mathcal{L}, V)$ to be a point $P \in C$ such that $s$ vanishes at $P$ for every $s \in V$, and we say that a linear series is **basepoint free** if it has no base points.

We conclude that basepoint-free $g^r_d$s on $C$ are in bijection with nondegenerate maps $C \to \mathbb{P}^r_k$ of degree $d$, considered up to automorphism of $\mathbb{P}^r_k$. As we will see, there is a projective moduli space parametrizing $g^r_d$s on $C$, so we can think of linear series as describing a compactification of the parameter space of maps from $C$ to $\mathbb{P}^r_k$.

Thus, it is rather clear how linear series relate to our questions on the extrinsic geometry of $C$. The relation to Weierstrass points comes via the Riemann-Roch theorem, which relates the pole orders at $P$ of rational functions to the vanishing orders at $P$ of differential forms. Then, because the sheaf $\Omega^1_{C/k}$ of differentials on $C$ is a line bundle (Theorem II.8.15 of [Har77]), for $g > 0$ we have a $g^{g-1}_{2g-2}$ on $C$, called the **canonical linear series**, given by $(\Omega^1_{C/k}, \Gamma(\Omega^1_{C/k}))$. In fact it follows almost immediately from the Riemann-Roch theorem that this is the unique $g^{g-1}_{2g-2}$ on $C$ (see Example 2.2.2 below). Thus, questions on global differential forms on curves

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We will use bundle and locally free sheaf terminology interchangeably throughout.
can also naturally be addressed in the context of linear series, and it is not hard to check (see Exercise 2.3.7 below) that the pole orders considered for Weierstrass points fall into this framework. Finally, the relation to the geometry of \( \mathcal{M}_g \) is far subtler, but we can sketch it nonetheless: to analyze this geometry, it is useful to be able to produce effective divisors in \( \mathcal{M}_g \), which is to say conditions on curves that are satisfied in codimension one among all curves. One way to accomplish this is to consider curves of genus \( g \) which have a \( g^r_d \); if we choose \( r \) to be just barely too large in relation to \( g \) and \( d \) for every curve to have a \( g^r_d \), we can obtain a divisor on \( \mathcal{M}_g \), called a “Brill-Noether divisor.” It is precisely these divisors, and an additional refinement of them, which were considered by Eisenbud and Harris in their work on the geometry of \( \mathcal{M}_g \).

1.3. Degenerations

Many questions on curves can be answered by direct analysis, but others – particularly those for which the answer depends on the curve in question – require more sophisticated techniques. One of the most versatile techniques for such arguments is that of degenerations. The idea is simple enough: we expand our horizons beyond nonsingular curves, consider families of curves consisting mostly of nonsingular curves, but also containing a singular (typically nodal) curve. Over the complex numbers, such families can be pictured by choosing one or more disjoint loops on a nonsingular curve and contracting them down to a point. In some cases, the result remains irreducible, and in some cases it becomes reducible, but in either case, the effect is much the same: although the geometry on the singular curve is in some sense more complicated due to the singularities, it is in a different sense simpler, in that we can describe it as starting with one or more curves of smaller genus, and gluing them together transversally at some marked points.

Thus, if we can understand how the question of interest behaves in such families, and we can also analyze the gluing conditions that arise, we can frequently use such techniques to reduce questions about curves of general genus \( g \) to easier questions for genus 0 or 1.

Question 1.1.3 was raised and answered by Brill and Noether in the 1870’s, as follows:

**Theorem 1.3.1 (Brill-Noether).** Given \( g, r, d \), let

\[
\rho = g - (r + 1)(g - d + r).
\]

Then every curve of genus \( g \) has a \( g^r_d \) if and only if \( \rho \geq 0 \). Moreover, if \( \rho \geq 0 \), then the parameter space of \( g^r_d \)s on a general curve of genus \( g \) has dimension exactly \( \rho \).

\( \rho \) is called the **Brill-Noether number**. Brill and Noether gave a heuristic dimension count to justify their assertions, and for decades it was taken for granted that the theorem was proved. In the interim, Castelnuovo pioneered degeneration techniques of the sort discussed above in thinking about the following question:

**Question 1.3.2.** If we choose \( g, r, d \) such that a general curve of genus \( g \) has finitely many \( g^r_d \)s, then how many will it have?

Castelnuovo answered this question (again, not up to modern standards of proof) by considering degenerations to a rational irreducible nodal curve. Severi later realized that this approach had the potential to lead to a rigorous proof of the Brill-Noether theorem, but was unable to carry this through. This idea was
revived and developed by Kleiman and Laksov in the 1970's, and the final step of the program was carried out by Griffiths and Harris, leading to a proof of Theorem 1.3.1 a full century after it had originally been claimed by Brill and Noether. The technique of degenerating to an irreducible rational curve was developed further by Gieseker shortly thereafter, but then Eisenbud and Harris realized that for many situations, it was simpler to degenerate instead to certain kinds of reducible nodal curves, and they developed a general theory of “limit linear series” in this context, and applied it successfully to a range of situations. The purpose of this book is to describe the Eisenbud-Harris theory of limit linear series and its applications. We also describe new perspectives on and proofs of the foundational results which lead to a somewhat more transparent and robust theory, and also lend themselves well to generalization. Finally, while most texts in the subject restrict themselves to fields of characteristic 0, we will treat fields of positive characteristic (and even mixed-characteristic base rings) on an equal footing whenever possible.
CHAPTER 2

Linear series and ramification on nonsingular curves

In this chapter, we recall some of the most basic facts about linear series on curves. We also define and explore the concept of ramification of a linear series, which plays a vital role in the theory of limit linear series. The first two sections are largely contained in Hartshorne [Har77], but we include them for the sake of completeness. Throughout this chapter, $C$ will denote a projective, nonsingular curve over an algebraically closed field $k$.

2.1. Linear series and divisors

We begin by recalling some basic facts about line bundles and divisors. Most of the statements here may be found with proof in §II.6 of [Har77].

We will be primarily interested in global sections of line bundles, which – although we define a line bundle to be an invertible sheaf – can be described geometrically without reference to sheaves. One such description is the following, which motivates the terminology: a line bundle on a variety (or scheme) $X$ is a variety (or scheme) $Y$ with a morphism to $X$ such that on an open cover $\{U_i\}$ of $X$, we have $Y|_{U_i} \cong \mathbb{A}^1_{U_i}$, and the gluing maps on the intersections of the $U_i$ are linear, meaning in this case that they are given simply by multiplication by an invertible function on $U_i \cap U_j$. In order to go from this description to the sheaf description, we simply consider sections of $Y \to X$ defined on a given open subset $U$ of $X$, meaning morphisms $U \to Y$ such that the composition $U \to Y \to X$ is the identity. See Exercise II.5.18 of [Har77] for more on this perspective.

However, the description which is more immediately useful to us is in terms of divisors. Let $X$ be a nonsingular variety (or more generally, a connected regular scheme). Then recall that a divisor on $X$ is a formal finite sum

$$\sum a_i Z_i,$$

where each $Z_i$ is an irreducible closed subset in $X$ of codimension 1. A divisor is effective if all its coefficients are nonnegative. Note that it makes sense to restrict divisors to open subsets of $X$. If we have a nonzero rational function $f$ on $X$, there is divisor $\text{div } f$ associated to $f$, given by the orders of the zeroes and poles of $f$. If $D$ is a divisor on $X$, then we have the line bundle $\mathcal{O}_X(D)$, whose sections on $U$ consist of 0 together with rational functions $f$ on $X$ such that $(\text{div } f + D)|_U$ is effective. In particular, the global sections of $\mathcal{O}_X(D)$ consist of 0 together with nonzero rational functions $f$ on $X$ such that $\text{div } f + D$ is effective. Thus, if $D$ is positive along some $Z_i$, the line bundle $\mathcal{O}_X(D)$ allows poles (up to a prescribed
order) along \( Z_i \), while if \( D \) is negative at \( Z_i \), we impose zeroes (or at least a certain order) along \( Z_i \).

Note that both line bundles and divisors come with a natural group law. In the case of line bundles, it is induced by tensor product, which corresponds simply to multiplication of transition functions in the geometric realization. In the case of divisors, the group law is addition of coefficients. Both of these are abelian. Because multiplication of functions corresponds to addition on orders of zeroes and poles, we see that \( \mathcal{O}(D) \otimes \mathcal{O}(D') = \mathcal{O}(D + D') \).

Next, recall that divisors \( D \) and \( D' \) are said to be linearly equivalent if \( D - D' = \text{div } f \) for some rational function \( f \).

Finally, we have the following definition:

**Definition 2.1.1.** If \( L \) is a line bundle on \( X \), and \( s \in L(U) \) is a nonzero section defined on some open subset \( U \subseteq X \), then the divisor \( \text{div } s \) of zeroes and poles of \( s \) (with the poles contained in \( X \setminus U \)) is defined using the fact that on an open cover, \( L \) is isomorphic to \( \mathcal{O}_X \). Specifically, given \( U' \) and an isomorphism \( \varphi_U : L|U' \sim \mathcal{O}_X|U' \), then \( \varphi_U \) applied to \( s \) gives us a rational function \( f \) on \( X \), and we set \( \text{div } s|U' \) to be \( \text{div } f|U' \).

This gives a well-defined divisor globally on \( X \) because \( \varphi_U \) is unique up to multiplication by invertible functions on \( U' \), which doesn’t change orders of zeroes or poles (again, on \( U' \)).

We can now make the following statement summarizing the relationship between divisors and line bundles.

**Theorem 2.1.2.** The map from divisors on \( X \) to (isomorphism classes of) line bundles on \( X \) given by

\[
D \mapsto \mathcal{O}_X(D)
\]

is a surjective group homomorphism, with two divisors \( D \) and \( D' \) mapping to isomorphic line bundles if and only if they are linearly equivalent. Moreover, \( \mathcal{O}_X(D) \cong L \) if and only if \( D \) occurs as \( \text{div } s \) for some rational section \( s \) of \( L \), and \( s \) is a global section if and only if \( \text{div } s \) is effective.

Finally, if \( X \) is a projective variety, then a rational section \( s \) is determined up to multiplication by nonzero scalar by its divisor \( \text{div } s \).

Another way of stating the theorem is that there is a bijection between divisors on \( X \) and (suitably defined) isomorphism classes of pairs \((L, s)\) where \( L \) is a line bundle and \( s \) is a nonzero rational section. This is induced by \( D \mapsto (\mathcal{O}_X(D), 1) \).

This is essentially the content of Corollary II.6.16 of [Har77], albeit under an unnecessary separatedness hypothesis. See Corollary 21.6.10 of [GD66] for the general case. In the case of a projective variety, a section is determined up to scalar by its divisor because \( \Gamma(X, \mathcal{O}_X) = k \) (Theorem I.3.4 of [Har77]).

**Warning 2.1.3.** Note that \( \text{div } s \) depends very much on what line bundle we consider \( s \) to be a (rational) section of. For instance, as a section of \( \mathcal{O}_X \), the function 1 has \( \text{div } 1 = 0 \), since it has no zeroes or poles. However, considered as a rational section of \( \mathcal{O}_X(D) \), we see that 1 has \( \text{div } 1 = D \). Similarly, if a rational function \( f \) vanishes to order \( a > 0 \) along some subvariety \( Z \), then by definition as a section of \( \mathcal{O}_X \) we have that \( f \) vanishes along \( Z \). However, as a section of \( \mathcal{O}_X(-aZ) \), we find that \( f \) is nonvanishing along \( Z \).

The next definition is useful in a range of contexts.
2.2. The Riemann-Roch theorem and initial applications

The Riemann-Roch theorem is the starting point for nearly all analysis of linear series on curves. Before stating it, we note that if $D$ is a divisor on a nonsingular
curve $C$ over a field $k$, then because $\Omega^1_C$ is a line bundle, $\Omega^1_C(D)$ makes sense (Definition 2.1.4).

Also, we observe that for any line bundle $\mathcal{L}$, the global sections $\Gamma(C, \mathcal{L})$ have a natural $k$-vector space structure. If $C$ is projective, this space is in fact finite-dimensional (Theorem II.5.19 of [Har77]).

Then we have:

**Theorem 2.2.1 (Riemann-Roch).** Let $C$ be a projective, nonsingular curve of
genus $g$ over an algebraically closed field $k$. Given any line bundle $\mathcal{L}$ of degree $d$
on $C$, we have

$$\dim \Gamma(C, \mathcal{L}) - \dim \Gamma(C, \Omega^1_C \otimes \mathcal{L}^{-1}) = d + 1 - g.$$ 

Equivalently, given any divisor $D$ of degree $d$ on $C$, we have

$$\dim \Gamma(C, \mathcal{O}(D)) - \dim \Gamma(C, \Omega^1_C(-D)) = d + 1 - g.$$ 

See Theorem IV.1.3 of [Har77].

**Example 2.2.2.** We see that $\deg \Omega^1_C = 2g - 2$. Indeed, if we apply the Riemann-Roch theorem with $\mathcal{L} = \Omega^1_C$, we have

$$d + 1 - g = \dim \Gamma(C, \Omega^1_C) - \dim \Gamma(C, \Omega^1_C \otimes (\Omega^1_C)^{-1})$$

$$= \dim \Gamma(C, \Omega^1_C) - \dim \Gamma(C, \mathcal{O}_C)$$

$$= g - 1,$$

giving $d = 2g - 2$.

Thus, we see that the canonical linear series is a $g^1_{2g-2}$. In fact, it is the only one: if $\mathcal{L}$ has degree $2g - 2$, then the Riemann-Roch theorem gives

$$\dim \Gamma(C, \mathcal{L}) - \dim \Gamma(C, \Omega^1_C \otimes \mathcal{L}^{-1}) = 2g - 2 + 1 - g = g - 1.$$ 

Now, $\Omega^1_C \otimes \mathcal{L}^{-1}$ has degree $0$, so Example 2.1.6 says that $\dim \Gamma(C, \Omega^1_C \otimes \mathcal{L}^{-1})$ is 0 or 1, with the latter occurring only if $\Omega^1_C \otimes \mathcal{L}^{-1} \cong \mathcal{O}_C$, or equivalently, $\mathcal{L} \cong \Omega^1_C$. We thus see that $\dim \Gamma(C, \mathcal{L}) \leq g$, with equality if and only if $\mathcal{L} \cong \Omega^1_C$, giving the desired statement.

We also see that the canonical linear series is basepoint free. Indeed, if $P \in C$ is any point, we have $P$ a base point of the canonical linear series if and only if $\Gamma(C, \Omega^1_C(-P)) = \Gamma(C, \mathcal{O}_C)$, which by the Riemann-Roch theorem implies $\Gamma(C, \mathcal{O}_C) \subseteq \Gamma(C, \mathcal{O}_C(P))$. The latter implies that $C$ has a $\mathfrak{g}^1_1$, which is impossible by Example 2.1.7 because $g > 0$.

**Example 2.2.3.** If $d := \deg \mathcal{L} > 2g - 2$, then $\dim \Gamma(C, \mathcal{L}) = d + 1 - g$. Indeed, in this case $\Omega^1_C \otimes \mathcal{L}^{-1}$ has negative degree, so the second term in the Riemann-Roch theorem vanishes by Example 2.1.5.

**Example 2.2.4.** If $g = 0$, then according to Example 2.2.3 we have $\dim \Gamma(C, \mathcal{L}) = d + 1$ for any line bundle $\mathcal{L}$ of degree $d$. It follows from Example 2.1.7 that $C \cong \mathbb{P}^1$.

**Example 2.2.5.** We see that if $g = 0, 1$ or 2, then $C$ always has a $\mathfrak{g}^1_1$. Indeed, if $g \leq 1$, then by Example 2.2.3, every line bundle $\mathcal{L}$ of degree 2 has $\dim \Gamma(C, \mathcal{L}) = 2 + 1 - g \geq 2$. On the other hand, if $g = 2$ then the canonical linear series is a $\mathfrak{g}^2_2$.

We next want to consider when a linear series defines a closed immersion into projective space. It will be helpful to consider more generally what happens to a linear series when we impose vanishing along an effective divisor.
NOTATION 2.2.6. Let \((\mathcal{L}, V)\) be a linear series on \(C\). Given an effective divisor \(D = \sum_i a_i P_i\) on \(C\), write
\[
V(-D) = V \cap \Gamma(C, \mathcal{L}(-D)).
\]

That is, \(V(-D)\) is the subspace of \(V\) consisting of sections which vanish to order at least \(a_i\) at \(P_i\) for all \(i\), when considered as sections of \(\mathcal{L}\). Bear in mind that, as per Warning 2.1.3, order of vanishing at a point depends on the choice of which line bundle the section is in.

The basic observation on orders of vanishing is then the following.

PROPOSITION 2.2.7. Given a linear series \((\mathcal{L}, V)\) and a effective divisors \(D, D'\) on \(C\), we have
\[
\dim V(-D) - \dim V(-D - D') \leq \deg D'.
\]

PROOF. The proof is by induction on \(\deg D'\), with the base case \(D' = 0\) being trivial. Let \(P\) be a point contained in \(D'\); then by the induction hypothesis, it is enough to show that \(\dim V(-(D + D' - P)) - \dim V(-(D - D')) \leq 1\). But if \(V(-(D + D')) \neq V(-(D + D' - P))\), and \(s \in V(-(D + D' - P)) \setminus V(-(D + D'))\), we see that \(s\) spans \(V(-(D + D' - P)) \setminus V(-(D + D'))\). Indeed, given any \(s' \in V(-(D + D' - P)) \setminus V(-(D + D'))\), since both sections vanish to the same order at \(P\), we can subtract a scalar multiple of \(s\) from \(s'\) to obtain a section vanishing to strictly higher order at \(P\), which gives the desired statement.

We can now state a useful criterion for closed immersions. We already know when a linear series \((\mathcal{L}, V)\) defines a morphism to projective space: when it is basepoint free. Being basepoint free is equivalent to saying that \(\dim V(-P) = \dim V - 1\) for all \(P \in C\). It is easy to see from the definitions that if \((\mathcal{L}, V)\) is basepoint-free, the associated morphism is injective if and only if \(\dim V(-P - Q) = \dim V - 2\) for all distinct \(P, Q \in C\). With a little more work, we can extend this to a criterion for when the morphism is a closed immersion:

PROPOSITION 2.2.8. A linear series \((\mathcal{L}, V)\) on \(C\) of dimension \(r\) defined a closed immersion \(C \rightarrow \mathbb{P}^r\) if and only if for all (not necessarily distinct) \(P, Q \in C\), we have
\[
\dim V(-P - Q) = \dim V - 2.
\]

The idea is that once we know that the morphism is injective, it is enough to check that it is also injective on tangent spaces, which corresponds to the case \(P = Q\). For the details, see Proposition IV.3.1 of [Har77]; this is phrased in terms of complete linear series, but the general case follows exactly the same argument.

We can thus obtain some statements on which linear series define immersions. Recall that a linear series is **complete** if it is of the form \((\mathcal{L}, \Gamma(C, \mathcal{L}))\) for some line bundle \(\mathcal{L}\) on \(C\). The following is then an immediate consequence of Proposition 2.2.8 together with Example 2.2.3.

COROLLARY 2.2.9. A complete linear series of degree at least \(2g + 1\) defines a closed immersion into projective space.

We can also study when the canonical linear series defines a closed immersion.

PROPOSITION 2.2.10. The canonical linear series on \(C\) defines a closed immersion into \(\mathbb{P}^{g-1}\) if and only if \(C\) does not have any \(g_2^1\).
Thus, in light of Example 2.2.5 we have that the canonical linear series on $C$ defines a closed immersion if and only if $C$ has genus at least 2 and is not hyperelliptic.²

**Proof.** Given any $P, Q \in C$, the Riemann-Roch theorem gives us
\[
\dim \Gamma(C, \Omega_C^1(-P - Q)) = \dim \Gamma(C, \mathcal{O}(P + Q)) - 2 - 1 + g.
\]
First, suppose $(\mathcal{L}, V)$ is a $\mathcal{g}_1^2$ on $C$. Then in particular, $\mathcal{L} = \mathcal{O}(D)$ for an effective divisor $D$ of degree 2, say $D = P + Q$. Then by the above, we have
\[
\dim \Gamma(C, \Omega_C^1(-P - Q)) \geq g - 1,
\]
so applying Proposition 2.2.8 to the canonical linear series, we see that it does not define a closed immersion.

Conversely, if $C$ does not have a $\mathcal{g}_1^2$, then for any $P, Q \in C$, we have
\[
\dim \Gamma(C, \mathcal{O}(P + Q)) = 1,
\]
so the above equation gives us
\[
\dim \Gamma(C, \Omega_C^1(-P - Q)) = g - 2,
\]
and Proposition 2.2.8 says that the canonical linear series does define a closed immersion. □

As a consequence of Proposition 2.2.7 together with the Riemann-Roch theorem, we can draw some elementary restrictions on dimensions of spaces of global sections of line bundles.

**Corollary 2.2.11.** Let $\mathcal{L}$ be a line bundle of degree $d$, with $0 \leq d \leq 2g - 2$. Then
\[
\max\{d + 1 - g, 0\} \leq \dim \Gamma(C, \mathcal{L}) \leq \min\{d + 1, g\}.
\]
**Proof.** The first inequality is immediate from the Riemann-Roch theorem. For the second, let $D$ be an effective divisor of degree $d + 1$, and set $\mathcal{M} = \mathcal{L}(-D)$. Then by Example 2.1.5, we have $\dim \Gamma(C, \mathcal{M}) = 0$. At the same time, by Proposition 2.2.7, we have
\[
\dim \Gamma(C, \mathcal{L}) \leq \dim \Gamma(C, \mathcal{M})(-D) + d + 1 = \dim \Gamma(C, \mathcal{M}) + d + 1 = d + 1.
\]
On the other hand, if instead we set $D$ to be an effective divisor of degree $2g - 1 - d$, and set $\mathcal{M} = \mathcal{L}(D)$, then by Example 2.2.3 we have $\Gamma(C, \mathcal{M}) = g$, and $\Gamma(C, \mathcal{L}) \subseteq \Gamma(C, \mathcal{M})$, so we obtain the second inequality as well. □

**Remark 2.2.12.** We remark in passing that a slightly more careful version of the argument for Corollary 2.2.11 shows that the given lower bound is sharp, but the upper bound is far from optimal. Indeed, Clifford’s theorem (§III.1 of [ACGH85]) asserts that if $\mathcal{L}$ is a line bundle of degree $d$, with $0 \leq d \leq 2g - 1$, then
\[
\dim \Gamma(C, \mathcal{L}) \leq \frac{d}{2} + 1,
\]
with equality occurring only in one of the following three cases: $\mathcal{L} = \mathcal{O}_C$; $\mathcal{L} = \Omega_C^1$; or $\mathcal{L}$ is a tensor power of a line bundle of degree 2 with a 2-dimensional space of global sections (in particular, in this case $C$ is hyperelliptic).

### 2.3. Ramification of linear series

As mentioned previously, the concept of ramification plays an important role in the theory of limit linear series. However, it is equally important for its own sake, as it simultaneously encapsulates many phenomena of classical algebraic geometry.

Proposition 2.2.7 implies in particular that if we consider a sequence of the form
\[
V, V(-P), V(-2P), \ldots, V(-(d + 1)P)
\]
the dimension can drop by at most 1 at each step, starting at $r + 1$, and ending at 0. We can thus make the following definition.

²For a bit more on hyperelliptic curves, see Exercise IV.1.7 and Example IV.5.5.5 of [Har77].
2.3. RAMIFICATION OF LINEAR SERIES

Definition 2.3.1. Given a linear series \((\mathcal{L}, V)\) and a point \(P \in C\), the **vanishing sequence** \(a_0(P), \ldots, a_r(P)\) of \((\mathcal{L}, V)\) at \(P\) is defined to be the strictly increasing sequence of orders of vanishing at \(P\) of sections of \(V\). The non-decreasing **ramification sequence** \(\alpha_0(P), \ldots, \alpha_r(P)\) is defined by \(\alpha_i(P) = a_i(P) - i\). If \(\alpha_i(P) = 0\) for all \(i\), we say \((\mathcal{L}, V)\) is **unramified** at \(P\), otherwise \((\mathcal{L}, V)\) is **ramified** at \(P\), or \(P\) is a **ramification point** of \((\mathcal{L}, V)\).

Thus, by definition the vanishing sequence is bounded between 0 and \(d\), and the ramification sequence between 0 and \(d - r\).

When a linear series defines a morphism \(C \to \mathbb{P}^r\), we can think of the vanishing sequence geometrically in terms of the possible multiplicities at \(P\) of preimages of hyperplanes in \(\mathbb{P}^r\). We now discuss some basic examples of ramification.

Example 2.3.2. We see that \((\mathcal{L}, V)\) is basepoint free, and hence corresponds to a map to \(\mathbb{P}^r\), when \(a_0(P) = 0\) for all \(P\). More generally, a point \(P\) is a base point if and only if \(a_0(P) > 0\). Since any given non-zero section can have only finitely many zeroes, we see immediately that a linear series can have at most finitely many base points.

Example 2.3.3. If \(r = 1\), and \((\mathcal{L}, V)\) is basepoint free, then we obtain a map \(C \to \mathbb{P}^1\), so we have the usual notion of ramification of a map of curves (§IV.2 of [Har77]). In this case, the two notions of ramification coincide, although there is some ambiguity: in classical algebraic geometry, the ramification index is typically set to be \(\alpha_1(P) = a_1(P) - 1\), while in arithmetic algebraic geometry, it is set to \(a_1(P)\).

Example 2.3.4. If \(r = 2\) and \((\mathcal{L}, V)\) is basepoint free, suppose also that the induced map \(f : C \to \mathbb{P}^2\) is birational onto its image. Then a point with \(\alpha_1(P) > 0\) corresponds to a cusp-type singularity in \(f(C)\), since the preimage of any line through \(f(P)\) must contain \(P\) with multiplicity larger than 1.

On the other hand, if \(f(P)\) is a nonsingular point of \(f(C)\) (so that in particular, \(\alpha_1(P) = 0\)), a point with \(\alpha_2(P) > 0\) corresponds to an inflection point (also called a flex point). Indeed, this condition says that the preimages of some lines through \(f(P)\) contain \(P\) exactly once, but that a line through \(f(P)\) whose preimage contains \(P\) at least twice (i.e., the tangent line to \(f(C)\) at \(f(P)\)) must have its preimage containing \(P\) at least three times.

Exercise 2.3.7 below provides another example, showing that a ramification point of the canonical linear series is the same as a Weierstrass point.

It is helpful to have a good notion of multiplicity of a ramification point, which turns out to be the simplest idea one might imagine: we simply consider the sum of the entries in the ramification sequence. A basic formula for the total ramification of a linear series is the following Plucker formula (see Proposition 1.1 of [EH86]):

---

3A ramification point is sometimes also called an inflection point or generalized Weierstrass point.

4Both \(\alpha_1(P)\) and \(a_1(P)\) arise naturally in results on nonconstant maps of curves: the former in the Riemann-Hurwitz formula, and the latter in the formula for the number of points in each fiber of the map.
Theorem 2.3.5. Let \((\mathcal{L}, V)\) be a \(g^r_d\) on a nonsingular projective curve \(C\) of genus \(g\), and suppose the characteristic of the base field \(k\) is either 0 or \(p > d\). Then
\[
\sum_{P \in C} \sum_{i=0}^{r} \alpha_i(P) = (r+1)d + \binom{r+1}{2}(2g - 2).
\]
In particular, \((\mathcal{L}, V)\) has only finitely many ramification points.

Example 2.3.6. In the case \(r = 1\), if we assume our linear series is basepoint free, we get
\[
\sum_{P \in C} \alpha_1(P) = 2d + 2g - 2,
\]
which recovers the Riemann-Hurwitz formula (Corollary IV.2.4 of [Har77]) for the case that the target curve is \(\mathbb{P}^1\).

Exercise 2.3.7. Suppose the base field has characteristic 0. Given a nonsingular projective curve \(C\) of genus \(g\), and a point \(P \in C\), define the Weierstrass semigroup \(H_P\) to be the set of \(n \geq 0\) such that there exists a rational function \(f\) on \(C\) with a pole of order \(n\) at \(P\), and no other poles. Define the Weierstrass gap sequence \(G_P\) to be \(\mathbb{Z} \geq 0 \setminus H_P\), so that by definition, a point is a Weierstrass point if and only if \(G_P \neq \{1, \ldots, g\}\). Show the following:

(a) \(H_P\) is a semigroup under addition;
(b) \(G_P = \{a_i(P) + 1 : i = 0, \ldots, g-1\}\), where \(a_i(P)\) is the vanishing sequence of the canonical linear series at \(P\). In particular, \(P\) is a Weierstrass point if and only if it is a ramification point of the canonical linear series.
(c) If we define the weight of a Weierstrass point \(P\) to be \((\sum_{n \in G_P} n) - \binom{g+1}{2}\), then the number of Weierstrass points on \(C\), counted by weight, is \(g(g+1)(g-1)\).
(d) The weight of a Weierstrass point is at most \((\frac{g}{2})\), with equality if and only if \(2 \in H_P\).
(e) For \(g \geq 2\), the curve \(C\) is hyperelliptic if and only if it has a Weierstrass point \(P\) of weight \((\frac{g}{2})\). Moreover, in this case, \(C\) has precisely \(2g + 2\) Weierstrass points, and for each of them we have equality in (d).

When the base field \(k\) has characteristic \(p \leq d\), then Theorem 2.3.5 does not hold as stated. Just as in the case with the Riemann-Hurwitz formula, problems arise in two ways: with wild ramification, and inseparability.

Definition 2.3.8. Given a linear series \((\mathcal{L}, V)\) on \(C\), and a ramification point \(P\) of \((\mathcal{L}, V)\), we say that \((\mathcal{L}, V)\) is tamely ramified at \(P\) if the either the characteristic of the base field is 0, or if the characteristic is \(p\) and the \(a_i(P)\) are maximally distributed modulo \(p\). Otherwise, \((\mathcal{L}, V)\) is wildly ramified at \(P\).

By the \(a_i(P)\) being maximally distributed modulo \(p\), we mean that the differences \(a_j(P) - a_i(P)\) are multiples of \(p\) as infrequently as possible; formally, that
\[
\prod_{0 \leq i < j \leq r} \frac{a_j(P) - a_i(P)}{j - i}
\]
is nonzero modulo \(p\). Note that this actually requires maximal distribution mod powers of \(p\) as well. See also Exercise 2.3.15 below.

Note that the sequence \(0, 1, 2, \ldots, r\) is maximally distributed modulo \(p\), so we can think of unramified points as being tamely ramified.
We say that a linear series is **inseparable** if every point is a ramification point. Otherwise, it is **separable**.

We can now state the more general version of Theorem 2.3.5:

**Theorem 2.3.10.** Let \((L, V)\) be a \(g^r_d\) on a nonsingular projective curve \(C\) of genus \(g\), and suppose that \((L, V)\) is separable. Then

\[
\sum_{P \in C} \sum_{i=0}^{r} \alpha_i(P) \leq (r + 1)d + \binom{r + 1}{2}(2g - 2),
\]

with equality if and only if \((L, V)\) is nowhere wildly ramified. In particular, \((L, V)\) has only finitely many ramification points.

Furthermore, if the characteristic of the base field is equal to 0 or is strictly greater than \(d\), every linear series is separable.

**Remark 2.3.11.** Our terminology of wild ramification and inseparability is not standard in the context of linear series; we have borrowed it from the established terminology for maps of curves. In the case \(r = 1\), our terminology agrees precisely with the usual terminology for the associated map \(C \to \mathbb{P}^1\), considered as a map of curves. In general, perhaps some justification can be offered for the inseparability terminology due to the relationship between inseparability in our sense and inseparability of higher Gauss maps.

**Example 2.3.12.** For the case \(r = 1\), wild ramification and inseparability are both very well studied. A linear series on \(C\) is inseparable if and only if the corresponding map \(C \to \mathbb{P}^1\) is inseparable, which occurs if and only if the map factors through the Frobenius morphism on \(C\). Wild ramification is in some respects harder to control, but not unusual. It occurs precisely when a point has ramification index which is a multiple of \(p\). A prototypical example is the Artin-Shreier map \(\mathbb{P}^1 \to \mathbb{P}^1\) given by

\[x \mapsto x^p - x\]

in characteristic \(p\). This is a degree \(d\) map which is unramified on \(\mathbb{A}^1\), but wildly ramified at \(\infty\).

**Example 2.3.13.** The following example is Exercise IV.2.4 of [Har77]: the plane quartic curve

\[x^3y + y^3z + z^3x = 0\]

in characteristic 3 is a (canonically imbedded) nonsingular genus 3 such that every point is an inflection point. Hence, for this curve we see that the canonical linear series is inseparable. Not coincidentally, the natural map from the curve to its dual curve is inseparable.

**Remark 2.3.14.** If \(k\) is not algebraically closed, the natural context in which to define ramification sequence is for \(k\)-valued points of \(C\). In this case, Proposition 2.2.7 still holds via the same argument, so one can still define the vanishing and ramification sequences. Moreover, vanishing sequences will be invariant under extension of \(k\), so one can use the algebraically closed case to study the general case.

However, the Plucker formula does not hold, because some ramification could occur at non-\(k\)-rational closed points of \(C\).
Exercise 2.3.15. Show that the fraction in (2.3.1) is always an integer, by realizing it as the determinant of the matrix whose $i,j$th entry is $(a_j(P))$.

2.4. Moduli spaces of line bundles

We now discuss the Picard variety of a nonsingular projective curve $C$, whose points parametrize line bundles of fixed degree on $C$. We summarize the main theorem as follows:

**Theorem 2.4.1.** Given a projective nonsingular curve $C$ of genus $g$ over an algebraically closed field $k$, and an integer $d$, there exists a variety $\text{Pic}^d(C)$, called the $d$th **Picard variety** of $C$, with the following properties:

(i) There is a line bundle $\mathcal{L}$ on $\text{Pic}^d(C) \times_k C$, called the **Poincare line bundle**, such that for all points $Q \in \text{Pic}^d(C)$, the restriction $\mathcal{L}|_{\{Q\} \times_k C}$ is a line bundle of degree $d$. Furthermore, such restriction induces a bijection between $\text{Pic}^d(C)$ and the set of isomorphism classes of line bundles of degree $d$ on $C$.

(ii) For any variety (or scheme) $X$ over $k$, and a line bundle $L$ on $X \times_k C$ such that every fiber is a line bundle of degree $d$ on $C$, there is an induced morphism $\varphi_L : X \to \text{Pic}^d(C)$, with the property that $L$ is isomorphic to $\varphi^* \mathcal{L}$ after restriction to every point of $X$.

(iii) $\text{Pic}^d(C)$ is nonsingular and projective, of dimension $g$.

(iv) In the case $d = 0$, the group structure on $\text{Pic}^0(C)$ arising from tensor product of line bundles is algebraic, meaning that the multiplication map $\text{Pic}^0(C) \times_k \text{Pic}^0(C) \to \text{Pic}^0(C)$ and inverse map $\text{Pic}^0(C) \to \text{Pic}^0(C)$ are both morphisms.

(v) Given an effective divisor $D$ of degree $d'$, tensor product by $\mathcal{O}(D)$ induces an isomorphism

$$\text{Pic}^d(C) \cong \text{Pic}^{d+d'}(C).$$

Note that while $\text{Pic}^d(C)$ is unique up to isomorphism, the Poincare line bundle is not. Nonetheless, the first condition implies that the points of $\text{Pic}^d(C)$ are in bijection with line bundles of degree 0 on $C$. In the case $d = 0$, one usually refers to $\text{Pic}^0(C)$ as the **Jacobian** of $C$. As a projective group variety, the Jacobian is the prototypical example of an abelian variety.

Having introduced the Picard varieties, it is natural to discuss the Abel-Jacobi map. This map will not play an important role in our story, but it is nonetheless a basic tool for studying linear series. First, we have the following definition:

**Definition 2.4.2.** Given a smooth curve $C$, the $d$th **symmetric product** of $C$, written $S^dC$, is the quotient of the $d$-fold product $C^d$ by the natural permutation action of $S_d$.

**Proposition 2.4.3.** $S^dC$ has the structure of a nonsingular, $d$-dimensional variety, with a natural quotient map $C^d \to S^dC$, and it also has the property that a map $C^d \to X$ factors through $S^dC$ if and only if it is invariant under the permutation action of $S_d$.

Now, we see that $S^dC$ can be thought of as parametrizing effective divisors of degree $D$ on $C$. Moreover, we have a tautological morphism $C^d \to \text{Pic}^d(C)$ induced
by $(P_1, \ldots, P_d) \mapsto \mathcal{O}(P_1 + \cdots + P_d)$, and since this map is $S_d$-invariant, we see that we obtain a morphism

\[(2.4.1) \quad A_d(C) : S^d C \to \text{Pic}^d(C),\]

induced by sending an effective divisor $D$ to $\mathcal{O}(D)$. This map is called the $d$th Abel-Jacobi map.

According to Theorem 2.1.2, we conclude that the fibers of $A_d(C)$ consist of all effective divisors linearly equivalent to a given one. Thus, we can think of them as being described by complete linear series, from the divisor point of view. This means that they ought to be projective spaces, and in fact they are:

**Proposition 2.4.4.** The fiber of $A_d(C)$ over a point of Pic$^d(C)$ corresponding to a line bundle $\mathcal{L}$ is the set of effective divisors associated to the complete linear series $(\mathcal{L}, \Gamma(C, \mathcal{L}))$. It is a projective space of dimension $\dim \Gamma(C, \mathcal{L}) - 1$, and every linear series $(\mathcal{L}, V)$ of dimension $r$ and with underlying line bundle $\mathcal{L}$ corresponds to an $r$-dimensional linear subspace of the fiber.

**Example 2.4.5.** In the case of $g = d = 1$, the Abel-Jacobi map $A_1(C)$ induces an isomorphism from $C$ to Pic$^1(C)$. Indeed, by Example 2.2.3, if $D$ is a divisor of degree 1 then $\Gamma(C, \mathcal{O}(D))$ is 1-dimensional, so $D$ is linearly equivalent to some effective divisor, which must then be $P$ for some $P \in C$. Thus, $A_1(C)$ is surjective. On the other hand, its fibers are projective spaces by Proposition 2.4.4, and they must be 0-dimensional since $A_1(C)$ is a surjective map of curves, so we conclude that $A_1(C)$ has degree 1, and must be an isomorphism.

Since Pic$^d(C) \cong$ Pic$^1(C)$ for all $d$, we conclude that Pic$^d(C) \cong C$ for all $d$.

### 2.5. Brill-Noether theory

We now move on to formulating the basic statements of Brill-Noether theory more precisely. We begin by defining the loci in Pic$^d(C)$ studied by Brill and Noether.

**Definition 2.5.1.** Given $r, d$ nonnegative integers, the **Brill-Noether locus** $W^r_d(C)$ is the subset of Pic$^d(C)$ consisting of points corresponding to line bundles $\mathcal{L}$ with $\dim \Gamma(C, \mathcal{L}) \geq r + 1$.

Thus, we have nested sequence

$$\text{Pic}^d(C) \supseteq W^0_d(C) \supseteq W^1_d(C) \supseteq \ldots.$$ 

The basic observation about $W^r_d(C)$ is the following.

**Proposition 2.5.2.** $W^r_d(C)$ is a closed subset of Pic$^d(C)$, and if it is nonempty, every irreducible component has dimension at least $\rho$ (as in (1.3.1)) as long as $\rho \leq g$.

**Proof.** Choose an effective divisor $D$ of degree at least $2g - 1 - d$. Denote by $D'$ the divisor Pic$^d(C) \times_k D$ on Pic$^d(C) \times_k C$. Then every fiber of $\mathcal{L}(D')$ has degree $d + \deg D \geq 2g - 1$, so by Example 2.2.3, every fiber has a $(d + \deg D + 1 - g)$-dimensional space of global sections. Thus, if we push $\mathcal{L}(D')$ forward to Pic$^d(C)$ under the first projection $p_1$, by Corollary III.12.9 of [Har77] we obtain a vector bundle on Pic$^d(C)$ of rank $d + \deg D + 1 - g$, whose fibers at a point are identified with the global sections of the restriction of $\mathcal{L}(D')$ to that point.
We want to describe the sections of $p_1\cdot\mathcal{L}(D')$ which come from sections of $\mathcal{L}$ itself, rather than $\mathcal{L}(D')$. In order to do this, we consider the restriction $\mathcal{L}(D')|_{D'}$. We have an exact sequence
\[ 0 \to \mathcal{L} \to \mathcal{L}(D') \to \mathcal{L}(D')|_{D'} \to 0, \]
which we think of as saying that $\mathcal{L}(D')$ allows poles along points of $D'$, but requiring that a section maps to $0$ in $\mathcal{L}(D')|_{D'}$ is the same thing as requiring that it did not in fact have poles in the first place. Now, because $\mathcal{L}(D')|_{D'}$ is only supported at finitely many points on each fiber (in fact, $\deg D$ points if counted with multiplicity), we have that $p_1\cdot\mathcal{L}(D')|_{D'}$ is also a vector bundle, of rank $\deg D$.

Then $W_d^r(C)$ is precisely the locus on $\text{Pic}^d(C)$ on which the map
\[ p_1\cdot\mathcal{L}(D') \to p_1\cdot\mathcal{L}(D')|_{D'} \]
has kernel of dimension at least $r+1$, or equivalently, has rank less than or equal to $d + \deg D + 1 - g - (r + 1) = d + \deg D - g - r$. If $d \geq g + r$, this is automatically satisfied, since the target vector bundle has rank only $\deg D$, so in this case $W_d^r(C) = \text{Pic}^d(C)$, and we see that we have $\rho \geq g$. Otherwise, this condition is locally cut out by the $(d + \deg D - g - r + 1) \times (d + \deg D - g - r + 1)$ minors of $(d + \deg D + 1 - g) \times \deg D$ matrix, so it is closed as claimed, and by the classical theory of determinantal loci, it has every component of codimension at most
\[ (d + \deg D + 1 - g - (d + \deg D - g - r)) \cdot (\deg D - (d + \deg D - g - r)) = (r + 1)(g + r - d), \]
so must have dimension at least $g - (r + 1)(g + r - d)$, as claimed. \hfill \Box

**Remark 2.5.3.** The reason for the necessity of the trick of twisting up by $D'$ is that otherwise the pushforward doesn’t have the desired behavior. For instance, if most line bundles of degree $d$ on $C$ have no nonzero global sections, the pushforward of $\mathcal{L}$ will be equal to $0$, even if some line bundles did have non-zero global sections. We express this by saying that pushforward does not commute with restriction to points in general, and the construction we have used is a standard way of getting around this; a special and explicit case of the theory of cohomology and base change (§III.12 of [Har77]; in particular, we are producing an explicit version of the complex of Proposition III.12.2).

**Remark 2.5.4.** The description in the proof of Proposition 2.5.2 of $W_d^r(C)$ as being cut out by determinantal conditions inside of $\text{Pic}^d(C)$ also can be used to give $W_d^r(C)$ a scheme structure, which one can check is independent of the choice of the divisor $D$ used in the construction. In fact, although $W_d^r(C)$ is a moduli space, it is not so easy to describe it canonically in terms of a functor of points, so this is the easiest way to endow it with a scheme structure.

Famously, in the 1870’s Brill and Noether asserted that for a “general” curve, $W_d^r(C)$ should have dimension exactly $\rho$, and while this turns out to be true, it took a century to produce a proof. However, the resulting theorem is still inevitably called the “Brill-Noether theorem.” It is natural to wonder not only about the dimension of $W_d^r(C)$, but also about other associated properties. We summarize the basic known results as follows, in non-chronological order.

**Theorem 2.5.5.** Let $C$ be a nonsingular projective curve of genus $g$. Then:
(i) if $\rho \geq 0$, we have $W^r_d(C)$ nonempty, with every component of dimension at least $\rho$ as long as $\rho \leq g$;
(ii) if $\rho \geq 1$, we have $W^r_d(C)$ connected;
(iii) if $C$ is general, then $W^r_d(C)$ has pure dimension $\min\{\rho, g\}$, and in particular is empty if $\rho < 0$;
(iv) if $C$ is general and $\rho = 0$, then $W^r_d(C)$ consists of
$$g! \prod_{i=0}^r \frac{i!}{(g - d + r + i)!}$$
points.
(v) if $C$ is general, then $W^r_d(C)$ is nonsingular on the complement of $W^{r+1}_d(C)$.

The classical Brill-Noether theorem is (i) and (iii); (i) was proved by Kempf, Kleiman and Laksov, while (iii) was proved by Griffiths and Harris, together with (iv), based on ideas of Severi and Castelnuovo. (v) was proved by Gieseker based on ideas of Petri, and (ii) was proved by Fulton and Lazarsfeld [FL81]. One of our main goals will be to give proofs of (i), (iii), (iv) and (v) using limit linear series.

Remark 2.5.6. Now is as good a time as any to discuss what is meant by “general curve.” The idea is simple enough: there should be a moduli space $M_g$ whose points naturally parametrize nonsingular projective curves of genus $g$, and if a statement is asserted to be true for a “general curve,” it should mean that there is a dense open subset $U \subseteq M_g$ such that the statement in question is true for every curve corresponding to a point of $U$. In fact, $M_g$ should be irreducible, so that $U$ is automatically dense if it is nonempty. It should then be enough to show that the property holds for a (possibly empty) open subset of $M_g$, and also that it holds for a single curve.

To formalize these statements on $M_g$ is difficult because the non-uniform behavior of automorphisms of curves (i.e., different curves have different automorphism groups) means that if one forces $M_g$ to be a variety, or even a scheme, it will not be well behaved as a moduli space, and we would have to introduce the machinery of coarse moduli spaces to describe it adequately. The other option is to treat $M_g$ as an algebraic stack, which is better behaved from a moduli point of view, but requires introducing an even larger machinery generalizing our basic geometric objects of study.

Nonetheless, we can easily formalize what needs to be done in order to prove a statement holds for a general curve: we first show that it is an open condition, in the sense for if we have a family of curves over a base $T$, then the set of points of $T$ on which the statement holds is an open subset (possibly empty); we then show that the statement is true for at least one curve. Thus, rather than developing either formalism of $M_g$, we will opt instead to state and prove our results in ways such as the above which are fully rigorous, do not reference $M_g$ directly, but, once a theory of $M_g$ is developed, immediately imply the desired statements about it.

Remark 2.5.7. Despite working over complex varieties throughout, and making extensive use of singular cohomology in their connectedness argument, Fulton and Lazarsfeld point out in Remark 2.8 of [FL81] that their argument can be adapted to arbitrary characteristic via the use of etale cohomology.

Remark 2.5.8. Fulton and Lazarsfeld used Cohen-Macaulayness of determinantal loci and a connectedness theorem of Hartshorne to prove that (ii) and (v) of
Theorem 2.5.5 together imply that if \( C \) is general and \( \rho \geq 1 \), then \( W_d(C) \) is in fact irreducible. However, following [ACGH85] we will give a more straightforward deduction of this fact using moduli of linear series in Corollary 2.6.5 below.

## 2.6. Moduli of linear series

We can think of the loci \( W_d'(C) \) as being a moduli space for line bundles \( \mathcal{L} \) of degree \( d \) such that there exists an \((r+1)\)-dimensional space of global sections of \( \mathcal{L} \). The modern point of view of moduli theory is that one is likely to get a better-behaved moduli space if one instead considers line bundles \( \mathcal{L} \) of degree \( d \) together with an \((r+1)\)-dimensional space of global sections – that is, a moduli space of \( g_d^r \)’s. We will put off a formal definition of this moduli space until we discuss curves in families, but as we did with the space \( W_d'(C) \), we will describe how to construct it, and deduce some basic properties. For this construction, we will need the relative Grassmannian. Informally, if we have a vector bundle \( \mathcal{E} \) of rank \( d \) on \( X \), for \( r \) between 0 and \( d \) the relative Grassmannian \( G(r, \mathcal{E}) \) is a moduli space for \( r \)-dimensional subspaces of fibers of \( \mathcal{E} \). It maps to \( X \), and the fiber over a point \( x \in X \) is the Grassmannian \( G(r, d) \) obtained as the space of \( r \)-dimensional subspaces of the fiber \( \mathcal{E}|_x \).

**Proposition 2.6.1.** Given a nonsingular projective curve \( C \) of genus \( g \), and \( r, d \) positive integers, there is a space \( G^r_d(C) \) parametrizing \( g_d^r \)’s on \( C \). It is projective, and has the property that every component has dimension at least \( \rho \) (as in (1.3.1)).

Unlike \( \text{Pic}^d(C) \), for \( G^r_d(C) \) there will in general be a difference whether it is considered as a variety or a scheme (that is, it may naturally be nonreduced), so it is better to consider it as a scheme. However, the statements of Proposition 2.6.1 do not involve the scheme structure, and the construction we describe makes sense in either context.

**Proof.** The argument is similar to the construction of \( W_d'(C) \) in Proposition 2.5.2. Again let \( D \) be an effective divisor of degree at least \( 2g-1-d \), and denote by \( D' \) the divisor \( \text{Pic}^d(C) \times_k D \) on \( \text{Pic}^d(C) \times_k C \). We will cut out \( G^r_d(C) \) inside the relative Grassmannian \( G(r+1, p_1*\mathcal{L}(D')) \) over \( \text{Pic}^d(C) \), which parametrizes choices of \((r+1)\)-dimensional subspaces of global sections of fibers of \( \mathcal{L}(D') \). Since \( \text{Pic}^d(C) \) itself parametrizes line bundles of degree \( d \) on \( C \), we see that \( G(r+1, p_1*\mathcal{L}(D')) \) parametrizes pairs \((\mathcal{L}, V)\), where \( \mathcal{L} \) is a line bundle of degree \( d \), and \( V \) is an \((r+1)\)-dimensional space of global sections of \( \mathcal{L}(D) \). Thus, \( G^r_d(C) \) is cut out inside \( G(r+1, p_1*\mathcal{L}(D')) \) by the condition that \( V \) in fact is contained in \( \mathcal{L} \) rather than \( \mathcal{L}(D) \). To impose this condition, we use the same method as before: let 

\[
g : G(r+1, p_1*\mathcal{L}(D')) \to \text{Pic}^d(C)
\]

be the structure map, and

\[
\forall : g^*p_1*\mathcal{L}(D')
\]

be the tautological subbundle on \( G(r+1, p_1*\mathcal{L}(D')) \), so that the spaces \( V \) as above are obtained by restricting \( \forall \) to its fibers at different points. Then \( G^r_d(C) \) consists precisely of the points where the composed map

\[
\forall : g^*p_1*\mathcal{L}(D') \to p_1*\mathcal{L}(D')|_{D'}
\]
is equal to zero. Since this is a map of vector bundles, this is a closed condition, with the number of local equations given by the product of the ranks. Since Pic$^d(C)$ is projective, and relative Grassmannians are projective over their base, we see that $G^r_d(C)$ is projective, as claimed.

It remains to consider the dimension. Recalling that Pic$^d(C)$ is nonsingular of dimension $g$, and $G(r + 1, p_1, \tilde{L}(D'))$ is smooth over Pic$^d(C)$ with fibers isomorphic to $G(r + 1, D + 1 - g)$, we see that $G(r + 1, p_1, \tilde{L}(D'))$ is nonsingular of dimension

\[ g + (r + 1)(d + \deg D - g - (r + 1)) = g + (r + 1)(d + \deg D - g - r). \]

Then $G^r_d(C)$ is locally cut out by $(r + 1)(\deg D)$ equations, so we conclude by the Krull principal ideal theorem that every component of $G^r_d(C)$ has dimension at least

\[ g + (r + 1)(d + \deg D - g - r) - (r + 1)(\deg D) = g + (r + 1)(d - g - r) = \rho, \]

as desired. \(\Box\)

We can restate the Brill-Noether theorem and its refinements in the context of the $G^r_d$ spaces.

**Theorem 2.6.2.** Let $C$ be a nonsingular projective curve of genus $g$. Then:

(i) if $\rho \geq 0$, we have $G^r_d(C)$ nonempty, with every component of dimension at least $\rho$;

(ii) if $\rho \geq 1$, we have $G^r_d(C)$ connected;

(iii) if $C$ is general, then $G^r_d(C)$ has pure dimension $\rho$, and in particular is empty if $\rho < 0$;

(iv) if $C$ is general and $\rho = 0$, then $G^r_d(C)$ consists of

\[ g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!} \]

points.

(v) if $C$ is general, then $G^r_d(C)$ is nonsingular.

Note that the statement of Theorem 2.6.2 is already nicer than that of Theorem 2.5.5, in that it gives the correct dimension whether or not $\rho \geq g$, and that nonsingularity holds on all of $G^r_d(C)$, as opposed to on the complement of $W^{r+1}_d(C)$ in $W^r_d(C)$. In particular, we immediately deduce the following corollary from (ii) and (v) of Theorem 2.6.2 (compare Remark 2.5.8):

**Corollary 2.6.3.** If $C$ is general, and $\rho \geq 1$, then $G^r_d(C)$ is irreducible.

By construction, the space $G^r_d(C)$ has a forgetful morphism to Pic$^d(C)$, and it is clear that the image of this map is precisely $W^r_d(C)$. Thus, at least on a set-theoretical level we have a surjective forgetful morphism

\[ G^r_d(C) \to W^r_d(C). \]

Using this, it is not difficult to see that each of the two versions of parts (i)-(iv) of the Brill-Noether theorem are equivalent. We will discuss (v) further below.

**Proposition 2.6.4.** Statements (i)-(iv) of Theorem 2.5.5 are each equivalent to statements (i)-(iv) of Theorem 2.6.2.
We have already proved the lower bounds on dimension while constructing $W_d^r(C)$ and $G_d^r(C)$, so the only new part in (i) is the nonemptiness statement. The equivalence of the two versions of this follows immediately from the surjectivity of (2.6.1). Likewise, for (ii) the surjectivity means that connectedness of $G_d^r(C)$ implies connectedness of $W_d^r(C)$, and conversely, since the fibers of (2.6.1) are Grassmannians, they are connected, and since $G_d^r(C)$ is projective, (2.6.1) is a closed map, so connectedness of $W_d^r(C)$ implies connectedness of $G_d^r(C)$.

Next, given the known dimension lower bounds, statement (iii) in each case amounts to an upper bound. Again by surjectivity of (2.6.1), it follows immediately from the dimension upper bound for $G_d^r(C)$ implies the same bound for $W_d^r(C)$. Conversely, we can stratify $W_d^r(C)$ by $S_i := W_d^r(C) \setminus W_d^{r+1}(C)$ for $i \geq r$. Then the restriction of $G_d^r(C)$ to $S_i$ is a $G(r+1, i+1)$-bundle, so its dimension is $\dim S_i + (r+1)(i+1) = \dim S_i + (r+1)(i-r)$. Assuming statement (iii) for the $W_d^r(C)$ spaces, we have $\dim S_i = g - (i+1)(g-d+i)$ whenever $S_i \neq \emptyset$, or equivalently, whenever $g+i \geq d$. Thus, the restriction of $G_d^r(C)$ to $S_i$ has dimension $g - (i+1)(g-d+i) + (r+1)(i-r) = g - (r+1)(g-d+r) - (i-r)(g-d+i) \leq \rho$, because $i \geq r$ and $g+i \geq d$. We thus conclude the equivalence of the two versions of (iii), as desired.

Finally, because the fibers of (2.6.1) are connected, statement (iv) for $G_d^r(C)$ implies statement (iv) for $W_d^r(C)$. Conversely, statements (iv) for $W_d^r(C)$ together with statement (iii) imply that the points in question are contained in the complement of $W_d^{r+1}(C)$, where (2.6.1). Thus, we conclude statement (iv) for $G_d^r(C)$. □

In addition, the surjectivity of (2.6.1) together with Corollary 2.6.3 implies:

**Corollary 2.6.5.** *If $C$ is general, and $\rho \geq 1$, then $W_d^r(C)$ is irreducible.*

### 2.7. Moduli of linear series with imposed ramification

One advantage of working with the space $G_d^r(C)$ rather than $W_d^r(C)$ is that it is also allows us to consider imposed ramification at specified points. For convenience, we set the following terminology, which is slightly modified from [EH83], in line with the usage of [EH87].

**Definition 2.7.1.** Given nonnegative integers $r \leq d$, a **Schubert index**\(^5\) of type $(r, d)$ is a nondecreasing sequence $\alpha = \alpha_0, \ldots, \alpha_r$ of nonnegative integers bounded by $d-r$.

We thus make the following definition.

**Definition 2.7.2.** Let $C$ be a projective nonsingular curve, and $P_1, \ldots, P_n$ distinct points of $C$. Given $d, r$, for $i = 1, \ldots, n$, let $\alpha'^i := \alpha'^i_0, \ldots, \alpha'^i_r$ be a Schubert index of type $(r, d)$. Then

$$G_d^r(C, \{(P_i, \alpha'^i_i)\}) \subseteq G_d^r(C)$$

\(^5\)We have chosen this terminology over “ramification index” as used in [EH86] partly because the phrase ramification index is already used in a closely related but slightly different way for morphisms of curves, and partly to emphasize that we will typically use Schubert indices to impose ramification which is at least the given sequence at a point, but do not require equality. The disadvantage is that same the terminology is used with slightly different indexing in [EH83], which follows [Ful98].
consists of points corresponding to linear series which have ramification sequence greater than or equal to $\alpha^i$ at $P_i$ for each $i$.

By one sequence being greater than or equal to another, we mean that the inequality holds in every index. Building on the construction of the $G'_d$ space, we can prove some basic properties of the new space, as well.

**Proposition 2.7.3.** In the situation of Definition 2.7.2, we have that $G'_d(C, \{(P_i, \alpha^i)\}_i)$ is closed in $G'_d(C)$, hence projective. Furthermore, if $g$ is the genus of $C$, then every irreducible component of $G'_d(C, \{(P_i, \alpha^i)\}_i)$ has dimension at least

$$\rho := g - (r + 1)(g - d + r) - \sum_{i=1}^n \sum_{j=0}^r \alpha^i_j.$$  

(2.7.1)

Because the $\rho$ of Equation (2.7.1) is a strict generalization of that of Equation (1.3.1), we do not use different notation or terminology for them.

Before giving the proof of the proposition, we describe an equivalent way of thinking about vanishing sequences which is frequently useful.

**Definition 2.7.4.** Given $r$ and $d$, and a strictly increasing nonnegative integer sequence $a_0, \ldots, a_r$ bounded by $d$, define $b_0, \ldots, b_{d+1}$ by

$$b_j = \# \{i : a_i \geq j\}.$$  

(2.7.2)

Thus, we always have $b_0 = r + 1$ and $b_{d+1} = 0$. It is easy to verify that the data in the $b_j$ is equivalent to that of the $a_i$; specifically, we have the following.

**Proposition 2.7.5.** In the situation of Definition 2.7.4, the $a_i$ can be recovered from the $b_j$ as the set of indices in which $b_j$ decreases. That is,

$$\{a_0, \ldots, a_r\} = \{j : b_{j+1} < b_j\}.$$

This induces a bijection between strictly increasing nonnegative sequences $a_0, \ldots, a_r$ bounded by $d$, and nonincreasing integer sequences $b_0, \ldots, b_{d+1}$ starting at $r+1$ and ending at 0, and with $b_{j+1} - b_j \leq 1$ for $j = 0, \ldots, d$.

**Proof of Proposition 2.7.3.** For each $i$, let $a^i_j = \alpha^i_j + j$ for $j = 0, \ldots, r$; thus, to say a linear series has ramification sequence at least $\alpha^i$ at $P_i$ is the same as to say it has vanishing sequence at least $a^i$. On the other hand, if we construct sequences $b^i$ as in Definition 2.7.4, we see that to say that $(\mathcal{L}, V)$ has vanishing sequence at least $a^i$ at $P_i$ is the same as saying that for $j = 0, \ldots, d$ we have

$$\dim V(-jP_i) \geq b^i_j.$$  

(2.7.3)

Next, returning to the proof of Proposition 2.6.1, we have the universal vector bundle $\mathcal{V}$ on $G'_d(C)$, as well as the Poincare line bundle $\tilde{\mathcal{L}}$ on $G'_d(C) \times C$. Given $P \in C$, the points of $G'_d(C)$ where the corresponding pair $(\mathcal{L}, V)$ has $\dim V(-jP) \geq b$ are precisely those where the induced map

$$\mathcal{V} \rightarrow p_{1*} \left( \tilde{\mathcal{L}}|_{jP} \right)$$

has kernel of dimension at least $b$, or equivalently, rank less than or equal to $r+1-b$. Thus, $G'_d(C, \{(P_i, \alpha^i)\})$ is cut out by an intersection of determinantal conditions in $G'_d(C)$, and in particular, it is closed, as asserted.
To obtain the dimension estimate, we use that for each $i$, the condition at $P_i$ is a sequence of rank conditions coming from composed maps

$$\mathcal{V} \to p_{1*} \left( \widetilde{\mathcal{L}} |_{(d+1)P} \right) \to p_{1*} \left( \widetilde{\mathcal{L}} |_{dP} \right) \to \ldots \to p_{1*} \left( \widetilde{\mathcal{L}} |_{jP} \right).$$

Thus, each of these is a generalized Schubert condition associated to the flag of vector bundles given by the $p_{1*} \left( \widetilde{\mathcal{L}} |_{jP} \right)$. Thus, the standard codimension upper bound for Schubert conditions applies, giving precisely that the condition imposed at each $P_i$ can reduce dimension by at most $\sum_{j=0}^{r} \alpha_j^i$, as desired. \hfill \Box

**Remark 2.7.6.** When $g = 0$, we see that $G_r^d(C)$ is simply the Grassmanian $G(r + 1, d + 1)$. Thus, most of Brill-Noether theory is rather trivial for this case. Indeed, it is not hard to see that each ramification condition is described by a Schubert cycle in the Grassmannian, with the relevant flag being the osculating flag to the rational normal curve in $\mathbb{P}^d$.

Thus, even if the points are chosen to be general on $C$, the corresponding flags are osculating flags based at general points of the rational normal curve, and these are far from being general flags in $\mathbb{P}^d$. Thus, general theorems on transversality do not apply. In characteristic 0, Eisenbud and Harris gave a simple argument that the intersections in question have the expected dimension, and it is a much deeper and more recent theorem of Mukhin, Tarasov and Varchenko [MTV09] that if the points are general, the intersection is indeed transverse. However, in positive characteristic, even the dimension statement fails; see Remark 2.7.8.

Eisenbud and Harris proved that, at least in characteristic 0, the dimension portions of the Brill-Noether theorem can be generalized to the case in which one imposes ramification at fixed points. One can generalize further to positive characteristic as follows.

**Theorem 2.7.7.** Let $C$ be a nonsingular projective curve of genus $g$, and $P_1, \ldots, P_n$ distinct points on $C$. Given $r, d$, and $n$ Schubert indices $\alpha^j$ of type $(r, d)$, suppose that one of the following holds:

(I) $\text{char } k = 0$ or $d < \text{char } k$;

(II) $n \leq 2$.

Then if $C$ and $P_1, \ldots, P_n$ are general, we have that $G_{r}^d(C, \{(P_i, \alpha^i)\})$ has pure dimension $\rho$ if it is nonempty.

**Remark 2.7.8.** As stated, Theorem 2.7.7 fails in positive characteristic for $n \geq 3$, even in the case $n = 3$ and $g = 0$. The reason is the presence of inseparable linear series, which by definition have arbitrarily large amounts of ramification. According to Remark 2.7.6, this means that the osculating flags to the rational normal curve are not sufficiently general to obtain even the correct dimension for the intersection of the corresponding Schubert cycles.

However, it may be the case that the theorem remains correct if one restricts to separable linear series, and indeed this is true for $r = 1$, by arguments involving the deformation theory of branched covers; see Theorem 1.2 of [Oss05]. To the best of my knowledge, the general case remains open.
2.8. Spaces of linear series with prescribed ramification in low genus

We now calculate some examples of Theorem 2.7.7 in genus 0 and 1. These results will also serve as base cases for an inductive proof of the theorem using limit linear series.

**Proposition 2.8.1.** Let \( d, r \) be nonnegative integers, \( P_1, \ldots, P_n \) distinct points on \( \mathbb{P}_k^1 \), and \( \alpha \) a Schubert index of type \( (r, d) \). Suppose that (at least) one of the following holds:

(I) \( \text{char } k = 0 \) or \( d < \text{char } k \);  
(II) \( n \leq 2 \).

Then

\[ G_r^d(\mathbb{P}_k^1, \{(P_i, \alpha^i)\}) \]

is pure of the expected dimension \( \rho \) if it is nonempty; in particular, it is empty if \( \rho < 0 \).

**Proof.** Since we know by Proposition 2.7.3 that the dimension is at least \( \rho \), the desired statement is equivalent to having dimension at most \( \rho \).

For case (I), the Plucker formula Theorem 2.3.5 implies that the sum of all ramification indices of a \( g_r^d \) on \( \mathbb{P}_k^1 \) is equal to \( (r + 1)(d - r) \) and \( \rho = (r + 1)(d - r) - \sum_i \sum_j \alpha_i^j \). In particular, if \( \rho < 0 \), the space \( G_r^d(\mathbb{P}_k^1, \{(P_i, \alpha^i)\}) \) is empty. We prove the proposition by induction on \( \rho \), with the case \( \rho < 0 \) as the base case.

The basic claim is the following: if \( G_r^d(\mathbb{P}_k^1, \{(P_i, \alpha^i)\}) \) has positive dimension, and if we add a point \( P_{n+1} \) with \( \alpha^{n+1} = 0, \ldots, 0, 1 \), then the new \( G_r^d(\mathbb{P}_k^1, \{(P_i, \alpha^i)\}) \) is nonempty, and the dimension drops by at most 1. Given the claim, if we know the statement up to \( \rho = m - 1 \), and we want to prove it for \( \rho = m \), either \( m = 0 \) and \( G_r^d(\mathbb{P}_k^1, \{(P_i, \alpha^i)\}) \) is 0-dimensional, in which case there is nothing to prove, or \( m > 0 \) and \( G_r^d(\mathbb{P}_k^1, \{(P_i, \alpha^i)\}) \) is positive-dimensional. But then, adding a point as in the claim, we preserve nonemptiness, reduce the dimension by at most 1, and reduce \( \rho \) to \( m - 1 \). Thus, by the induction hypothesis the new space has dimension \( m - 1 \) (and in particular \( m > 0 \)), and the original space must have had dimension \( m \), as desired.

Now, the claim follows from the observation that the condition of having ramification at least \( \alpha^{n+1} \) at \( P_{n+1} \) is a codimension-1 Schubert cycle in the Grassmannian of \( (r+1) \)-dimensional subspaces of \( \Gamma(\mathbb{P}_k^1, \mathcal{O}(d)) \). Such Schubert cycles are unique up to linear equivalence, and correspond to the hyperplane section under the Plucker imbedding of the Grassmannian; in particular, they are ample divisors, and have nonempty intersection with every positive-dimensional subvariety of the Grassmannian. The fact that the dimension drops by at most 1 is just a consequence of the fact that they are divisors. Thus, the claim and the first case of the proposition are proved.

Next, for case (II), we might as well assume \( n = 2 \), since we can add additional points as necessary with trivial ramification conditions without affecting the statement. In this case, the flags defining the two Schubert cycles correspond to spaces of polynomials vanishing to different orders at a pair of points on the line. Such conditions are completely independent of one another, so the two flags in question are transverse, in the sense that any space in the first flag will meet any space in the second flag transversely. In particular, we can find a basis of the ambient space \( \Gamma(\mathbb{P}_k^1, \mathcal{O}(d)) \) which is compatible with both flags – namely, polynomials of degree
d\) vanishing only at the two points in question, letting the orders of vanishing at each point vary over all possibilities adding up to \(d\). Thus, up to change of basis of the ambient space, every such pair of flags is equivalent, and we conclude that the flags are general, and the corresponding Schubert cycles must intersect in the correct dimension, by Kleiman’s theorem.\(^6\) Thus we have the desired statement.

We now move on to the case of genus 1, considering two ramification points. This is more involved, and is the first case in which a generality hypothesis on the points arises, although it is still quite explicit.

**Proposition 2.8.2.** Suppose that \(C\) has genus 1, and let \(d,r\) be positive integers, \(P_1, P_2\) points of \(C\), and \(\alpha^1, \alpha^2\) Schubert indices of type \((r,d)\). Assume also that \(P_1 - P_2\) is not \(m\)-torsion for any \(m \leq d\). Then \(G^\alpha_d(C, (P_1, \alpha^1), (P_2, \alpha^2))\) is pure of the expected dimension \(\rho\) if it is nonempty; in particular, it is empty if \(\rho < 0\). Moreover, we have that \(G^\alpha_d(C, (P_1, \alpha^1), (P_2, \alpha^2))\) is nonempty if and only

\[\alpha^1_1 + \alpha^2_{r-j} \leq d - r\]

for all \(j\), with equality holding for at most one value of \(j\).

Note that \(C\) has only finitely many \(m\)-torsion points for any \(m\) (Proposition IV.4.10 of [Har77] for characteristic \(\neq 2\)), so for any choice of \(P_1\), the stated condition is satisfied for all but finitely many choices of \(P_2\).

**Proof.** Again, by Proposition 2.7.3 we know every component has dimension at least \(\rho\), so for the dimension part it is enough to prove that the dimension is at most \(\rho\). It will be convenient to set the following notation for vanishing sequences: \(a^i_j := a^i_j + j\) for all \(i, j\). We begin by reformulating our generality hypothesis on \(P_1, P_2\) into the form in which we actually want it: that we do not have \(\mathcal{O}_C(aP_1 + (d - a)P_2) \cong \mathcal{O}_C(a'P_1 + (d - a')P_2)\) for any distinct \(a, a'\) between 0 and \(d\). Indeed, if we had such an isomorphism, we’d have \(\mathcal{O}_C((a - a')P_1 + (a' - a)P_2) \cong \mathcal{O}_C\). This is the same as saying that \(P_1 - P_2\) is an \([a - a']\)-torsion point in \(C\), which violates our generality hypothesis.

We now make the following observation: if \((\mathcal{L}, V)\) is a \(g^r_d\) with vanishing sequence (at least) \(a^1_j\) at \(P_i\) for \(i = 1, 2\), then for any \(j = 0, \ldots, r\), there exists \(s \in V\) vanishing to order at least \(a^1_j\) at \(P_1\) and \(a^2_{r-j}\) at \(P_2\). Indeed, the dimension of \(V(-a^1_jP)\) is at least \(r + 1 - j\), and the dimension of \(V(-a^2_{r-j}P)\) is at least \((r + 1) - (r - j)\), so the intersection must have dimension at least 1.

We can conclude the emptiness assertion as follows: we immediately see that we must have \(a^1_j + a^2_{r-j} \leq d\) for all \(j\), since the total order of vanishing of any section of \(\mathcal{L}\) is equal to \(d\). On the other hand, if \(a^1_j = a\) and \(a^2_{r-j} = d - a\), we also conclude that we can only have a \(g^r_d\) with this vanishing if the underlying line bundle is \(\mathcal{O}_C(aP_1 + (d - a)P_2)\). By our generality hypothesis on \(P_1, P_2\) a given line bundle can be of this form for at most one \(a\), so we conclude the asserted emptiness condition for \(G^\alpha_d(C, (P_1, \alpha^1), (P_2, \alpha^2))\). In particular, from now on we assume that the \(\alpha^1, \alpha^2\) satisfy the asserted condition for nonemptiness.

For the dimension assertion, it is enough to prove an upper bound, working one point of \(\text{Pic}^d(C)\) at a time. By Example 2.4.5, we know that \(\text{Pic}^d(C) \cong C\), and

\(^6\)Although the transversality part of Kleiman’s theorem fails in positive characteristic in general, the dimensionality statement holds in any characteristic.
moreover every line bundle of degree $d$ on $C$ is isomorphic to $\mathcal{O}_C((d-1)P+Q)$ for a unique $Q \in C$. We assume $d > 0$, since the case $d = 0$ is essentially trivial in light of Example 2.1.6. We first observe that for a line bundle $\mathcal{L}$ of degree $d$, by Example 2.2.3 we have $\Gamma(C,\mathcal{L})$ of dimension $d$, and the vanishing sequence of the corresponding complete linear series begins $0, 1, \ldots, d-2$. Now, if $\mathcal{L} \not\cong \mathcal{O}_C(dp)$, then $\mathcal{L}(-dP) \not\cong \mathcal{O}_C$, so has no nonzero global sections, and the last entry in the vanishing sequence must be $d - 1$. On the other hand, if $\mathcal{L} = \mathcal{O}_C(dp)$, we have a section vanishing to order $d$ at $P$, so the final entry in the vanishing sequence must be $d$, rather than $d - 1$.

There are two types of line bundles to consider: first, if $\mathcal{L} \not\cong \mathcal{O}_C(aP_1 + (d-a)P_2)$ for any $0 \leq a \leq d$, we see that the flags obtained by imposing vanishing at $P_1$ and $P_2$ are transverse, hence general as before, so we conclude that for such an $\mathcal{L}$, if the corresponding fiber of $G^d_2(C,(P_1,\alpha^1),(P_2,\alpha^2))$ is nonempty, it has dimension

$$(r+1)(d-1-r) - \sum_{i=1}^{2} \sum_{j=0}^{r} \alpha^i_j = \rho - 1.$$  

Since the space of such line bundles $\mathcal{L}$ has dimension 1, we conclude that $G^d_2(C,(P_1,\alpha^1),(P_2,\alpha^2))$ can have dimension at most $\rho$ over the open subset of $\text{Pic}^d(C)$ corresponding to line bundles of this type.

Next, suppose that $\mathcal{L} \cong \mathcal{O}_C(aP_1 + (d-a)P_2)$ for some $a$, and recall that by our generality hypothesis, there can be at most one such $a$ for a given $\mathcal{L}$. We first consider the case that $a = 0$ or $d$. Here, we find that the flags obtained in $H^0(C,\mathcal{L})$ from vanishing at $P_1$ or $P_2$ are still transverse, and thus the Schubert cycles intersect transversely, but the indexing of the spaces in the flag may no longer precisely matches the vanishing sequence at $P$. If $a = d$ and $a^1 < d$, this makes no difference, but if $a^1 = d$, there is a discrepancy. Note that in this case, we must still have $a^1_{r-1} < d - 1$, the vanishing sequence, so we have one index in which $\alpha^1 = 1$ is off by 1 from the flag indexing of the relevant Schubert cycle. The same happens if $a = 0$ and $a^2 = d$. In either case, the dimension of at most Schubert cycle increases by at most 1, so we have that the dimension of the corresponding fiber is still at most

$$(r+1)(d-1-r) - \sum_{i=1}^{2} \sum_{j=0}^{r} \alpha^i_j + 1 = \rho.$$  

Finally, if $0 < a < d$, we find that the flag indexing corresponds to the Schubert cycle indexing, but the flags fail to be transverse at the spaces corresponding to vanishing of order $a$ at $P_1$ and $d-a$ at $P_2$. This only affects the dimension of intersection if for some $j$, we have $a^1_j = a$ and $a^2_{r-j} = d-a$. We now claim that if $j > 0$ and $G^d_2(C,(P_1,\alpha^1),(P_2,\alpha^2))$ is nonempty, we must have $a^1_{j-1} \leq a - 2$. Indeed, $a^2_{r-j+1} \geq d - a + 1$, so if $a^1_{j-1} = a - 1$, we have $a^1_{j-1} + a^2_{r-j+1} \geq d$, contrary to our hypothesis that at most one such pair adds up to $d$. Thus, regardless of whether or not $j > 0$, we can still obtain a valid vanishing sequence if we replace $a^1_j$ by $a - 1$. In this case, we have already seen that the Schubert cycles still intersect in the expected dimension, which is now

$$(r+1)(d-1-r) - \sum_{i=1}^{2} \sum_{j=0}^{r} \alpha^i_j + 1 = \rho$$  

because we changed $a^1_j$ by 1. This gives us the desired dimension statement.
It remains to check the asserted nonemptiness statement, which does not require any generality hypothesis. If we have $a_1, a_2$ with $a_1 + a_2 \leq d - 1$, then for any line bundle $\mathcal{L}$ on $C$ of degree $d$ we have a section vanishing to order exactly $a_1$ at $P_1$ and $a_2$ at $P_2$ as long as we do not have $a_1 + a_2 = d - 1$ and $\mathcal{L} \cong (a_1 P_1 + (d - a_1) P_2)$ or $\mathcal{L} \cong \mathcal{O}((d - a_2) P_1 + a_2 P_2)$. Thus, if $a_1 a_2^2 - a_1 \leq d - 1$ for all $j$, we can choose any $\mathcal{L} \not\cong \mathcal{O}(a P_1 + (d - a) P_2)$ for $a = 0, \ldots, d$, and then construct the desired $g^{a_i}_{a_k}$ via a basis of sections as above. Finally, if $a_1^2 + a_2^2 = d$ for some $j$, and $a_1^2 + a_2^2 \leq d - 1$ for $j' \neq j$, set $\mathcal{L} = \mathcal{O}(a_j^1 P_1 + a_{j'}^2 P_2)$, and since we cannot have $a_{j'} = a_j^1 - 1$ or $a_{j'} = a_{j'}^2 - 1$ for any $j'$, we are still able to find a basis of sections with precisely the desired vanishing. \qed
CHAPTER 3

Limit linear series

We now move from discussing linear series on nonsingular curves to limit linear series on reducible curves. By considering the behavior of linear series in families where the underlying curve degenerates, we will arrive at a definition of limit linear series on a reducible curve. By studying limit linear series on reducible curves, we will be able to prove theorems about linear series on nonsingular curves.

3.1. Curves of compact type

We begin by discussing the class of singular curves which will be our primary focus as we develop the theory of limit linear series. We will focus our attention on nodal curves – that is, curves whose singular points are formally locally isomorphic to the origin on the plane curve $xy = 0$. For a nodal curve $X_0$, the normalization $\tilde{X}_0$ is particularly simple to describe: it is obtained by “unghuing” the nodes, so that there are two distinct points lying above the nodes, and distinct irreducible components of $X_0$ become distinct connected components of $\tilde{X}_0$. The functions on $X_0$ inject into the functions on $\tilde{X}_0$, and the image of this injection is precisely the functions whose values agree at each pair of points lying above a node.

Although there is a more sophisticated approach, for the moment we use a more ad hoc definition of the genus of a nodal curve, which is in any case better suited to our needs.

**Definition 3.1.1.** Let $X_0$ be a projective nodal curve, with normalization $\tilde{X}_0$. Let $Y_1, \ldots, Y_n$ be the connected (equivalently, irreducible) components of $\tilde{X}_0$, and let $g_i$ be the genus of $Y_i$ for $i = 1, \ldots, n$. Finally, let $\delta$ be the number of nodes of $X_0$. Then the genus of $X_0$ is

$$1 + \delta + \sum_{i=1}^{n} (g_i - 1).$$

We will also need a definition of degree of a line bundle on a nodal curve. On a nodal (and especially on a reducible) curve $X_0$, the relationship between line bundles and divisors is more complicated than in the nonsingular case which we have discussed. We do not define divisors in general, but remark that a special case is a finite formal sum $D = \sum_i a_i P_i$ where all the $P_i$ are nonsingular points of $X_0$. In this case, the line bundle $\mathcal{L}_{X_0}(D)$ still makes sense more or less as we have previously defined it.\(^2\) We then have the following:

\[^1\text{This is also frequently called the arithmetic genus.}\]
\[^2\text{The only difference is that one has to explicitly impose that the rational functions in question be regular at all nodes in the given open subset.}\]
Proposition 3.1.2. Let $X_0$ be a projective nodal curve, and $\mathcal{L}$ a line bundle on $X_0$. Then there is a divisor $D = \sum a_i P_i$ with all $P_i$ being nonsingular points of $X_0$ such that $\mathcal{L} \cong \mathcal{O}_{X_0}(D)$. Moreover, the degree of $D$ depends only on $\mathcal{L}$. Furthermore, for any irreducible component $Y$ of $X_0$, the degree of $D|_Y$ depends only on $\mathcal{L}$.

Thus, we can define:

Definition 3.1.3. Given a projective nodal curve $X_0$, and a line bundle $\mathcal{L}$ on $X_0$, the degree $\deg \mathcal{L}$ of $\mathcal{L}$ is the degree of a divisor $D = \sum a_i P_i$ with all $P_i$ being nonsingular points of $X_0$, such that $\mathcal{L} \cong \mathcal{O}_{X_0}(D)$. Given such a $D$, the multidegree of $\mathcal{L}$ is the tuple of integers assigning to each irreducible component $Y$ of $X_0$ the degree of $D|_Y$.

It is clear from the definition that the entries of the multidegree always add up to the degree.

Exercise II.6.9 of [Har77] describes how the Picard group of a singular curve is related to that of its normalization. One finds that while $\text{Pic}^d(X_0)$ is always smooth of dimension $g$, it need not be projective, because it is not necessarily proper.

Example 3.1.4. If $X_0$ is an irreducible nodal curve of genus $g$ over $k$, suppose further that $X_0$ is rational — that is, its normalization is isomorphic to $\mathbb{P}^1$. We then see that $\text{Pic}^d(X_0)$ is isomorphic to $(k^*)^g$. This follows formally from the aforementioned exercise in [Har77], but the intuition is clear: there is a unique line bundle $\mathcal{O}(d)$ of degree $d$ on the normalization $\tilde{X}_0 \cong \mathbb{P}^1$, so a line bundle on $X_0$ is determined by how we choose to glue the fibers of $\mathcal{O}(d)$ over each pair of points lying over a given node of $X_0$. Each such gluing is an isomorphism of a line with itself, so the possibilities differ by nonzero scalar, and contribute one copy of $k^*$.

By the genus formula, there are $g$ nodes, so we see that $\text{Pic}^d(X_0) \cong (k^*)^g$.

We will be interested in those curves for which (each connected component of) $\text{Pic}^d(X_0)$ is projective. The most basic such example, which will also guide our discussion throughout, is the following.

Example 3.1.5. Let $X_0$ be a curve obtained from projective nonsingular curves $Y_1$ and $Y_2$ by gluing them together at a single node $P$. If $Y_1$ and $Y_2$ have genera $g_1$ and $g_2$ respectively, then $X_0$ has genus $g_1 + g_2$.

In contrast to Example 3.1.4, we have that line bundles on $X_0$ are in bijection with pairs of line bundles on $Y_1$ and $Y_2$, with no extra moduli coming from gluing. The reason for this is that although there is a choice of gluing at $P$, the line bundles on $Y_1$ and $Y_2$ have independent automorphism groups, each of which is isomorphic to $k^*$ (corresponding to global multiplication by a nonzero scalar). If we change the gluing by a scalar, this is equivalent to applying an automorphism to one of the two line bundles on the $Y_i$, so doesn’t change the isomorphism class on $X_0$. Thus, we find that
\[
\text{Pic}^d(X_0) \cong \prod_{d_1, d_2: d_1 + d_2 = d} \text{Pic}^{d_1}(Y_1) \times \text{Pic}^{d_2}(Y_2),
\]
where the disjoint union arising because of the different possibilities for the multidegree of a line bundle of degree $d$. This is not technically projective because it has infinitely many connected components, but each connected component is projective.
More generally, we have the following definition.

**Definition 3.1.6.** A projective nodal curve $X_0$ is of **compact type** if its dual graph is a tree.

Recall that the **dual graph** of a nodal curve $X_0$ is the graph (where we allow multiple edges and loops) having its vertices indexed by irreducible components of $X_0$, and its edges indexed by nodes of $X_0$. We have the following basic characterization of curves of compact type, which again follows from Exercise II.6.9 of [Har77].

**Proposition 3.1.7.** Let $X_0$ be a projective nodal curve. Then the following are equivalent:

(a) $X_0$ is of compact type;
(b) for every node $P \in X_0$, we have $X_0 \setminus \{ P \}$ disconnected;
(c) every connected component of $\text{Pic}^d(X_0)$ is projective;
(d) every connected component of $\text{Pic}^d(X_0)$ is proper.

In addition, if $X_0$ is of compact type, and its irreducible components are $Y_1, \ldots, Y_n$, then

$$\text{Pic}^d(X_0) \cong \prod_{d_1, \ldots, d_n} \text{Pic}^{d_1}(Y_1) \times \cdots \times \text{Pic}^{d_n}(Y_n),$$

where the $d_j$ run over all tuples of integers adding up to $d$.

Note that the lack of loops in the dual graph implies that every irreducible component of a curve of compact type must be nonsingular, so that the normalization is simply the disjoint union of the irreducible components. In addition, the dual graph being a tree implies that the number of nodes is one less than the number of irreducible components, so we conclude:

**Corollary 3.1.8.** If $X_0$ is a curve of compact type, with irreducible components $Y_1, \ldots, Y_n$, then each $Y_i$ is nonsingular, and the genus of $X_0$ is the sum of the genera of the $Y_i$.

### 3.2. Linear series in a degenerating family

We now motivate the definition of limit linear series by considering the behavior of linear series in certain families of curves. We will make our arguments more precise in a later chapter, but for the moment we settle for laying out the main ideas.

Let $B$ be a nonsingular curve,$^3$ and $\pi : X \to B$ a family of projective curves over $B$.$^4$ Suppose that we have some point $b_0 \in B$, such that $\pi$ is smooth except possibly over $b_0$, and set $X_0 = X_{b_0}$. Write $U$ for the complement of $b_0$, and $X_U$ for $X|_{\pi^{-1}(U)}$, and suppose we have a family of $g^d$’s on $X_U$. We want to understand how such a family might extend over $b_0$. It turns out that if we have an extension of the line bundle of degree $d$, there always exists a unique extension of the space of global sections, so initially we focus our attention on extending the line bundle.

In this situation, there is a moduli space $\text{Pic}^d(X/B)$ over $B$ which parametrizes line bundles of degree $d$ on the fibers of $\pi$. The fiber of $\text{Pic}^d(X/B)$ over a point $b \in B$ is $\text{Pic}^d(X_b)$. As one might expect, over the complement of $b_0$, the map

---

$^3$Or more generally, a regular one-dimensional scheme such as the spectrum of a DVR.

$^4$Formally, $\pi$ should be flat and proper, with 1-dimensional geometrically reduced fibers.
Pic^d(X/B) \to B is smooth and proper. It turns out that Pic^d(X/B) is smooth over all of B, but properness over b_0 depends on the geometry of X_0. If X_0 is also nonsingular, then of course Pic^d(X/B) is proper. In this case, by the valuative interpretation of properness, we know that we can extend our family of line bundles uniquely over b_0, so we likewise find that our family of \( g_d^r \)'s extends uniquely over b_0. However, if X_0 is singular, this is not the case.

One might first take X_0 to be an irreducible nodal curve. This is the situation that was first considered by Castelnuovo, studied further by Kleiman and Laksov [Kle76] based on an idea of Severi, and finally used by Griffiths and Harris [GH80] to complete the first proof of the Brill-Noether theorem. In this case, a number of technical difficulties arise, due mainly to the fact that gluings between components are more complicated. For instance, we have seen that in this case, Pic^d(X_0) is isomorphic to \((k^*)^d\), so is in particular not proper. In addition, gluing complications also carry over to understanding spaces of sections of line bundles.

The insight of Eisenbud and Harris was that it is much easier to construct a robust theory if instead of irreducible nodal curves, one considers curves of compact type. In this case, Pic^d(X/B) is not proper for the technical reason that Pic^d(X_0) is a disjoint union of infinitely many projective varieties, indexed by multidegree. So Pic^d(X/B) is horribly non-separated (think the affine line with not only two origins, but infinitely many origins). However, line bundles over U will still extend over Pic^d(X/B), just not uniquely. To examine the picture in more detail, let’s consider the simplest special case, where X_0 is the union of two smooth components Y_1 and Y_2 at a node P. Further suppose that the total space of X, which is two-dimensional, is nonsingular. In this case, Y_1 and Y_2 are each effective divisors on X.

Suppose that \( \mathcal{L}_U \) is a line bundle on X_U such that the restriction to each fiber X_b for b \in U has degree d. We have asserted that \( \mathcal{L}_U \) extends to a line bundle \( \mathcal{L} \) on all of X such that \( \mathcal{L}|_{X_U} = \mathcal{L}_U \). Assuming this, we see that \( \mathcal{L} \) is not unique with this property. Indeed, if we consider the line bundle \( \mathcal{L}(Y_1) \), since Y_1 is disjoint from X_U, we see that \( \mathcal{L}(Y_1)|_{X_U} \cong \mathcal{L}|_{X_U} \cong \mathcal{L}_U \). However, \( \mathcal{L}(Y_1) \) is different from \( \mathcal{L} \) on X_0. To analyze this, it is convenient to assume that the divisor b_0 on B is linearly equivalent to 0, or equivalently, that there is a function on B which vanishes only at b_0, to order exactly 1. This is always true in an open neighborhood of b_0 in B, and for the purposes of our discussion there is no harm in restricting B to such a neighborhood. In this case, we see that Y_1 + Y_2 is also linearly equivalent to 0 as a divisor on X, since it is just the preimage of b_0. Thus \( \mathcal{O}_X(Y_1) \cong \mathcal{O}_X(-Y_2) \). Now, returning to \( \mathcal{L}(Y_1) \), if we restrict to Y_2 we see that we get

\[
\mathcal{L}(Y_1)|_{Y_2} \cong \mathcal{L}|_{Y_2}(Y_1 \cap Y_2) = \mathcal{L}|_{Y_2}(P).
\]

Thus, the effect of twisting by Y_1 is that the restriction to Y_2 gets twisted by P. In particular, the degree goes up by 1. On the other hand, since \( \mathcal{O}_X(Y_1) \cong \mathcal{O}_X(-Y_2) \), we can perform the same analysis on Y_1:

\[
\mathcal{L}(Y_1)|_{Y_1} \cong \mathcal{L}(-Y_2)|_{Y_1} \cong \mathcal{L}|_{Y_1}(-Y_1 \cap Y_2) = \mathcal{L}|_{Y_1}(-P).
\]

Thus, on Y_1 we have lowered the degree by 1, by twisting by \(-P\). In this way, we see that no matter what multidegree \( \mathcal{L} \) started with, we can get to any other multidegree (adding up to d). In fact, if for d_1, d_2 with d_1 + d_2 = d, we define Pic^{d_1,d_2}(X/B) to parametrize families of line bundles of degree d over U and multidegree \((d_1, d_2)\) over b_0, we resolve the aforementioned nonseparatedness issue.
with \( \text{Pic}^d(X/B) \), and find that \( \text{Pic}^{(d_1,d_2)}(X/B) \) is proper. That is, every family of line bundles of degree \( d \) over \( U \) extends uniquely over \( b_0 \) to one with multidegree \( (d_1,d_2) \). As we’ve already mentioned, vector spaces of global sections always extend uniquely, so we conclude that if we have a family of \( g_U^d \)'s on \( X_U \), and we specify a multidegree \( (d_1,d_2) \), the family extends uniquely to a \( g_{X_U}^d \) on \( X_0 \). Put differently, given such a family of \( g_U^d \)'s on \( X_U \), we get an infinite collection of \( g_{X_U}^d \)'s extending the given family to \( X_0 \), indexed by the multidegree of the extension. Denote these extensions on \( X \) by \( (\mathcal{L}_{(d_1,d_2)}, \mathcal{V}_{(d_1,d_2)}) \), and their restrictions to \( X_0 \) by \( (\mathcal{L}_{(d_1,d_2)}, \mathcal{V}_{(d_1,d_2)}) \).

Eisenbud and Harris had the idea that we should focus on two particular such extensions: those with multidegree \((d,0)\), and \((0,d)\). Because the line bundle \( \mathcal{L}_{(d,0)} \) has degree 0 on \( Y_2 \), if we restrict to \( Y_1 \), we get an injection on global sections, and similarly for \( \mathcal{L}_{(0,d)} \). In particular, if we write \( \mathcal{L}^{Y_1} := \mathcal{L}_{(d,0)}|_{Y_1}; \mathcal{L}^{Y_2} := \mathcal{L}_{(0,d)}|_{Y_2} \), and \( V^{Y_1} := V_{(d,0)}|_{Y_1}; V^{Y_2} := V_{(0,d)}|_{Y_2} \), then we see that \( (\mathcal{L}^{Y_1}, \mathcal{L}^{Y_2}) \) is a \( g_{U}^d \) on \( Y_i \) for \( i = 1, 2 \). To recap, given a family of \( g_{U}^d \)'s on \( X_U \), we have produced a pair of \( g_{Y_i}^d \)'s, one on each component of \( X_0 \). However, not every pair of \( g_{U}^d \)'s on \( X_0 \) can arise in this fashion, so we now turn to considering what additional conditions arise from the manner in which we produced the pair.

To examine this, we first construct some basic maps relating the different \( \mathcal{L}_{(d_1,d_2)} \). Although each \( \mathcal{L}_{(d_1,d_2)} \) (and consequently each \( \mathcal{L}_{(d_1,d_2)} \)) is unique up to isomorphism, it is helpful to set a convention fixing the relationship between them more rigidly. We thus fix a choice of \( \mathcal{L}_{(d_0,0)} \), and declare that \( \mathcal{L}_{(d_1,d_2)} = \mathcal{L}_{(d_0,0)(d_2Y_1)} \) for each \((d_1,d_2)\). Then because \( Y_1 \) is an effective divisor, we have a natural inclusion

\[
\mathcal{L}_{(d_1,d_2)} \hookrightarrow \mathcal{L}_{(d_1-1,d_2+1)} = \mathcal{L}_{(d_1,d_2)}(Y_1)
\]

which is an isomorphism away from \( Y_1 \), and vanishes to order 1 along \( Y_1 \). If we consider what happens when we restrict to \( X_0 \), we obtain a map (no longer injective)

\[
(3.2.1) \quad \mathcal{L}_{(d_1,d_2)} \to \mathcal{L}_{(d_1-1,d_2+1)},
\]

which vanishes identically on \( Y_1 \), and is the natural inclusion on \( Y_2 \) (under the identification \( \mathcal{L}_{(d_1-1,d_2+1)}|_{Y_2} = \mathcal{L}_{(d_1,d_2)}|_{Y_2}(P) \)).

Now, we fix a choice of isomorphism \( \mathcal{O}_X(Y_1) \simeq \mathcal{O}_X(-Y_2) \). This then by the same argument maps

\[
\mathcal{L}_{(d_1-1,d_2+1)} \to \mathcal{L}_{(d_1,d_2)}
\]

and

\[
(3.2.2) \quad \mathcal{L}_{(d_1-1,d_2+1)} \to \mathcal{L}_{(d_1,d_2)},
\]

vanishing on \( Y_2 \) instead of on \( Y_1 \). Composing such maps together, and then restricting to the appropriate \( Y_i \), for any \((d_1,d_2)\) we obtain maps

\[
(3.2.3) \quad \mathcal{L}_{(d_1,d_2)} \to \mathcal{L}_{(d_0,0)} \to \mathcal{L}^{Y_1} \quad \text{and} \quad \mathcal{L}_{(d_1,d_2)} \to \mathcal{L}_{(0,d)} \to \mathcal{L}^{Y_2}.
\]

Moreover, it is not hard to see that the spaces \( V_{(d_1,d_2)} \) map into one another under the maps \((3.2.1)\) and \((3.2.2)\), and consequently, that \( V_{(d_1,d_2)} \) maps into \( \mathcal{L}^{Y_1} \) and \( \mathcal{L}^{Y_2} \) under \((3.2.3)\).

Put more intuitively, we are saying that \( V_{(d_1,d_2)} \), which consists of global sections of \( \mathcal{L}_{(d_1,d_2)} \) on \( X_0 \), can be obtained by gluing together sections chosen from \( V^{Y_1} \) on \( Y_1 \) and sections chosen from \( V^{Y_2} \) on \( Y_2 \). The fact that it is possible to construct an \((r + 1)\)-dimensional space of global sections of \( \mathcal{L}_{(d_1,d_2)} \) in this manner
for each \((d_1, d_2)\) turns out to be precisely equivalent to the additional condition imposed by Eisenbud and Harris, which we now describe.

**Definition 3.2.1.** Let \(X_0\) be the union of two nonsingular projective curves \(Y_1\) and \(Y_2\) at a single node \(P\). A pair consisting of \(g_{sY}^2(\mathcal{L}^{V_{Y_1}}, V^{Y_1})\) and \((\mathcal{L}^{V_{Y_2}}, V^{Y_2})\) on \(Y_1\) and \(Y_2\) respectively is a limit linear series if we have the following inequality on vanishing sequences:

\[
(3.2.4) \quad a_i(\mathcal{L}^{V_{Y_1}}, V^{Y_1})(P) + a_{i-1}(\mathcal{L}^{V_{Y_2}}, V^{Y_2})(P) \geq d \quad \text{for } i = 0, \ldots, r.
\]

We wish to prove that condition (3.2.4) is equivalent to the condition that we can always produce an \((r + 1)\)-dimensional space of global sections of \(\mathcal{L}(d_1, d_2)\) by gluing together sections from \(V^{Y_1}\) and \(V^{Y_2}\). We first have to make this latter condition more precise. Taking global sections and quotients in (3.2.3), we obtain a map

\[
(3.2.5) \quad \Gamma(X_0, \mathcal{L}(d_1, d_2)) \to \Gamma(Y_1, \mathcal{L}^{V_{Y_1}})/V^{Y_1} \oplus \Gamma(Y_2, \mathcal{L}^{V_{Y_2}})/V^{Y_2},
\]

whose kernel we can take to be the formal definition of the space of global sections of \(\mathcal{L}(d_1, d_2)\) obtained by gluing together sections of \(V^{Y_1}\) and \(V^{Y_2}\). However, it will also be convenient to have a different description, which we describe as follows: assuming that \(d_1, d_2\) are nonnegative, there is a map

\[
(3.2.6) \quad V^{Y_1}(-(d - d_1)P) \oplus V^{Y_2}(-(d - d_2)P) \to \mathcal{L}(d_1, d_2)|_P \cong k
\]

defined by a gluing condition at \(P\). Specifically, our maps \(\mathcal{L}(d_1, d_2) \to \mathcal{L}^{V_{Y_1}}\) and \(\mathcal{L}(d_1, d_2) \to \mathcal{L}^{V_{Y_2}}\) factor through \(\mathcal{L}(d_1, d_2)|_{Y_1}\) and \(\mathcal{L}(d_1, d_2)|_{Y_2}\) respectively, and induce isomorphisms \(\mathcal{L}(d_1, d_2)|_{Y_i} \sim \mathcal{L}^{V_{Y_i}}(-(d - d_i)P)\) for \(i = 1, 2\). Thus, we can use our maps to consider \(V^{Y_i}(-(d - d_i)P)\) as spaces of global sections of \(\mathcal{L}(d_1, d_2)|_{Y_i}\) for \(i = 1, 2\). Elements of these spaces can then be evaluated in the fiber \(\mathcal{L}(d_1, d_2)|_P\), and if we change one of the signs, we obtain the map in (3.2.6), whose kernel consists of pairs \((s_1, s_2) \in V^{Y_1}(-(d - d_1)P) \oplus V^{Y_2}(-(d - d_2)P)\) which agree in the fiber at \(P\). We then have:

**Proposition 3.2.2.** In the situation of Definition 3.2.1, the following are equivalent:

(a) The pair \((\mathcal{L}^{V_{Y_1}}, V^{Y_1})\) and \((\mathcal{L}^{V_{Y_2}}, V^{Y_2})\) is a limit linear series.

(b) For all nonnegative \(d_1, d_2\) with \(d_1 + d_2 = d\), the kernel of (3.2.5) has dimension at least \(r + 1\).

(c) For all nonnegative \(d_1, d_2\) with \(d_1 + d_2 = d\), the kernel of (3.2.6) has dimension at least \(r + 1\).

To prove this, the first order of business is the following.

**Lemma 3.2.3.** In the situation of Definition 3.2.1, the kernels of (3.2.5) and (3.2.6) are isomorphic via the map induced by (3.2.3).

**Proof.** First we check that we indeed get an induced map from the kernel of (3.2.5) to that of (3.2.6). Certainly, if \(s \in \Gamma(X_0, \mathcal{L}(d_1, d_2))\) is in the kernel of (3.2.5), the maps of (3.2.3) give elements of \(V^{Y_1}\) and \(V^{Y_2}\). Moreover, because \(\mathcal{L}(d_1, d_2)|_{Y_i}\) is identified with \(\mathcal{L}^{V_{Y_i}}(-(d - d_i)P)\) for \(i = 1, 2\), we see that these elements actually lie in \(V^{Y_1}(-(d - d_1)P)\) and \(V^{Y_2}(-(d - d_2)P)\), respectively, so we just need to see
that they lie in the kernel of (3.2.6). But this is immediate from the construction of (3.2.6) and the exact sequence

\[ 0 \to \mathcal{L}_{(d_1,d_2)} \to \mathcal{L}_{(d_1,d_2)}|_{Y_1} \oplus \mathcal{L}_{(d_1,d_2)}|_{Y_2} \to \mathcal{L}_{(d_1,d_2)}|_P \to 0. \]

The same exact sequence also gives surjectivity of our map from the kernel of (3.2.5) to that of (3.2.6). Finally, since a section of \( \mathcal{L}_{(d_1,d_2)} \) which vanishes on both \( Y_1 \) and \( Y_2 \) must be the zero section, we get injectivity as well, proving the lemma. \( \square \)

We now prove the proposition.

**Proof of Proposition 3.2.2.** According to Lemma 3.2.3 it is enough to prove that (a) is equivalent to (c). This is almost entirely combinatorial. Indeed, we see that the dimension of the kernel of (3.2.6) is

\[ \dim V^{Y_1}(-(d-d_1)P) + \dim V^{Y_2}(-(d-d_2)P) \alpha - \epsilon, \]

where \( \epsilon = 0 \) or 1 is the rank of (3.2.6). Now, the map

\[ V^{Y_1}(-(d-d_1)P) \to \mathcal{L}_{(d_1,d_2)}|_P \]

is nonzero if and only if there is a section in \( V^{Y_1}(-(d-d_1)P) \) which is nonvanishing at \( P \) as a section of \( \mathcal{L}_{(d_1,d_2)}|_{Y_1} \), or equivalently, which vanishes to order precisely \( d-d_1 \) as a section of \( \mathcal{L}^{Y_1} \). That is, the map is nonzero if and only if \( d-d_1 \) occurs in the vanishing sequence at \( P \) of \( (\mathcal{L}^{Y_1}, V^{Y_1}) \). Similarly,

\[ V^{Y_2}(-(d-d_2)P) \to \mathcal{L}_{(d_1,d_2)}|_P \]

is nonzero if and only if \( d-d_2 \) occurs in the vanishing sequence at \( P \) of \( (\mathcal{L}^{Y_2}, V^{Y_2}) \).

Thus, we get that \( \epsilon = 1 \) if and only if either \( d-d_1 = a_j^{(\mathcal{L}^{Y_1}, V^{Y_1})}(P) \) for some \( j \), or \( d-d_2 = a_j^{(\mathcal{L}^{Y_2}, V^{Y_2})}(P) \) for some \( j \). For notational convenience, set \( a_j^{(\mathcal{L}^{Y_1}, V^{Y_1})}(P) = a_j^{(\mathcal{L}^{Y_1}, V^{Y_1})}(P) \) for \( i = 1, 2 \) and \( j = 0, \ldots, r \).

First, suppose (a) is satisfied, so that \( a_1 + a_2 > d \) for all \( i = 0, \ldots, r \). Given \( (d_1, d_2) \), we wish to prove that the kernel of (3.2.6) has dimension at least \( r+1 \). Choose \( i_0 \) minimal with \( a_{i_0} > d-d_1 \); if \( a_{i_0}^{(2)} \leq d - d_1 \), set \( i_0 = r+1 \). Thus, \( \dim V^{Y_1}(-(d-d_1)P) \geq r+1-i_0 \), with equality if and only if \( d-d_1 \) does not appear in the sequence \( a_1 \). If \( i_0 = 0 \), we observe that we can only have \( \dim V^{Y_2}(-(d-d_2)P) = 0 \) if \( d-d_2 \) does not occur in the sequence \( a_2 \), and we thus conclude that the kernel of (3.2.6) has dimension at least \( r+1 \), as desired. On the other hand, if \( i_0 > 0 \), then \( a_{i_0-1}^{(2)} \leq d - d_1 \), again with equality if and only if \( d-d_1 \) does not appear in the sequence \( a_1 \), so we conclude from (a) that \( a_2^{(r+1-i_0)} \geq d - d_2 \), with equality only if \( a_1 = d - d_1 \). Thus we have \( \dim V^{Y_2}(-(d-d_2)P) \geq i_0 \), with equality only if \( a_2^{(r-i_0)} < d - d_2 \). Summing, we find that

\[ \dim V^{Y_1}(-(d-d_1)P) + \dim V^{Y_2}(-(d-d_2)P) \geq r+1, \]

with equality possible only if \( a_2^{(r-i_0)} < d - d_2 \) and \( a_1^{(i_0-1)} < d - d_1 \). But in the case of equality, then we also have from the above that we must have \( a_2^{(r+1-i_0)} > d - d_2 \), so we have that the \( \epsilon \) of (3.2.7) is equal to 0. We thus conclude that in either case the dimension of the kernel of (3.2.6) is at least \( r+1 \), as desired.

Conversely, suppose that (c) is satisfied, and we wish to show that given \( i = 0, \ldots, r \) we have \( a_1 + a_2 > d \). Setting \( d_1 = d - a_1^1 \) (and \( d_2 = d - d_1 \), we have...
\[ \dim V^{Y_1}(-d - d_1)P = r + 1 - i. \] Moreover, in this case the \( \epsilon \) of (3.2.7) is equal to 1, so we conclude that the dimension of the kernel of (3.2.6) is
\[ r + 1 - i + \dim V^{Y_2}(-(d - d_2)P) - 1 = r - i + \dim V^{Y_2}(-(d - d_2)P). \]
We conclude that \( \dim V^{Y_2}(-(d - d_2)P) \geq i + 1 \), so that
\[ a_{r-1}^2 \geq d - d_2 = d - a_1^1, \]
as desired. \( \square \)

The utility of Proposition 3.2.2 lies not only in providing a simple justification for the Eisenbud-Harris definition of limit linear series, but also in leading to robust construction of a moduli space of such limit linear series (as we will discuss in the next chapter), as well as generalizations to situations such as curves not of compact type, and vector bundles of higher rank.

**Remark 3.2.4.** Notice that condition (c) of Proposition 3.2.2 is nontrivial even at the extreme multidegrees \((d,0)\) and \((0,d)\). Indeed, in the case \((d,0)\) we have \( \dim V^{Y_1}(-(d - d_1)P) = \dim V^{Y_1} = r + 1 \), but if there is a section in \( V^{Y_1} \) which is nonvanishing at \( P \), the \( \epsilon \) of (3.2.7) is nonzero, so in order for the kernel of (3.2.6) to have dimension at least \( r + 1 \), we need \( V^{Y_2}(-(d - d_2)P) = V^{Y_2}(-dP) \neq 0 \); that is, we need \( a_{r-1}^{(d^{-1},0)}(P) = d \). Thus, the degree \((d,0)\) case is precisely responsible for ensuring that (3.2.4) is satisfied for \( i = 0 \) when \( a_{0}^{(d^{-1},0)}(P) = 0 \). Similarly, the \((0,d)\) case gives (3.2.4) for \( i = 0 \) when \( a_{0}^{(d^{-1},d)}(P) = 0 \).

**Remark 3.2.5.** In our motivating discussion of the behavior of linear series in families, we glossed over an important technical point. Namely, when we think of families of line bundles over \( U \), we have to consider not only line bundles on \( X_{U} \), but line bundles which may only be defined after some base change \( U' \to U \). This is an issue even when working over an algebraically closed field, and it is the real reason why we cannot assert that line bundles extend in the non-compact-type case. Indeed, when \( X \) is regular, as we assumed, line bundles on \( X_{U} \) always extend to all of \( X \) simply because divisors on \( X_{U} \) can be extended to divisors on \( X \). Thus, if we impose our regularity hypothesis and do not consider base change, in fact every line bundle on \( X_{U} \) extends over \( X_{0} \) regardless of whether or not \( X_{0} \) is of compact type. However, if we consider a base change \( X_{B'} \to X \), the regularity will not typically be preserved. Thus, the real distinction between the compact type case and the general case is that in the compact type case, line bundles always extend, even allowing for base change.

By the same token, our regularity hypothesis has nothing to do with being able to extend line bundles, but instead is used to guarantee that \( Y_1 \) and \( Y_2 \) are divisors on \( X \), giving us line bundles to twist by. Given a base change \( X_{B'} \to X \), we can pull back the line bundles from \( X \) to \( X_{B'} \), so while it is important that we start with \( X \) regular, it doesn’t matter that regularity will not be preserved under base change.

The great utility of the limit linear series theory of Eisenbud and Harris is that a limit linear series can be described in terms of linear series on the individual components, with very little interaction between the components. This sets up simple inductive arguments to prove foundational results such as the Brill-Noether theorem. To make the situation more inductive, note that we can make the following definition:
DEFINITION 3.2.6. Let \( X_0 \) be as in Definition 3.2.1, and suppose we have \((\mathcal{L}^{Y_1}, V^{Y_1})\) and \((\mathcal{L}^{Y_2}, V^{Y_2})\) a pair of \(g_s\)'s giving a limit linear series on \( X_0 \). Given a nonsingular point \( Q \) of \( X_0 \), we say the limit linear series has **vanishing sequence** \( a_0, \ldots, a_r \) (respectively, **ramification sequence** \( a_0, a_1 - 1, \ldots, a_r - r \)) if the vanishing sequence of \((\mathcal{L}^{Y_1}, V^{Y_1})\) at \( Q \) is equal to \( a_0, \ldots, a_r \), where \( i \) is uniquely determined by the condition that \( Q \in Y_i \).

We also introduce some notation for moduli spaces of limit linear series.

**NOTATION 3.2.7.** With \( X_0 \) as in Definition 3.2.1, denote by
\[
G_d^r(X_0) \subseteq G_d^r(Y_1) \times G_d^r(Y_2)
\]
the subset of points giving limit linear series. Additionally, given nonsingular points \( Q_1, \ldots, Q_n \) on \( X_0 \) and Schubert indices \( \alpha^1, \ldots, \alpha^n \) of type \((r, d)\), denote by
\[
G_d^r(X_0, \{(Q_i, \alpha^i)\}) \subseteq G_d^r(X_0)
\]
the subset consisting of points giving limit linear series which have ramification sequence at least \( \alpha^i \) at \( Q_i \) for \( i = 1, \ldots, n \).

The following proposition describes the basic structure of spaces of limit linear series.

**PROPOSITION 3.2.8.** With \( X_0 \) as in Definition 3.2.1, and given also nonsingular points \( Q_1, \ldots, Q_n \) on \( X_0 \) and Schubert indices \( \alpha^1, \ldots, \alpha^n \) of type \((r, d)\), suppose that the \( Q_i \) are ordered so that \( Q_1, \ldots, Q_m \) lies on \( Y_1 \), and \( Q_{m+1}, \ldots, Q_n \) lies on \( Y_2 \). Then
\[
\bigcup_{\alpha^{Y_1}, \alpha^{Y_2}} G_d^r(Y_1, \{(Q_i, \alpha^i)_{i=1,\ldots,m}, (P, \alpha^{Y_1})\}) \times G_d^r(Y_2, \{(Q_i, \alpha^i)_{i=m+1,\ldots,n}, (P, \alpha^{Y_2})\}),
\]
where \( \alpha^{Y_1}, \alpha^{Y_2} \) run over all pairs of Schubert indices of type \((r, d)\) such that
\[
\alpha_i + \alpha_{j-1} = d - r
\]
for all \( i = 0, \ldots, r \).

In particular, \( G_d^r(X_0, \{(Q_i, \alpha^i)\}) \) is closed in \( G_d^r(Y_1) \times G_d^r(Y_2) \) and hence projective.

**PROOF.** This is almost immediate from the definition of limit linear series. The first observation is that the identity
\[
\alpha_i + \alpha_{j-1} = d - r
\]
is equivalent to
\[
\alpha_i + \alpha_{j-1} = d
\]
if we set \( a_Y = \alpha_Y + i \) for \( i = 0, \ldots, r \) and \( j = 1, 2 \). Thus, the given union is contained in the limit linear series space, and if we have a limit linear series given by \((\mathcal{L}^{Y_1}, V^{Y_1})\) and \((\mathcal{L}^{Y_2}, V^{Y_2})\) and we set \( \alpha^{Y_1} \) equal to the ramification sequence of \((\mathcal{L}^{Y_1}, V^{Y_1})\) at \( P \), and \( \alpha^{Y_2} = d - r - \alpha^{Y_1}_{i-1} \) for \( i = 0, \ldots, r \), then by (3.2.4) the chosen limit linear series lies in
\[
G_d^r(Y_1, \{(Q_i, \alpha^i)_{i=1,\ldots,m}, (P, \alpha^{Y_1})\}) \times G_d^r(Y_2, \{(Q_i, \alpha^i)_{i=m+1,\ldots,n}, (P, \alpha^{Y_2})\}).
\]
This proves the proposition. \( \square \)
We can thus describe the dimension of spaces of limit linear series. The below corollary includes the observation that the inequalities of (3.2.4) lead to an “additivity of Brill-Noether number” which plays an essential role in the theory.

**Corollary 3.2.9.** In the situation of Proposition 3.2.8, if $X_0$ has genus $g$, then every component of $G^r_d(X_0, \{(Q_i, \alpha^i)\})$ has dimension at least the $\rho$ of (2.7.1).

Furthermore, if $Y_1$ and $Y_2$ have genus $g_1$ and $g_2$ respectively, given $\alpha^{Y_1}$ and $\alpha^{Y_2}$ as in Proposition 3.2.8, set

$$\rho_1 = g_1 - (r + 1)(g_1 - d + r) - \sum_{i=1}^{m} \sum_{j=0}^{r} \alpha_j^{Y_1} - \sum_{j=0}^{r} \alpha_j^{Y_1},$$

and

$$\rho_2 = g_2 - (r + 1)(g_2 - d + r) - \sum_{i=m+1}^{n} \sum_{j=0}^{r} \alpha_j^{Y_2} - \sum_{j=0}^{r} \alpha_j^{Y_2}.$$

Then $G^r_d(X_0, \{(Q_i, \alpha^i)\})$ (if nonempty) has dimension exactly $\rho$ if and only if for all such $\alpha^{Y_1}$ and $\alpha^{Y_2}$ such that

$$G^r_d(Y_1, \{(Q_i, \alpha^i)_{i=1,\ldots,m}, (P, \alpha^{Y_1})\}) \neq \emptyset$$

and

$$G^r_d(Y_2, \{(Q_i, \alpha^i)_{i=m+1,\ldots,n}, (P, \alpha^{Y_2})\}) \neq \emptyset,$$

we have

$$\dim G^r_d(Y_1, \{(Q_i, \alpha^i)_{i=1,\ldots,m}, (P, \alpha^{Y_1})\}) = \rho_1$$

and

$$\dim G^r_d(Y_2, \{(Q_i, \alpha^i)_{i=m+1,\ldots,n}, (P, \alpha^{Y_2})\}) = \rho_2.$$

**Proof.** In light of Propositions 3.2.8 and 2.7.3, it is enough to verify that $\rho_1 + \rho_2 = \rho$, which follows from the observation that

$$\sum_{j=0}^{r} \alpha_j^{Y_1} + \sum_{j=0}^{r} \alpha_j^{Y_2} = (r + 1)(d - r).$$

\[\square\]

### 3.3. Basic theory and first applications of limit linear series

We now state the two fundamental results of limit linear series theory, in the following situation.

**Situation 3.3.1.** Let $B$ be a nonsingular curve, and $\pi : X \to B$ a family of projective curves over $B$. Suppose that we have some point $b_0 \in B$, such that $\pi$ is smooth except over $b_0$. Assume further that the fiber $X_0 = \pi^{-1}(b_0)$ is a curve obtained as the union of two nonsingular projectives curves $Y_1, Y_2$ glued to one another at a single node $P$. Finally, assume that the total space of the surface $X$ is nonsingular.

Let $P_1, \ldots, P_n$ be disjoint sections of $\pi$ with image contained in the points of $X$ on which $\pi$ is smooth.

The first proposition is a specialization statement; we have essentially already proved the nonemptiness aspect of it in motivating the definition of limit linear series.
Proposition 3.3.2. In Situation 3.3.1, given \( r, d \) and Schubert indices \( \alpha^1, \ldots, \alpha^n \) of type \((r, d)\), suppose that the spaces \( G^\alpha_{r,d}(X, \{((P_i)|_{b_i}, \alpha^i)\}) \) are nonempty of dimension at least \( m \) for infinitely many \( b \) in \( B \). Then the space \( G^\alpha_{r,d}(X, \{((P_i)|_{b_i}, \alpha^i)\}) \) is nonempty of dimension at least \( m \).

The complementary statement, which is considerably more difficult, is the following smoothing theorem.

Theorem 3.3.3. In Situation 3.3.1, given \( r, d \) and Schubert indices \( \alpha^1, \ldots, \alpha^n \) of type \((r, d)\), suppose that the space \( G^\alpha_{r,d}(X, \{((P_i)|_{b_i}, \alpha^i)\}) \) is nonempty of dimension exactly \( \rho \). Then there is a nonempty open subset \( U \) of \( B \) such that for all \( b \in U \), the space \( G^\alpha_{r,d}(X, \{((P_i)|_{b_i}, \alpha^i)\}) \) is nonempty of dimension exactly \( \rho \).

Both of these results will follow from a foundational theorem generalizing Proposition 2.7.3 to families as in Situation 3.3.1 in which we consider a moduli space parametrizing usual linear series on \( X_b \) for \( b \neq b_0 \), and limit linear series on \( X_0 \).

The proof of these results (and in particular Theorem 3.3.3) is rather difficult, so we defer it until after we have demonstrated some of its applications.

In order to apply the foundational results on limit linear series, we will also need a theorem due to Winters \[\text{Win74}\] on existence of families of curves. This is proved via deformation theory, and we will use it without proof.

Theorem 3.3.4. Let \( X_0 \) be a projective nodal curve. Then there exists a nonsingular curve \( B \), a family of projective curves \( \pi : X \to B \) parametrized by \( B \), and a point \( b_0 \in B \) such that:

(i) the fiber \( X|_{b_0} \) is isomorphic to \( X_0 \);
(ii) \( \pi \) is smooth except over \( b_0 \);
(iii) the surface \( X \) is regular.

Moreover, if \( P_1, \ldots, P_n \) are distinct nonsingular points of \( X_0 \), we can choose \( X, B \) and \( \pi \) so that there exist disjoint sections \( P_1, \ldots, P_n \) of \( \pi \) with each \( P_i \) specializing to \( P_i \) in \( X_0 \).

Our first application of limit linear series will be the proof of the dimension and nonemptiness portions of the Brill-Noether theorem, or at least as much as we can prove without having discussed constructions in families. More precisely, assuming Proposition 3.3.2 and Theorem 3.3.3, together with Theorem 3.3.4, we can prove the following result.

Theorem 3.3.5. Fix a base field \( k \). Given \( g, r, d \) and \( n \), suppose that (at least) one of the following holds:

(I) \( \text{char } k = 0 \) or \( d < \text{char } k \);
(II) \( n \leq 2 \).

Then there exists a nonsingular projective curve \( C \) of genus \( g \) and \( P_1, \ldots, P_n \) distinct points on \( C \) such that for any Schubert indices \( \alpha^1, \ldots, \alpha^n \) of type \((r, d)\), the space \( G^{\alpha^1}_{d}(C, \{((P_i)|_{\alpha^i})\}) \) has pure dimension \( \rho \) if it is nonempty.

Moreover, if \( n = 2 \), we have \( G^{\alpha^1}_{d}(C, \{((P_i)|_{\alpha^i})\}) \) nonempty if and only if

\[
\sum_{j; \alpha^1_j + \alpha^2_{r-j} \geq d-r+1-g} \alpha^1_j + \alpha^2_{r-j} - (d - r - g) \leq g.
\]
Remark 3.3.6. If we take for granted the basic properties of the moduli space of curves, as per Remark 2.5.6, a standard argument which we will give in Corollary 4.1.4 below shows that the property of $G_d^r(X, \{(P_i, \alpha^i)\})$ being either empty or nonempty of dimension $\rho$ is open in families of nonsingular curves, so from Theorem 3.3.5 we can conclude the more familiar statement that the conclusions of the theorem hold for a general curve $C$ with general points $P_1, \ldots, P_n$.

Remark 3.3.7. The difference between the righthand side and the lefthand side of (3.3.1) is precisely
\[
\rho - \sum_{j: \alpha^i_j + \alpha^r_{r-j} \leq d-r-g} d-r-g - (\alpha^i_j + \alpha^r_{r-j}) \leq \rho,
\]
so if (3.3.1) is satisfied, we immediately find that $\rho \geq 0$. It is not true in general that if $\rho > 0$ we necessarily have (3.3.1) satisfied, but we can analyze the relationship further.

First, we that we have equality above precisely when we have $\alpha^i_j + \alpha^r_{r-j} \geq d-r-g$ for $j = 0, \ldots, r$. In particular, if $d-r-g \leq 0$, then we see that regardless of the $\alpha^i_j$, we have (3.3.1) satisfied if and only if $\rho \geq 0$, so in this case nonemptiness is still equivalent to $\rho \geq 0$.

On the other hand, if $d-r-g > 0$, consider the case that $\alpha^i_j = 0$ for all $i, j$, so that we have not imposed any ramification. Then $\alpha^i_j + \alpha^r_{r-j} \leq d-r-g$ for all $j$, so (3.3.1) is automatically satisfied, and we thus recover the statement of the classical Brill-Noether theorem (Theorem 2.6.2 (i)) that we have nonemptiness when $\rho > 0$ and no ramification is imposed.

The proof of Theorem 3.3.5 is now straightforward, although it is lengthened somewhat by treating dimension and nonemptiness arguments together.

Proof of Theorem 3.3.5. The proof is by induction on the genus $g$, the base case of $g = 0$ having been handled already in Proposition 2.8.1.

If $n < 2$, we may assume $n = 2$ by adding points $P_i$ with $\alpha^i = 0, \ldots, 0$, as necessary. Let $Y_1, Y_2$ be projective nonsingular curves of genus $g-1$ and 1 respectively, chosen via the induction hypothesis and Proposition 2.8.2 so that there exist points $P_1, \ldots, P_{n-1}, Q_1$ on $Y_1$ and $P_n, Q_2$ on $Y_2$ satisfying the conclusions of the theorem. Note that we need only consider $\max\{n, 2\}$ points on $Y_1$ and $Y_2$, so whether we are in case (I) or case (II) the hypotheses are preserved for the induction. Let $X_0$ be the nodal curve obtained by gluing $Q_1$ on $Y_1$ to $Q_2$ on $Y_2$. Then by construction and by Corollary 3.2.9, we have that $G_d^r(X_0, \{((P_i, \alpha^i))\}_i)$ has dimension exactly $\rho$ if it is nonempty.

Now, let $\pi : X \to B$ be a family as in Theorem 3.3.4, and $P_1, \ldots, P_n$ disjoint sections of $\pi$ specializing to $P_1, \ldots, P_n$, respectively. For a fixed choice of the $\alpha^i$, by Proposition 3.3.2 we now conclude that $G_d^r(X|_b, \{((P_i)|_b, \alpha^i)\})$ can be nonempty of dimension strictly greater than $\rho$ on only finitely many points of $b$, so that there is an open subset $U$ of $B$ such that either $G_d^r(X|_b, \{((P_i)|_b, \alpha^i)\})$ is empty for all $b \in U$, or $\dim G_d^r(X|_b, \{((P_i)|_b, \alpha^i)\}) = \rho$ for all $b \in U$. Since there are only finitely many choices of the $\alpha^i$, the intersection of the open subsets obtained above is still a nonempty open subset, so choosing any $b$ in this intersection, we find that $X|_b$ together with the $(P_i)|_b$ has the desired dimension property: that is, $\alpha^1, \ldots, \alpha^n$ of type $(r, d)$, the space $G_d^r(X|_b, \{((P_i)|_b, \alpha^i)\})$ has pure dimension $\rho$ if it is nonempty.
It remains to analyze the nonemptiness assertion in case (II), so that \( n = 2 \). Again, by construction and by Corollary 3.2.9 we have that \( G_d^r(X_0, (\bar{P}_1, \alpha^1), (\bar{P}_2, \alpha^2)) \) is non-empty if and only if there exist Schubert indices \( \alpha^{Y_1}, \alpha^{Y_2} \) of type \((r, d)\) such that
\[
\alpha^{Y_1}_{r-i} + \alpha^{Y_2}_{r-i} = d - r
\]
for all \( i = 0, \ldots, r \) with both \( G_d^r(Y_1, (\bar{P}_1, \alpha^1), (Q_1, \alpha^{Y_1})) \) and \( G_d^r(Y_2, (\bar{P}_2, \alpha^2), (Q_2, \alpha^{Y_2})) \) nonempty. By hypothesis, these two spaces are nonempty if and only if
\[
\sum_{j: \alpha^1_j + \alpha^{Y_1}_{r-j} \geq d-r+1-(g-1)} \alpha^1_j + \alpha^{Y_1}_{r-j} - (d - r - (g - 1)) \leq g - 1
\]
and
\[
\alpha^2_j + \alpha^{Y_2}_{r-j} \leq d - r
\]
for all \( j \), with equality holding for at most one \( j \).

We claim that there exist \( \alpha^{Y_1}, \alpha^{Y_2} \) satisfying these last two conditions if and only if (3.3.1) holds. First suppose \( \alpha^{Y_2} \) satisfies (3.3.3). Then \( \alpha^{Y_1}_j \geq \alpha^2_j \) for all \( j \), with equality holding for at most one value of \( j \). We thus conclude
\[
\sum_{j: \alpha^1_j + \alpha^{Y_1}_{r-j} \geq d-r+1-(g-1)} \alpha^1_j + \alpha^{Y_1}_{r-j} - (d - r - (g - 1)) \geq \left( \sum_{j: \alpha^1_j + \alpha^{Y_2}_{r-j} \geq d-r+1-g} \alpha^1_j + \alpha^{Y_2}_{r-j} - (d - r - g) \right) - 1.
\]
In particular, if there exist \( \alpha^{Y_1} \) and \( \alpha^{Y_2} \) satisfying (3.3.2) and (3.3.3) respectively, then (3.3.1) must hold. Conversely, suppose first that (3.3.1) is satisfied with equality. It is enough to see that we can choose \( \alpha^{Y_1}, \alpha^{Y_2} \) so that (3.3.3) is satisfied, and (3.3.4) is satisfied with equality. Since \( g > 0 \), we must have at least some indices \( j \) such that \( \alpha^1_j + \alpha^{Y_2}_{r-j} \geq d - r + 1 - g \); let \( j_0 \) be the maximal such index. Then we set \( \alpha^{Y_2}_j = d - r - 1 - \alpha^{Y_2}_{r-j} \) for all \( j \neq j_0 \), and \( \alpha^{Y_2}_{j_0} = d - r - \alpha^{Y_2}_{r-j_0} \). Note that this gives a valid Schubert index because of the maximality of \( j_0 \). If we then set \( \alpha^{Y_1}_j = d - r - \alpha^{Y_1}_{r-j} \) for all \( j \), we will achieve equality in (3.3.4), as desired. Lastly, suppose that (3.3.1) is satisfied with strict inequality. Then we can set \( \alpha^{Y_2}_j = d - r - 1 - \alpha^{Y_2}_{r-j} \) and \( \alpha^{Y_1}_j = d - r - \alpha^{Y_1}_{r-j} \) for all \( j \), and we will still have (3.3.2). This proves the claim, and we conclude that \( G_d^r(X_0, ((\bar{P}_1, \alpha^1), (\bar{P}_2, \alpha^2))) \) is nonempty if and only if (3.3.1) is satisfied.

Finally, returning to our smoothing family \( \pi : X \to B \), for a fixed choice of \( \alpha^1 \) and \( \alpha^2 \), if (3.3.1) is satisfied, we now conclude from Theorem 3.3.3 that there is a nonempty open subset \( U \) of \( B \) such that \( G_d^r(X_\beta, \{(\bar{P}_i)\}_\beta, \{(\bar{P}_i)\}_\beta, \alpha^1) \) is nonempty of dimension exactly \( \rho \). On the other hand, if (3.3.1) is violated, we conclude from Proposition 3.3.2 that we can only have \( G_d^r(X_\beta, \{(\bar{P}_i)\}_\beta, \{(\bar{P}_i)\}_\beta) \) nonempty for finitely many \( \beta \in B \), which is to say \( G_d^r(X_\beta, \{(\bar{P}_i)\}_\beta, \{(\bar{P}_i)\}_\beta) \) is empty on a nonempty open subset \( U \) of \( B \). Since there are only finitely many choices of \( \alpha^1, \alpha^2 \), the intersection of the open subsets obtained above is still a nonempty open subset, so choosing any \( \beta \) in this intersection, we find that \( X_\beta \) together with \( (\bar{P}_1)_\beta \) and \( (\bar{P}_2)_\beta \) has \( G_d^r(X_\beta, \{(\bar{P}_i)\}_\beta, \{(\bar{P}_i)\}_\beta) \) nonempty exactly when (3.3.1) is satisfied, as desired. \( \Box \)
CHAPTER 4

Families and formal constructions

We begin by giving the construction of \( \mathcal{G}_d^r \) spaces in the case of families of smooth curves. We then discuss the structure of families of nodal curves, and explain how to construct limit linear series spaces for families of curves of compact type. This construction will also yield proofs of the foundational theorems on limit linear series theory.

In this chapter, we extensively use the material from Appendix A on representable functors and moduli spaces.

4.1. Moduli of linear series for families

We begin by revisiting the constructions described for nonsingular curves in Sections 2.6 and 2.7, generalizing them to families and giving rigorous definitions and statements. We will be able to show that in order to prove the Brill-Noether theorem, it is enough to produce a single nonsingular curve satisfying its hypotheses, and the construction also serves as a template for the more complicated case of limit linear series.

The first step is to define a moduli functor for \( \mathcal{G}_d^r \) on a nonsingular curve, or family of nonsingular curves.

**Definition 4.1.1.** Let \( \pi : X \to B \) be a smooth, proper family of curves of genus \( g \) having a section. Given \( r, d \geq 0 \), the functor \( \mathcal{G}_d^r(X/B) \) is defined by associating to each scheme \( T \) over \( B \) the set of equivalence classes of pairs \( (\mathcal{L}, \mathcal{V}) \), where \( \mathcal{L} \) is a line bundle on \( X \times_B T \) having degree \( d \) on all fibers, and \( \mathcal{V} \subseteq p_2^*(\mathcal{L}) \) is a subbundle of rank \( r+1 \). Here \( (\mathcal{L}, \mathcal{V}) \) and \( (\mathcal{L}', \mathcal{V}') \) are equivalent if there exists a line bundle \( M \) on \( T \) and an isomorphism \( \varphi : \mathcal{L} \to \mathcal{L}' \otimes p_2^*M \) such that \( p_2^*\varphi \) maps \( \mathcal{V} \) into \( \mathcal{V}' \).

Note that the last condition makes sense because \( p_2^*(\mathcal{L}' \otimes p_2^*M) \) is canonically isomorphic to \( (p_2^*\mathcal{L}') \otimes M \) (Exercise II.5.1(d) of [Har77]), and although there is no natural morphism \( p_{2*}\mathcal{L}' \to (p_{2*}\mathcal{L}') \otimes M \), sections up to invertible scalar multiplication are canonically identified, so subbundles of the two sheaves are likewise canonically identified. The reason we need the equivalence relation in the definition of \( \mathcal{G}_d^r(X/B) \) is that, just as in the case of \( \mathcal{P}ic^d(X/B) \) of Definition A.4.11, our functor would otherwise not form a Zariski sheaf.

We also define a moduli functor with imposed ramification as follows:

**Definition 4.1.2.** In the situation of Definition 4.1.1, suppose we are given also sections \( P_1, \ldots, P_n \) of \( \pi \), and \( \alpha^1, \ldots, \alpha^n \) Schubert indices of type \( (r, d) \). Then the subfunctor

\[
\mathcal{G}_d^r(X/B, \{(P_i, \alpha^i)\}_i) \subseteq \mathcal{G}_d^r(X/B)
\]

consists of pairs \( (\mathcal{L}, \mathcal{V}) \) satisfying the additional condition that

\[
\text{rk} (\mathcal{V} \to p_{2*}(\mathcal{L}|_{P_i})) \leq r + 1 - b_i^j
\]
for each \( i = 1, \ldots, n \) and \( j = 0, \ldots, d \), where
\[
b_j^i := \#\{\ell : \alpha_\ell^i + \ell \geq j\}.
\]

The main structural result is the following:

**Theorem 4.1.3.** In the situation of Definition 4.1.1, the functor \( G_d^r(X/B) \) is represented by a scheme \( G_d^r(X/B) \) which is proper over \( B \). In the situation of Definition 4.1.2, \( G_d^r(X/B, \{(P_i, \alpha_i^r)\}_i) \) is represented by a closed subscheme \( G_d^r(X/B, \{(P_i, \alpha_i^r)\}_i) \) of \( G_d^r(X/B) \), and is therefore likewise proper over \( B \). Moreover, \( G_d^r(X/B, \{(P_i, \alpha_i^r)\}_i) \) has relative dimension at least
\[
\rho := g - (r + 1)(g - d + r) - \sum_{i=1}^n \sum_{j=0}^r \alpha_j^i
\]
over \( B \).

Note that \( G_d^r(X/B) \) is a special case of \( G_d^r(X/B, \{(P_i, \alpha_i^r)\}_i) \), either by taking \( n = 0 \), or by setting all \( \alpha_j^i = 0 \). The proof of Theorem 4.1.3 follows the construction sketched in Sections 2.6 and 2.7. Before giving it, we observe the following corollary, which explains why it is enough to produce a single nonsingular curve satisfying the Brill-Noether theorem.

**Corollary 4.1.4.** In the situation of Definition 4.1.2, the subset of \( B \) over which \( G_d^r(X/B, \{(P_i, \alpha_i^r)\}_i) \) is nonempty is a closed subset. For any \( m \geq 0 \), the set of \( b \in B \) over which the fiber of \( G_d^r(X/B, \{(P_i, \alpha_i^r)\}_i) \) is nonempty of dimension at least \( m \) is also a closed subset.

In particular, if for some \( b_0 \in B \) we have \( G_d^r(X|_{b_0}/b_0, \{(P_i|_{b_0}, \alpha_i^r)\}) \) empty, then the same holds in an open neighborhood of \( b_0 \). If for some \( b_0 \in B \) we have \( G_d^r(X|_{b_0}/b_0, \{(P_i|_{b_0}, \alpha_i^r)\}) \) nonempty of dimension \( \rho \), then the same holds in an open neighborhood of \( b_0 \), and moreover every point of \( G_d^r(X|_{b_0}/b_0, \{(P_i|_{b_0}, \alpha_i^r)\}) \) is the specialization of points of \( G_d^r(X/B, \{(P_i, \alpha_i^r)\}_i) \) supported away from \( b_0 \).

**Proof.** The assertion on nonemptiness is immediate from the properness of \( G_d^r(X/B, \{(P_i, \alpha_i^r)\}_i) \). The first statement on dimension follows from properness and Chevalley’s theorem on semicontinuity of fiber dimension (Theorem 14.8 of [Eis95]). The statement for the case of fiber dimension \( \rho \) then follows from the fact that \( G_d^r(X/B, \{(P_i, \alpha_i^r)\}_i) \) has relative dimension at least \( \rho \) over \( B \). \( \square \)

Thus, Corollary 4.1.4 says that in families of curves, the conditions of having no \( g_0^r \)’s (satisfying given ramification conditions) or of having the expected dimension of such \( g_0^r \)’s are both open. This is why statements in the Brill-Noether theorem asserted for general curves reduce to statements on the existence of a single curve; see Remarks 2.5.6 and 3.3.6.

We now proceed to the proof of the theorem.

**Proof of Theorem 4.1.3.** This is a matter of revisiting the proof of Propositions 2.6.1 and 2.7.3 and verifying that it works as claimed, and generalizes to families of curves. Let \( \sigma \) be a section of \( \pi \), and let \( D \) be the effective divisor on \( X \) obtained as \( m\sigma \) for a nonnegative \( m \) at least \( 2g - 1 - d \).\(^1\) Let \( \text{Pic}^d(X/B) \) be the relative Picard scheme, with Poincare line bundle \( \hat{L} \); see Section A.4. Denote by

\(^1\)This portion of the argument does not in fact require a section, but since we need a section in order to work with Pic, we might as well use it.
D' the divisor $D \times_B \text{Pic}^d(X/B)$ on $X \times_B \text{Pic}^d(X/B)$. We will cut out $G'_{d}(X/B)$ inside the relative Grassmannian $G(r + 1, p_{2*}\mathcal{L}(D'))$ over $\text{Pic}^d(X/B)$; see Section A.3.

From the definitions of $G(r + 1, p_{2*}\mathcal{L}(D'))$ and $\text{Pic}^d(X/B)$, we see that $G(r + 1, p_{2*}\mathcal{L}(D'))$ represents the functor of equivalence classes of pairs $(\mathcal{L}, \mathcal{V})$, where $\mathcal{L}$ is a line bundle on $X \times_B T$ having degree $d$ on fibers, and $\mathcal{V}$ is a rank-$(r+1)$ sub-bundle of $p_{2*}((\mathcal{L}(D \times_B T))$. Here the equivalence relation is as in Definition 4.1.1. The one subtlety is that if $f : T \to \text{Pic}^d(X/B)$ induced by $\mathcal{L}$, we want sub-bundles of $p_{2*}((\mathcal{L}(D \times_B T))$, which is the same as sub-bundles of $p_{2*}(\text{id} \times f)^*\mathcal{L}(D')$, but the definition of the relative Grassmannian gives us instead sub-bundles of $f^*p_{2*}\mathcal{L}(D')$. Thus, we use that having twisted by $D'$, push-forward commutes with base change.

To cut out a closed subscheme whose $T$-valued points are the pairs of Definition 4.1.1, let
\[
g : G(r + 1, p_{2*}\mathcal{L}(D')) \to \text{Pic}^d(X/B)
\]
be the structure map, and
\[
\mathcal{V} \to g^*p_{2*}\mathcal{L}(D')
\]
the tautological subbundle on $G(r + 1, p_{2*}\mathcal{L}(D'))$. Then we claim that the closed subscheme on which the composed map
\[
\mathcal{V} \to g^*p_{2*}\mathcal{L}(D') \to g^*p_{2*}\mathcal{L}(D')|_{D'}
\]
is equal to zero represents $G'_{d}(X/B)$. Indeed, for any $f : T \to G(r + 1, p_{2*}\mathcal{L}(D'))$ we can identify
\[
p_{2*}(\text{id} \times g \circ f)^*\mathcal{L}(D') = (g \circ f)^*p_{2*}\mathcal{L}(D')
\]
and
\[
p_{2*}(\text{id} \times g \circ f)^*\mathcal{L}(D')|_{D'} = (g \circ f)^*p_{2*}\mathcal{L}(D')|_{D'},
\]
so this amounts to the assertion that we have an exact sequence
\[
0 \to p_{2*}(\text{id} \times g \circ f)^*\mathcal{L}(D') \to p_{2*}(\text{id} \times g \circ f)^*\mathcal{L}(D') \to p_{2*}(\text{id} \times g \circ f)^*\mathcal{L}(D')|_{D'}.
\]
But because $\mathcal{L}(D')|_{D'}$ is flat over $\text{Pic}^d(X/B)$, the exact sequence
\[
0 \to \mathcal{L} \to \mathcal{L}(D') \to \mathcal{L}(D')|_{D'} \to 0
\]
remains exact under $(\text{id} \times g \circ f)^*$, and then $p_{2*}$ induces the desired exact sequence. We thus conclude the representability assertion for $\text{Pic}^d(X/B)$. Moreover, since $\text{Pic}^d(X/B)$ is proper over $B$, and $G(r + 1, p_{2*}\mathcal{L}(D'))$ is proper over $\text{Pic}^d(X/B)$, we conclude that $G'_{d}(X/B)$ is proper over $B$, as claimed.

Abusing notation slightly, we also use $g$ to denote the structure map of $G'_{d}(X/B)$, and
\[
\mathcal{V} \to g^*p_{2*}\mathcal{L}(D')
\]
the tautological subbundle on $G'_{d}(X/B)$.

Next, consider the ramification conditions. Since determinantal conditions (Definition A.3.9) commute with pullback, it is just a matter of chasing definitions to verify $G'_{d}(X/B, \{(P_i, \alpha_i^1)\})$ is represented by the closed subscheme of $G'_{d}(X/B)$ on which the composed maps
\[
\mathcal{V} \to p_{2*}(\text{id} \times g)^*p_{2*}\mathcal{L} \to p_{2*}(\text{id} \times g)^*\mathcal{L}|_{jP_i}
\]
have rank less than or equal to $r + 1 - b^i_j$ for each $i = 1, \ldots, n$ and $j = 0, \ldots, d$, where $b^i_j$ is as in Definition 4.1.2.
It remains to consider the dimension. Recalling that $\text{Pic}^d(X/B)$ is smooth of relative dimension $g$ over $B$, and $G(r+1, p_{2z}, \mathcal{L}(D'))$ is smooth over $\text{Pic}^d(X/B)$ of relative dimension $(r+1)((d+\deg D+1-g)-(r+1))$, we see that $G(r+1, p_{2z}, \mathcal{L}(D'))$ is smooth of relative dimension $g + (r+1)(d+\deg D - g - r)$ over $B$. Then $G^*_d(X/B)$ is locally cut out by $(r+1)(\deg D)$ equations, so we conclude by the Krull principal ideal theorem that $G^*_d(X/B)$ has relative dimension at least $g + (r+1)(d+\deg D - g - r) - (r+1)(\deg D) = g + (r+1)(d - g - r)$.

Next, for each $i = 1, \ldots, n$, the condition that $\tilde{V} \to p_{2z}(\text{id} \times g)^* p_{2z} \mathcal{L} \to p_{2z}(\text{id} \times g)^* \mathcal{L}|_{jP_i}$ have rank less than or equal to $r + 1 - b^i_j$ for $j = 0, \ldots, d$ imposes a Schubert condition, which can have codimension at most $\sum_{j=0}^r \alpha^i_j$. Thus, we find that $G^*_d(X/B, \{(P_i, \alpha^i)\}_i)$ has relative dimension at least $\rho$, as desired. □

**Remark 4.1.5.** Frequently, in Definition 4.1.2 one requires the sections $P_i$ to be disjoint, but this hypothesis is not necessary, so we omit it. However, if the sections are not disjoint in some fiber, then the ramifications conditions become redundant, and we cannot hope that the space of $g^r_d$s with the given ramification on the fiber in question will have the expected dimension.

### 4.2. Families of nodal curves

Not all families of nodal curves are well suited to limit linear series constructions. For brevity, we combine the most important conditions into a single definition as follows.

**Definition 4.2.1.** A morphism of schemes $\pi : X \to B$ constitutes a genus-$g$ smoothing family if:

(I) $B$ is regular and connected;

(II) $\pi$ is flat and proper;

(III) The fibers of $\pi$ are genus-$g$ curves of compact type;

(IV) $\pi$ has a section;

(V) The irreducible components of every fibers of $\pi$ are geometrically irreducible;

(VI) Each connected component of the non-smooth locus of $\pi$ maps isomorphically onto its scheme-theoretic image in $B$;

(VII) Any point in the nonsmooth locus of $\pi$ which is smoothed in the generic fiber is regular in the total space of $X$.

For convenience, we will impose one additional condition beyond those enumerated in Definition 4.2.1. The idea is that it is convenient to work with families where there is a unique “most degenerate” dual graph to consider. We therefore introduce a condition which we call “almost local,” which ensures that there is a maximal dual graph $\Gamma$ such that the dual graph in every fiber is (naturally and compatibly) a contraction of $\Gamma$. The remainder of this section is of a technical nature, and the proofs in particular could be skipped on a first reading.

We begin by considering the behavior of dual graphs in general smoothing families.
Lemma 4.2.2. Suppose that $\pi : X \to B$ is a smoothing family, and $b$ specializing to $b'$ are points of $B$. Then if $\Gamma_b$ and $\Gamma_b'$ denote the dual graphs of the fibers $X_b$ and $X_{b'}$ respectively, there is a unique contraction map

$$cl_{b,b'} : \Gamma_{b'} \to \Gamma_b$$

induced on vertices by associating to a component $Y'$ of $X_{b'}$ the component $Y$ of $X_b$ containing $Y'$ in its closure. The behavior of $cl_{b,b'}$ on edges is as follows: given an edge $e$ in $\Gamma_b'$ corresponding to a node $\Delta'$ in $X_{b'}$, if there is a node of $X_b$ specializing to $\Delta'$, then $cl_{b,b'}$ maps $e$ to the corresponding edge of $\Gamma_b$; otherwise, $e$ is contracted.

If also $b'$ specializes to some $b'' \in B$, then we have

$$cl_{b,b''} = cl_{b,b'} \circ cl_{b',b''}.$$  

Proof. Given $b$ specializing to $b'$ and a component $Y'$ of $X_{b'}$, we first need to see that there exists a unique component $Y$ of $X_b$ containing $Y'$ in its closure. Existence follows from flatness of $\pi$. To see uniqueness, let $Y$ and $Z$ be distinct components of $X_b$, and let $\Delta'$ be the connected component of the non-smooth locus of $\pi$ containing a node of $X_b$ separating $Y$ and $Z$. Let $\Delta$ be the image of $\Delta'$; since $b, b' \in \Delta$, we can check uniqueness of specialization after restricting to $\Delta$. By definition of a smoothing family, we have that $\pi^{-1}(\Delta)$ breaks into (not necessarily irreducible) components $Y_\Delta$ and $Z_\Delta$ with $Y_\Delta \cap Z_\Delta = \Delta'$. Then the generic point of any component of $X_b$ or $X_{b'}$ is contained in precisely one of $Y_\Delta$ and $Z_\Delta$, so it follows that $Y'$ can be in the closure of at most one of $Y$ and $Z$, as desired. This gives $cl_{b,b'}$ on the vertices of $\Gamma_{b'}$ and $\Gamma_b$; to see that we in fact obtain a contraction of graphs, we also need to check that if $Y'$ and $Z'$ are components of $X_{b'}$ meeting at a node, and $Y$ and $Z$ are the components of $X_b$ containing $Y$ and $Z$ respectively in their closure, then either $Y = Z$ or $Y$ and $Z$ also meet at a node. But arguing as above, if $Y$ and $Z$ are separated by two or more nodes of $X_b$, we see that the same is true for any components of $X_{b'}$ in their closures, giving the desired statement. By the same argument, if $Y \neq Z$ we find that the node $Y' \cap Z'$ is the specialization of the node $Y \cap Z$, and conversely that if $Y = Z$ then there can be no node of $X_b$ specializing to $Y' \cap Z'$, giving the desired description of $cl_{b,b'}$ on edges. Finally, associativity is clear from the definition.

Although the below definition is slightly complicated, the idea behind it is simple: it captures the desired condition of having a maximal dual graph.

Definition 4.2.3. We say a smoothing family $\pi : X \to B$ is almost local if the following condition is satisfied: if $\Delta_1', \ldots, \Delta_m'$ are the connected components of the non-smooth locus of $\pi$, with images $\Delta_1, \ldots, \Delta_m$ in $B$, then there exists a $b_0 \in B$ such that for all $S, S' \subseteq \{1, \ldots, m\}$, we have $\bigcap_{i \in S} \Delta_i$ non-empty, and for any irreducible components $Z$ of $\bigcap_{i \in S} \Delta_i$ and $Z'$ of $\bigcap_{i \in S'} \Delta_i$, every irreducible component of $Z \cap Z'$ contains $b_0$.

Remark 4.2.4. Observe that the almost local hypothesis is trivially satisfied in the case that $\pi : X \to B$ has connected non-smooth locus, as is the situation in [Oss06]. Likewise, it is always satisfied if $\Delta_i = B$ for each $i$.

Finally, it is always satisfied locally on $B$: indeed, given $b \in B$, we construct an open neighborhood on which $\pi$ is almost local simply by removing any irreducible component of each $\bigcap_{i \in S} \Delta_i$ which does not contain $b$, and doing likewise for intersections of pairs of components.
The following proposition says that an almost local smoothing family admits a maximal dual graph in a natural way; this is the reason for the hypothesis, as it will greatly simplify keeping track of dual graphs and multidegrees.

**Proposition 4.2.5.** Suppose that \( \pi : X \to B \) is an almost local smoothing family. Then there exists a graph \( \Gamma \), occurring as the dual graph of some fiber of \( \pi \), and, for every \( b \in B \), a contraction

\[
\cl_b : \Gamma \to \Gamma_b,
\]

where \( \Gamma_b \) is the dual graph of the fiber \( X_b \), satisfying the following condition: if \( b \) specializes to \( b' \), we have

\[
(4.2.1) \quad \cl_b = \cl_{b,b'} \circ \cl_{b'}.
\]

Moreover, up to automorphism of \( \Gamma \), we have that \( \Gamma \) and the contractions \( \cl_b \) are unique.

Note that the uniqueness assertion in particular implies that the data of \( \Gamma \) and the \( \cl_b \) contractions is independent of the choice of \( b_0 \) from Definition 4.2.3.

**Proof.** We first fix some notation. Let \( \Delta'_1, \ldots, \Delta'_m \) be the connected components of the non-smooth locus, and \( \Delta_1, \ldots, \Delta_m \) their images in \( B \). For \( b \in B \), set \( S_b \subseteq \{1, \ldots, m\} \) to be the subset of \( i \) such that \( b \in \Delta_i \). Observe that given \( b \) specializing to \( b' \), if \( S_b = S_{b'} \) the contraction \( \cl_{b',b} \) is a surjection of trees with the same number of edges, and hence is necessarily an isomorphism. Let \( b_0 \) be as in Definition 4.2.3. Set \( \Gamma = \Gamma_{b_0} \), with \( \cl_{b_0} \) being the identity. Now, for any \( b \in B \), let \( \tilde{b} \) be a generic point of \( \cap_{i \in S_b} \Delta_i \) which specializes to \( b \), so that \( \cl_{\tilde{b}} \) is an isomorphism. It follows from the definition of almost local that \( \tilde{b} \) specializes to \( b_0 \), so we can set

\[
(4.2.2) \quad \cl_b = \cl_{b,b'}^{-1} \circ \cl_{b,b_0} \circ \cl_{b_0}.
\]

It remains to check that given \( b \) specializing to \( b' \), we have (4.2.1). Let \( \tilde{b}' \) be the generization of \( b' \) used to define \( \cl_{b'} \), and let \( Z, Z' \) be the closures of \( \tilde{b} \) and \( \tilde{b}' \), respectively. Let \( \tilde{Z} \) be a component of \( Z \cap Z' \) containing \( b' \), and \( \tilde{b} \) its generic point. Then according to the almost local hypothesis, we have that \( \tilde{b} \) specializes to \( b_0 \).

Using associativity, we then have

\[
\cl_b = \cl_{b,b}^{-1} \circ \cl_{b,b_0} \circ \cl_{b_0} = \cl_{b,b}^{-1} \circ \cl_{b,b_0} \circ \cl_{b_0} \circ \cl_{b_0} = \cl_{b,b}^{-1} \circ \cl_{b,b_0} \circ \cl_{b_0} \circ \cl_{b_0} = \cdots = \cl_{b,b'} \circ \cl_{b',b_0} \circ \cl_{b_0} = \cl_{b,b'} \circ \cl_{b',b_0}.
\]

as desired.

For the uniqueness assertion, observe that \( \Gamma_{b_0} \) has \( m \) edges, the maximal number possible among any \( \Gamma_b \), so we must have \( \Gamma \cong \Gamma_{b_0} \). If we fix a choice of \( \cl_{b_0} \), which is defined precisely up to automorphism of \( \Gamma \), we then have that (4.2.1) implies that (4.2.2) must hold, so we have that \( \cl_{b} \) is uniquely determined for all \( b \), as asserted.

\[\square\]
4.2. Families of Nodal Curves

Corollary 4.2.6. Let \( \pi : X \to B \) be an almost local smoothing family, and \( \Gamma \) and \( \text{cl}_b \) for \( b \in B \) as given by Proposition 4.2.5. Then there is a bijection from \( E(\Gamma) \) to the connected components of the non-smooth locus of \( \pi \) induced by sending an edge \( e \in E(\Gamma) \) to

\[
\Delta'_e := \bigcup_{b \text{cl}_b \text{ does not contract } e} \Delta'_{\text{cl}_b(e)},
\]

where \( \Delta'_{\text{cl}_b(e)} \) denotes the node of \( X_b \) corresponding to \( \text{cl}_b(e) \).

Furthermore, if \( \Delta_\varepsilon \subseteq B \) denotes the image of \( \Delta'_e \), then given \( v \in V(\Gamma) \) adjacent to \( e \), there is a unique closed subset \( \Delta(\varepsilon, v) \subseteq \pi^{-1} \Delta'_{\varepsilon} \) such that for each \( b \in \Delta_\varepsilon \), the fiber \( (Y_{(e,v)})_b \) is equal to the union of the components of \( X_b \) corresponding to \( b \) that have fiber \( (\Gamma) \)

Thus, if \( v, v' \) are the two vertices adjacent to an edge \( e \), then \( Y_{(e,v)} \cup Y_{(e,v')} = \pi^{-1} \Delta_\varepsilon \), and \( Y_{(e,v)} \cap Y_{(e,v')} = \Delta'_{\varepsilon} \).

Proof. For the first assertion, we need to check that \( \Delta'_e \) is in fact a connected component of the non-smooth locus of \( \pi \), and that the induced map is a bijection. Let \( b_0 \) be as in the definition of almost local, and fix \( e \in E(\Gamma) \). Let \( \Delta' \) be the connected component of the non-smooth locus containing \( \Delta'_{\text{cl}_b(e)} \). We want to see that \( \Delta'_e = \Delta' \). Then given \( b \in B \), let \( \bar{b} \) be a point generizing \( b \) and \( b_0 \), and such that \( \text{cl}_{\bar{b}} \) is an isomorphism. First suppose that \( \text{cl}_b \) does not contract \( e \). Then we have that \( \Delta'_{\text{cl}_{\bar{b}}(e)} \) specializes to both \( \Delta'_{\text{cl}_{b_0}(e)} \) and to \( \Delta'_{\text{cl}_b(e)} \). Thus, since \( \Delta' \) is a connected component of the non-smooth locus, we conclude that \( \Delta'_{\text{cl}_{\bar{b}}(e)} \subseteq \Delta' \), so \( \Delta'_e \subseteq \Delta' \). For the opposite containment, suppose that \( b \) is in the image of \( \Delta'_e \). Then since \( \Delta' \) is a section of \( \pi \) over its image, we see that \( \Delta' \) has a point of \( X_b \) which specializes to \( \Delta'_{\text{cl}_{b_0}(e)} \). It follows that \( \text{cl}_b \) does not contract \( e \), and then neither does \( \text{cl}_{\bar{b}} \). Thus, \( b \) is in the image of \( \Delta'_e \), and since \( \Delta' \) is a section over its image and \( \Delta'_e \subseteq \Delta' \), we conclude that \( \Delta'_e = \Delta' \), as desired.

Next, to see that we have a bijection, we note that injectivity is trivial, since for \( e \neq e' \) we have that \( \Delta'_{\text{cl}_{b_0}(e)} \) and \( \Delta'_{\text{cl}_{b_0}(e')} \) are distinct points. On the other hand, the definition of almost local imposes that every connected component \( \Delta' \) of the non-smooth locus of \( \pi \) meets \( X_{b_0} \), so meets some \( \Delta'_{\text{cl}_{b_0}(e)} \). Since \( \Delta'_{\text{cl}_{b_0}(e)} \) is also a connected component of the non-smooth locus, we conclude \( \Delta' = \Delta'_{\text{cl}_{b_0}(e)} \), giving surjectivity.

For the assertion on \( Y_{(e,v)} \), we have by hypothesis that \( \pi^{-1} \Delta_\varepsilon \) decomposes as \( Y \cup Z \), with \( Y \cap Z = \Delta'_{\varepsilon} \); exactly one of \( Y \) or \( Z \) contains the component of \( X_{b_0} \) corresponding to \( \text{cl}_{b_0}(v) \), so if we set \( Y_{(e,v)} \) equal to this subset, we need only verify that it has the desired form on every fiber. It is evident from the construction that for each \( b \in \Delta_\varepsilon \), we have \( Y_{(e,v)}|_b \) equal to the union of components of \( X_b \) corresponding to a connected component of \( \Gamma_b \setminus \{e\} \), so we need only verify that the connected component in question is the one containing \( \text{cl}_b(v) \). If \( b \) specializes to \( b_0 \), this is immediate from the fact that \( \text{cl}_b = \text{cl}_{b_0} \circ \text{cl}_{b_0} \), and in the general case it follows by choosing some \( \bar{b} \) specializing to both \( b \) and \( b_0 \) as above.

We conclude by applying the above discussion to produce twisting line bundles on almost local smoothing families. We need the following basic fact:

Theorem 4.2.7. If \( \pi : X \to B \) is a smoothing family, and \( \Delta \) is the image in \( B \) of a connected component of the non-smooth locus of \( \pi \), then \( \Delta \) is either all of \( B \), or a divisor in \( B \).
Notation 4.2.8. Let \( \pi : X \to B \) be an almost local smoothing family, and \( \Gamma \) as in Proposition 4.2.5. For every pair \((e, v)\) of an edge \( e \in E(\Gamma) \) and an adjacent vertex \( v \), denote by \( \mathcal{O}_{(e, v)} \) the line bundle on \( X \) obtained as follows. Write
\[
X_{\Delta_e} = Y_{(e,v)} \cup Y_{(e,v')},
\]
where \( v' \) is the other vertex adjacent to \( e \).

Now, if \( \Delta_e \neq B \), we have \( Y_{(e,v)} \) a divisor in \( X \), and we set \( \mathcal{O}_{(e,v)} = \mathcal{O}_X(Y_{(e,v)}) \).

On the other hand, if \( \Delta_e = B \), then line bundles on \( X \) are uniquely determined by their restrictions to \( Y_{(e,v)} \) and \( Y_{(e,v')} \), and we define \( \mathcal{O}_{(e,v)} \) to be \( \mathcal{O}_{Y_{(e,v)}}(-\Delta'_e) \) on \( Y_{(e,v)} \) and to be \( \mathcal{O}_{Y_{(e,v')}}(\Delta'_e) \) on \( Y_{(e,v')} \).

4.3. Moduli of limit linear series, revisited

We now move to the technical heart of the book, which is the construction of a moduli space of limit linear series for families of curves of compact type. We will carry this out in two stages, beginning in this section by analyzing the situation for a single curve of compact type (and more generally, for families in which the dual graph is constant), and then in the next section considering families in which nodes may be smoothed. The main idea of the construction comes out of Proposition 3.2.2, with additional ingredients including a generalization of determinantal loci to pushforwards, and generalization from curves with two components to arbitrary curves of compact type; this forces us to introduce more notation but will not make the definitions themselves any harder. As suggested by Proposition 3.2.2, the point of view which emerges is that the space of Eisenbud-Harris limit linear series should be thought of as a hybrid between a \( G'_g \) space and a \( W'_g \) space, in that a limit linear series consists of some \( g'_d \)'s, but these must satisfy conditions of a determinantal nature.

We begin by generalizing Definition 3.2.1 from curves with two components to arbitrary curves of compact type.

Definition 4.3.1. Let \( X_0 \) be a curve of compact type over a field \( k \). Let \( \Gamma \) be the dual graph of \( X_0 \), and denote by \( Y_v \) for \( v \in V(\Gamma) \) the corresponding irreducible component of \( X_0 \), and by \( \Delta'_v \) for \( e \in E(\Gamma) \) the corresponding node. Given \( r, d \), a limit linear series on \( X_0 \) is a tuple \((\mathcal{L}^v, V^v)\) of \( g'_d \)'s on each \( Y_v \), satisfying the following compatibility condition on vanishing sequences: for any \( e \in E(\Gamma) \), with \( v, v' \) the adjacent vertices, we have
\[
a_j(\mathcal{L}^v, V^v)(\Delta'_e) + a_{r-j}(\mathcal{L}^{v'}, V^{v'})(\Delta'_e) \geq d
\]
for \( j = 0, \ldots, r \).

As we mentioned previously, to construct spaces of limit linear series in families where fibers may be smoothed, we will have to work with line bundles on the whole curve, rather than on each component. However, we will still want to know that what we obtain is equivalent to limit linear series as defined by Eisenbud and Harris, so we will begin by defining a moduli space closer to the Eisenbud-Harris definition for the case in which no nodes are smoothed – that is, \( \Delta_e = B \) for all \( e \in E(\Gamma) \). As one would expect, it will be a closed subscheme of a product of \( G'_g \) spaces on components. The notation is as follows.

Notation 4.3.2. Given an almost local smoothing family \( \pi : X \to B \), let \( \Gamma \) be as in Proposition 4.2.5, and suppose further that \( \Delta_e = B \) for all \( e \in E(\Gamma) \), so that
every fiber of $\pi$ has dual graph $\Gamma$. For each $v \in V(\Gamma)$, let $Y_v$ denote the smooth, irreducible component of $X$ corresponding to $v$. Given also $r \leq d \in \mathbb{Z}_{\geq 0}$, let
\[
A_{\pi,r,d} = \prod_{v \in V(\Gamma)} G^r_d(Y_v/B),
\]
where the product is fibered over $B$.

In order to describe the determinantal conditions cutting out the space of limit linear series inside $G^r_d(Y_v)$, we will need to work with line bundles globally on $X$, and we will need to introduce a substantial amount of additional notation to enable us to generalize the framework discussed previously in the case of curves with two components to arbitrary curves of compact type. In carrying out this generalization, perhaps the most obvious idea is to consider all possible nonnegative multidegrees adding up to $d$, but it turns out to be easier to work with a somewhat restricted set of multidegrees, parametrized by the vertices of a graph which we now define.

**Notation 4.3.3.** Let $\Gamma$ be a tree, and $d \in \mathbb{Z}_{\geq 0}$. Let $G$ be the directed graph defined as follows: $V(G)$ is the subset of $(d_v)_{v \in V(\Gamma)} \in \mathbb{Z}_{\geq 0}^{V(\Gamma)}$ satisfying

(I) $\sum_v d_v = d$;

(II) There exists $e \in E(\Gamma)$ such that, if $v$ and $v'$ are the vertices adjacent to $e$, then $d_{v''} = 0$ for all $v'' \neq v, v'$.

$E(G)$ contains an edge $\tilde{e}$ from $(d_v)_v$ to $(d'_v)_v$ if there is a pair $v, v' \in V(\Gamma)$ of adjacent vertices such that we have $d_{v''} = d_{v''}$ for all $v'' \neq v, v'$, but $d_v - d'_v = 1$ and $d_v - d'_v = -1$. In this case, write $v_{\tilde{e}} := v$, and let $e_{\tilde{e}}$ be the edge in $\Gamma$ connecting $v$ and $v'$.

Also, given $v \in \Gamma$, and $d \in \mathbb{Z}_{\geq 0}$, let $\tilde{d}^v$ be the multidegree which is equal to $d$ in index $v$ and 0 in all other indices.

Thus, $G$ is determined by its underlying undirected graph by always including edges in each direction, and (the underlying undirected graph of) $G$ is obtained from $\Gamma$ simply by subdividing each edge of $\Gamma$ into $d$ edges. However, from a bookkeeping standpoint $G$ is quite useful, as its vertices keep track of multidegrees, and its edges of possible maps induced by twists. The next order of business is to describe line bundles obtained by twisting a given one.

**Notation 4.3.4.** Given an almost local smoothing family $\pi : X \to B$, let $\Gamma$ be as in Proposition 4.2.5.

Now, fix a choice of $v_0 \in V(\Gamma)$, and set $\tilde{d}_0 := \tilde{d}^0$. Given $T$ over $B$, and a line bundle $\mathcal{L}$ on $X \times_B T$ of multidegree $\tilde{d}_0$, for $\tilde{d} = (d_v)_v \in G$, denote by $\mathcal{L}_{\tilde{d}}$ the line bundle of multidegree $\tilde{d}$ on $X \times_B T$ obtained as follows: let $(\tilde{e}_1, \ldots, \tilde{e}_n)$ be the unique minimal directed path from $\tilde{d}_0$ to $\tilde{d}$ in $G$, and set
\[
\mathcal{L}_{\tilde{d}} := \mathcal{L} \otimes \bigotimes_{i=1}^n \mathcal{O}_{\tilde{e}_i}^{\tilde{d}_{\tilde{e}_i}}.
\]

Given a fixed line bundle, we have thus associated a twist of it to each vertex in $G$; the next step is to associate maps between these line bundles for each edge (and more generally each directed path) in $G$. Note that for $e \in E(\Gamma)$, if $\Delta_e = B$, with adjacent vertices $v, v'$, then $\mathcal{O}_{(e,v)} \otimes \mathcal{O}_{(e,v')} \cong \mathcal{O}_X$ by construction. We use this as follows.
Situation 4.3.5. In the situation of Notation 4.3.4, suppose also that $\Delta_e = B$ for all $e \in E(\Gamma)$. For each pair $(e, v)$ of an edge and adjacent vertex in $\Gamma$, observe that we have a canonical section

$$\mathcal{O}_X \rightarrow \mathcal{O}_{(e,v)};$$

if $v'$ is the other vertex adjacent to $e$, the section is determined as the 0 map on $Y_{(e,v')}$, and the canonical inclusion on $Y_{(e,v')}$ coming from twisting by the effective divisor $\Delta'_e$. Next, for each $e$, let $v, v'$ be the adjacent vertices, and fix an isomorphism

$$\mathcal{O}_{(e,v)} \otimes \mathcal{O}_{(e,v')} \cong \mathcal{O}_X.$$

Using the above sections, these isomorphisms then induce maps

$$\mathcal{O}_{(e,v)} \rightarrow \mathcal{O}_X$$

for each pair $(e, v)$ as above.

The desired maps are then defined as follows.

Notation 4.3.6. Given any $\tilde{e} \in E(G)$, from $\tilde{d}$ to $\tilde{d}'$, and $\mathcal{L}$ of multidegree $\bar{d}_0$, we define a map

$$f_{\tilde{e}} : \mathcal{L}_{\tilde{d}} \rightarrow \mathcal{L}_{\tilde{d}'}$$

using the maps fixed in Situation 4.3.5 as follows: if $v'$ is the vertex other than $v_{\tilde{e}}$ adjacent to $e_{\tilde{e}}$, we have either $\mathcal{L}_{\tilde{d}'} = \mathcal{L}_{\tilde{d}} \otimes \mathcal{O}_{(e_{\tilde{e}}, v_{\tilde{e}})}$ or $\mathcal{L}_{\tilde{d}} = \mathcal{L}_{\tilde{d}} \otimes \mathcal{O}_{(e_{\tilde{e}}, v')}$; in the first case, $f_{\tilde{e}}$ is induced by the section $\mathcal{O}_X \rightarrow \mathcal{O}_{(e_{\tilde{e}}, v_{\tilde{e}})}$, and in the second case $f_{\tilde{e}}$ is induced by the map $\mathcal{O}_{(e_{\tilde{e}}, v')} \rightarrow \mathcal{O}_X$.

Given any $\tilde{d}, \tilde{d}' \in V(G)$, let

$$f_{\tilde{d}, \tilde{d}'} : \mathcal{L}_{\tilde{d}} \rightarrow \mathcal{L}_{\tilde{d}'}$$

be obtained as $f_{\tilde{e}_n} \circ \cdots \circ f_{\tilde{e}_1}$, where $(\tilde{e}_1, \ldots, \tilde{e}_n)$ is the unique minimal directed path from $\tilde{d}$ to $\tilde{d}'$.

We can now define the limit linear series functor. As in the two-component case, the underlying idea is that we should be able to construct an $(r + 1)$-dimensional space of global sections of each $\mathcal{L}_{\tilde{d}}$ by gluing together sections chosen from the various $\mathcal{L}_{\tilde{d}'}$. Before defining the functor, we generalize Proposition 3.2.2 to the case of arbitrary curves of compact type, showing that the definition we will give is indeed putting a moduli functor structure on the set of limit linear series inside $A_{\pi, r, d}$.

Proposition 4.3.7. In the situation of Definition 4.3.1, the following are equivalent:

(a) The tuple $(\mathcal{L}^v, V^v)_{v \in V(\Gamma)}$ is a limit linear series.

(b) For all $(d_v)_{v \in V(\Gamma)} \in V(G)$, the kernel of the map

$$\Gamma(X_0, \mathcal{L}_{\tilde{d}}) \rightarrow \bigoplus_{v \in V(\Gamma)} \Gamma(Y_v, \mathcal{L}_{\tilde{d}}^v)/V^v$$

induced by the $f_{\tilde{d}, \tilde{d}'}$ and restriction to $Y_v$ has dimension at least $r + 1$.

(c) For all $(d_v)_{v \in V(\Gamma)} \in V(G)$, the kernel of the map

$$\bigoplus_{v \in V(\Gamma)} V^v(-(d - d_v)\Delta'_e) \rightarrow \bigoplus_{e \in E(\Gamma)} \mathcal{L}_{\tilde{d}'}|_{\Delta'_e}$$
has dimension at least \( r + 1 \), where \( e_v \) is defined as follows: let \( e_0 \in E(\Gamma) \), with adjacent vertices \( v_0, v'_0 \) be such that \( d_v = 0 \) for all \( v \neq v_0, v'_0 \). Then \( e_{v_0} = e_{v'_0} = e_0 \), and for all \( v \neq v_0, v'_0 \), we set \( e_v \) to be the unique edge adjacent to \( v \) such that \( e_0 \) and \( v \) lie on different connected components of \( \Gamma \setminus \{ e \} \).

**Definition 4.3.8.** In Situation 4.3.5, let \( G^r_d(X/B) \) be the subfunctor of the functor of points of \( A_{\pi,r,d} \) defined as follows. Given a \( T \)-valued point \((L^v, \mathcal{V}^v)_{v \in V(\Gamma)}\) of \( A_{\pi,r,d} \), let \( \mathcal{L} \) be the line bundle on \( X \) induced under the isomorphism of Corollary A.4.7 by \( L^v_0 \) on \( Y^v_0 \), and by \( \mathcal{L}^v(\mathcal{O}(-d \Delta_e)) \) for \( v \neq v_0 \), where \( e \in E(\Gamma) \) is the unique edge adjacent to \( v \) such that \( v \) and \( v_0 \) lie on separate connected components of \( \Gamma \setminus \{ e \} \). Then \((\mathcal{L}^v, \mathcal{V}^v)_{v \in V(\Gamma)}\) is a point of \( G^r_d(X/B) \) if, for all \( \vec{d} \in V(G) \), the map
\[
p_2^* \mathcal{L}^\vec{d} \to \bigoplus_v (p_2^* \mathcal{L}^v) / \mathcal{V}^v
\]
induced by the \( f_{\vec{d},\vec{d}'} \) and restriction to \( Y^v \) has \((r + 1)\)st vanishing locus equal to all of \( T \).

Here, the \( n \)th vanishing locus is a canonical (closed sub-) scheme structure on the subset on which the kernel has dimension at least \( n \); this locus is compatible with base change, so we get a functor, and since it is a closed subscheme, we obtain representability. Moreover, our space is independent of the choices made.

**Proposition 4.3.9.** The functor \( G^r_d(X/B) \) is represented by a closed subscheme \( G^r_d(X/B) \) of \( A_{\pi,r,d} \), which is in particular proper over \( B \). Moreover, this scheme is independent of choice of \( v_0 \) and of the isomorphisms in Situation 4.3.5, and is compatible with any base change \( B' \to B \) to a regular, connected scheme \( B' \).

Proposition 4.3.7 says that in the case \( B \) is a point, we are simply putting a scheme structure on the set of limit linear series. For general \( B \), noting that in Proposition 4.3.9 base change to a point is in particular allowable, we see that the fibers of \( G^r_d(X/B) \) are precisely moduli spaces of limit linear series on each fiber \( X_b \) of \( X \) over \( B \).

**Remark 4.3.10.** The description of limit linear series spaces given in Proposition 3.2.8 generalizes immediately to arbitrary curves of compact type, and gives a natural scheme structure on the set of limit linear series. This description is very useful for computing spaces of limit linear series, but it is less useful from a technical standpoint: even without considering families of curves, it does not yield a moduli functor description, because it is typically not possible to describe the functor of points of a union in terms of the original functors of points.

### 4.4. Moduli of limit linear series in smoothing families

We now move on to the case of more general smoothing families, constructing a moduli space whose fibers over points corresponding to nonsingular curves are moduli spaces for usual linear series, and whose fibers over points corresponding to reducible curves are moduli spaces for limit linear series. The only additional observation needed for this construction is that in extremal multidegrees, we can work equally well with line bundles on the entire curve as with their restrictions to individual components, and the former fits more naturally into families of curves.
Our next task is to generalize these definitions to families in which some nodes may be smoothed.

Because the irreducible components \( Y_v \) we used to define our ambient space \( A_{\pi,r,d} \) are not defined over all of \( B \) when some nodes are smoothed, the first step is to replace \( G_{\pi,r,d} \) by an analogous space which does not require restricting to components. The points of \( A_{\pi,r,d} \) correspond to a line bundle \( \mathcal{L}^v \) of degree \( d \) for each \( v \in V(\Gamma) \), and a subbundle \( \mathcal{V}^v \) of \( \mathcal{L}^v \) for each \( v \in V(\Gamma) \). As in Definition 4.3.8, we can use the isomorphism of Corollary A.4.7 to replace the tuple of line bundles with a single line bundle \( \mathcal{L} \) on all of \( X \), of multidegree \( \vec{d}_0 \). We then define the \( \mathcal{V}_v \) to be subbundles of the twists \( \mathcal{L}_{\vec{d}_0}^v \). Thus, we make the definition as follows.

**Definition 4.4.1.** Given an almost local smoothing family \( \pi : X \to B \), let \( \Gamma \) be as in Proposition 4.2.5, and fix a choice of \( v_0 \in \Gamma \). Given also \( r \leq d \in \mathbb{Z}_{\geq 0} \), let \( \mathcal{A}^r_{\pi,r,d} \) be the functor which associates to \( T \) the set of equivalence classes of tuples \( (\mathcal{L}^v, (\mathcal{V}_v)_{v \in V(\Gamma)}) \) where \( \mathcal{L} \) is a line bundle of multidegree \( \vec{d}_0 \) on \( X \times_B T \), and for each \( v \in V(\Gamma) \), we have \( \mathcal{V}_v \subseteq \mathcal{L}_{\vec{d}_0}^v \), a subbundle of rank \( r + 1 \). The equivalence relation is as in Definition 4.1.1.

The main additional annoyance in the case that some nodes are smoothed is that we have only have \( \mathcal{O}_{(e,v)} \otimes \mathcal{O}_{(e,v')} \) isomorphic to \( \mathcal{O}_X \) locally on \( B \). Indeed, by Theorem 4.2.7, we have \( \Delta_\pi \) is a divisor on \( B \) when it is not all of \( B \), and in this case

\[
\mathcal{O}_{(e,v)} \otimes \mathcal{O}_{(e,v')} = \mathcal{O}_X(\mathcal{Y}_{(e,v)}) \otimes \mathcal{O}_X(\mathcal{Y}_{(e,v')}) = \mathcal{O}_X(\pi^{-1}\Delta_e) = \pi^*(\mathcal{O}_B(\Delta_e)),
\]

so if \( \mathcal{O}_B(\Delta_e) \) is trivialized on an open subset \( U \) of \( B \), then \( \mathcal{O}_{(e,v)} \otimes \mathcal{O}_{(e,v')} \) is likewise trivial on the preimage of \( U \).

We could simply impose that \( \mathcal{O}_{(e,v)} \otimes \mathcal{O}_{(e,v')} \cong \mathcal{O}_X \) in the definition of a smoothing family, but since this isn’t in fact necessary to define the functor, we will not do so. Instead, we will fix choices as follows.

**Situation 4.4.2.** In the situation of Notation 4.3.4, for each pair \( (e, v) \) of an edge and adjacent vertex in \( \Gamma \), we have a canonical section

\[
\mathcal{O}_X \to \mathcal{O}_{(e,v)};
\]

in the case that \( \Delta_e = B \), this is described in Situation 4.3.5, while in the case \( \Delta_e \neq B \), this is just the canonical inclusion, since \( \mathcal{O}_{(e,v)} = \mathcal{O}_X(\mathcal{Y}_{(e,v)}) \). Next, fix an open cover \( \{U_i\}_{i \in I} \) of \( B \) such that for each \( i \in I \) and \( e \in E(\Gamma) \), we have

\[
(\mathcal{O}_{(e,v)} \otimes \mathcal{O}_{(e,v')})|_{\pi^{-1}(U_i)} \cong \mathcal{O}_{\pi^{-1}(U_i)}
\]

where \( v, v' \) are the vertices adjacent to \( e \); fix a choice of such an isomorphism for each \( i \) and \( e \). Using the above sections, these isomorphisms then induce maps

\[
\mathcal{O}_{(e,v)}|_{\pi^{-1}(U_i)} \to \mathcal{O}_{\pi^{-1}(U_i)}
\]

for each pair \( (e, v) \) as above.

**Notation 4.4.3.** Given any \( \bar{e} \in E(G) \), from \( \bar{d} \) to \( \bar{d} \), for each \( i \in I \) we define a map

\[
f_{\bar{e},i} : \mathcal{L}_{\bar{d}}|_{\pi^{-1}(U_i)} \to \mathcal{L}_{\bar{d}}|_{\pi^{-1}(U_i)}
\]

using the maps fixed in Situation 4.4.2 just as in Notation 4.3.6.
Given any $\vec{d}, \vec{d}' \in V(G)$ and $i \in I$, let
\[ f_{\vec{d}, \vec{d}', i} : \mathcal{L}_{\vec{d}|\pi^{-1}(U_i)} \to \mathcal{L}_{\vec{d}'|\pi^{-1}(U_i)} \]
be obtained as $f_{\vec{e}_i, i} \circ \cdots \circ f_{\vec{e}_1, i}$, where $(\vec{e}_1, \ldots, \vec{e}_n)$ is the unique minimal directed path from $\vec{d}$ to $\vec{d}'$.

**Definition 4.4.4.** In Situation 4.4.2, let $\mathcal{G}_d(X/B)$ be the subfunctor of $\mathcal{A}_{\pi, r, d}$ defined as follows. Given $f : T \to B$, an element $(\mathcal{L}_v, (\mathcal{L}_v^i)_{v \in V(T)})$ in $\mathcal{A}_{\pi, r, d}(T)$ lies in $\mathcal{G}_d(X/B)(T)$ if, for all $\vec{d} \in V(G)$, and all $i \in I$ the map
\[ p_{2*}\mathcal{L}_{\vec{d}|(f_{op2})^{-1}(U_i)} \to \bigoplus_v (p_{2*}\mathcal{L}_{\vec{d}|(f_{op2})^{-1}(U_i)})/\mathcal{L}_v|f^{-1}(U_i) \]
induced by the $f_{\vec{d}_n, \vec{d}', i}$ has $(r + 1)$st vanishing locus equal to all of $T$.

The first point to check is that although we have given two different definitions of limit series functor, they agree with one another.

**Proposition 4.4.5.** In Situation 4.3.5, fix also the open cover $\{U_i\}$ and isomorphisms as in Situation 4.4.2. Then there is a natural isomorphism between the functors $\mathcal{G}_d(X/B)$ of Definition 4.3.8 and of Definition 4.4.4, induced by restriction to the components $Y_v$ of $X$.

We next investigate the basic properties of Definition 4.4.4.

**Proposition 4.4.6.** In Situation 4.4.2, the functor $\mathcal{A}'_{\pi, r, d}$ is represented by a scheme $A'_{\pi, r, d}$ which is proper over $\text{Pic}^{d_0}(X/B)$, and hence over $B$. Additionally, $\mathcal{G}_d(X/B)$ is represented by a closed subscheme $G'_d(X/B)$ of $A'_{\pi, r, d}$, which is therefore also proper over $B$. Moreover, both are independent of the choice of $\nu_0 \in V(\Gamma)$, of the cover $\{U_i\}$, and the isomorphisms of Situation 4.4.2. Finally, formation of $\mathcal{G}_d(X/B)$ is compatible with any base change $B' \to B$ which preserves the conditions of an almost local smoothing family.

As before, base change to any point preserves the almost-local smoothing conditions, but the situation is subtler than for Proposition 4.3.9, because now the base change may change the graph $\Gamma$. Thus, compatibility with base change requires more argument, and likewise yields more content: Proposition 4.4.6 in particular describes the fibers of $\mathcal{G}_d(X/B)$ over $b \in B$ as limit linear series spaces when $X_b$ is reducible, and as spaces of linear series when $X_b$ is nonsingular.

**Proof.** One can prove directly that $\mathcal{A}'_{\pi, r, d}$ is representable, following the arguments for representability of moduli spaces of linear series as in Theorem 4.1.3. Alternatively, we could observe that the proof of Theorem 4.1.3 goes through verbatim for smoothing families if one replaces $d$ with a multidegree $\vec{d}$, and then $A_{\pi, r, d}$ is simply a fibered product of various spaces $G'_{d_0}(X/B)$, fibered over $\text{Pic}^{d_0}(X/B)$ via the maps induced by forgetting the subbundles and twisting by suitable combinations of the $\mathcal{O}_{(e, v)}$ to go from each degree $\vec{d}_n$ to $\vec{d}_0$.

That $\mathcal{G}_d(X/B)$ is represented by a closed subscheme of $A'_{\pi, r, d}$ then immediate from the properties of the $(r + 1)$st vanishing locus, just as in Proposition 4.3.9. □

The next step is to discuss imposed ramification; this is handled largely as suggested by Definition 3.2.6.
4. FAMILIES AND FORMAL CONSTRUCTIONS

\textbf{Definition 4.4.7.} In Situation 4.4.2, suppose we are given also \(P_1, \ldots, P_n\) sections of \(\pi\) with image contained in the smooth locus of \(\pi\), and Schubert indices \(\alpha^1, \ldots, \alpha^n\) of type \((r, d)\). We define the subfunctor

\[ G^r_d(X/B, \{(P_i, \alpha^i)\}) \subseteq G^r_d(X/B) \]

as follows. Fix \(b_0 \in B\) such that \(X_{b_0}\) has dual graph \(\Gamma\), and for \(i = 1, \ldots, n\), let \(v_i \in V(\Gamma)\) be the vertex corresponding to the component of \(X_{b_0}\) containing \(P_i\). Then \(G^r_d(X/B, \{(P_i, \alpha^i)_i\})\) consists of tuples \((\mathcal{L}, \{\mathcal{V}_v\}_{v \in V(\Gamma)})\) such that, for \(i = 1, \ldots, n\), and \(j = 0, \ldots, r\), we have

\[ \text{rk}(\mathcal{V}_{v_i} \to p_{2*}(\mathcal{L}_{b_i/v_i} |_{P_i})) \leq r + 1 - b_j^i \]

where \(b_j^i\) is as in Definition 4.1.2.

Just as in the proof of Theorem 4.1.3, the conditions for imposed ramification are simply determinantal conditions, so we therefore conclude:

\textbf{Proposition 4.4.8.} In the situation of Definition 4.4.7, the functor \(G^r_d(X/B, \{(P_i, \alpha^i)_i\})\) is represented by a closed subscheme \(G^r_d(X/B, \{(P_i, \alpha^i)_i\})\) of \(G^r_d(X/B)\), which is in particular proper over \(B\).

This is already enough to prove one of our two foundational results on limit linear series, stated above – in a special case and without proof – as Proposition 3.3.2. Indeed, the following specialization result is an immediate consequence of the properness assertion of Proposition 4.4.8, together with semicontinuity of fiber dimension (Theorem B.1.9).

\textbf{Corollary 4.4.9.} In Situation 4.4.2, given \(r, d\) and Schubert indices \(\alpha^1, \ldots, \alpha^n\) of type \((r, d)\), the locus of \(b \in B\) such that the spaces \(G^r_d(X_{b_0}/b_0, \{(P_i)_{b_0}, \alpha_i\})\) are nonempty of dimension at least \(m\) is closed in \(B\). In particular, given \(b_0 \in B\), if the space \(G^r_d(X_{b_0}/b_0, \{(P_i)_{b_0}, \alpha_i\})\) is empty or has dimension less than or equal to \(m\), the same holds for all \(b \in B\) in an open neighborhood of \(b_0\).

Note that it is possible in general to have \(G^r_d(X_{b_0}/b_0, \{(P_i)_{b_0}, \alpha_i\})\) nonempty, but all the other fibers empty. However, we will use a dimension argument to show that this is not possible in an important special case. The following theorem is the key technical tool in the argument.

\textbf{Theorem 4.4.10.} In the situation of Definition 4.4.7, any open subset of the space \(G^r_d(X/B, \{(P_i, \alpha^i)_i\})\) has universal relative dimension over \(B\) at least the \(\rho\) of Theorem 4.1.3.

This lets us prove the second foundational result (see Theorem 3.3.3) on smoothing of limit linear series to linear series on nearby fibers.

\textbf{Corollary 4.4.11.} In the situation of Definition 4.4.7, suppose that for some \(b_0 \in B\), the space \(G^r_d(X_{b_0}/b_0, \{(P_i)_{b_0}, \alpha_i\})\) is nonempty of the expected dimension \(\rho\). Then the same holds for all \(b \in B\) in some open neighborhood \(V\) of \(b_0\), and \(G^r_d(X/V, \{(P_i)_{b_0}, \alpha_i\})\) is universally open on the preimage of \(V\).

Almost the same statement holds if we replace \(G^r_d(X/B, \{(P_i, \alpha^i)_i\})\) with any open subset \(U\); in this case, we find that for some possibly smaller open subset \(U'\) containing \(U \cap G^r_d(X_{b_0}/b_0, \{(P_i)_{b_0}, \alpha_i\})\), we have that \(U'\) is universally open over an open neighborhood \(V\) of \(b_0\), and all fibers of \(U'\) over \(V\) are nonempty of dimension \(\rho\).
Proof. In the more general situation, if $U \cap G_d^r(X_{b_0}/b_0, \{(P_i)_{b_0}, \alpha^i\})$ has dimension $\rho$, then Theorem 4.4.10 together with Proposition B.2.7 imply the existence of a $U'$ containing $U \cap G_d^r(X_{b_0}/b_0, \{(P_i)_{b_0}, \alpha^i\})$ on which all fibers have dimension $\rho$ (when nonempty), and further imply that any such $U'$ is universally open over $B$. Taking $V$ to be the image of $U'$ then gives the second statement of the corollary.

In the case $U$ is all of $G_d^r(X/B, \{(P_i, \alpha^i)\})$, by properness and semicontinuity of fiber dimension (Theorem B.1.9), if we set $V$ to be the complement of the locus with fiber dimension greater than $\rho$, we can set $U'$ to be the preimage of $V$, and obtain the first statement as well. □

Obviously, if $\rho < 0$, the statement of Corollary 4.4.11 never applies. However, we still want some smoothing results in this setting, for instance to prove existence of Weierstrass points, so we also state the following weaker, but more general variant of Corollary 4.4.11.

Corollary 4.4.12. In the situation of Definition 4.4.7, suppose that for some irreducible closed subset $B_0 \subseteq B$, every irreducible component $Z_0$ of $G_d^r(X/B, \{(P_i, \alpha^i)\})_{|B_0}$ satisfies

$$\dim g^{-1}(\eta_0) - \text{codim}_{B_0} g(Z_0) = \rho,$$

where $g : G_d^r(X/B, \{(P_i, \alpha^i)\}) \to B$ is the structure map, and $\eta_0$ is the generic point of $Z_0$. Then every irreducible component $Z$ of $G_d^r(X/B, \{(P_i, \alpha^i)\})$ likewise satisfies

$$\dim g^{-1}(\eta) - \text{codim}_B g(Z) = \rho,$$

where $\eta$ is the generic point of $Z$. In particular, if $B_0 \neq B$, then $Z$ cannot be supported over $B_0$.

The same is true if we replace $G_d^r(X/B, \{(P_i, \alpha^i)\})$ with any open subset.

This is immediate from Theorem 4.4.10 and Proposition B.2.8.

In the case that $B$ is of finite type over a field, the identities in Corollary 4.4.12 are equivalent to $\dim Z_0 - \dim B_0 = \rho$ and $\dim Z - \dim B = \rho$, respectively.

4.5. Weierstrass points

Using Corollary 4.4.12, we are now ready to explain the proof of the theorem of Eisenbud and Harris on existence of Weierstrass points. Recall (see Exercise 2.3.7) that we defined Weierstrass points in terms of the semigroup $H_P$ of orders of poles at $P$ of rational functions which are regular away from $P$, and we verified that Weierstrass points are the same as ramification points of the canonical linear series. We also saw that the condition that $H_P$ is a semigroup places substantial constraints on what sort of Weierstrass points can occur, in the sense of what possible vanishing sequences can occur for the canonical linear series. In 1893, Hurwitz asked if this is the only restriction. This remained open for almost 100 years, but the following example due to Buchweitz shows that the semigroup condition is not enough to characterize possible Weierstrass points.

Exercise 4.5.1. Show that for any curve of genus 16, the hypothetical ramification sequence

$$0, \ldots, 0, 6, 7, 9, 9$$

12 times
yields via the correspondence of Exercise 2.3.7 (b) an $H_P$ satisfying the semigroup condition, but that it cannot actually occur as the ramification sequence of the canonical linear series at any point on the curve. Hint: consider orders of vanishing of tensor products of pairs of differentials, and show that one would necessarily obtain more orders of vanishing than the dimension of $\Gamma(C, (\Omega^1_C)^{\otimes 2})$.

The question of what Weierstrass points can actually occur remains open and is the subject of current research. However, in 1987 Eisenbud and Harris were able to make substantial progress on the problem as follows:

**Theorem 4.5.2.** Let $\alpha := \alpha_0, \ldots, \alpha_{g-1}$ be a Schubert index of type $(g-1, 2g-2)$, and suppose that $\sum_i \alpha_i \leq g/2$. Then there exists a projective nonsingular curve $C$ of genus $g$ and a point $P \in C$ such that the ramification sequence of the canonical linear series at $P$ is equal to $\alpha$.

The proof is via limit linear series techniques, and we now describe it. As with the proof of the Brill-Noether theorem, a substantial part of the proof involves a careful analysis of linear series on nonsingular curves with prescribed ramification, and the limit linear series theory sets up an induction argument. However, the situation is rather different from what we have considered before, because every nonsingular curve has a unique $g_{g-1}$. Thus, for a smooth family $X$ over $B$, we have $G_{g-1}(X/B)$ isomorphic to $B$, and any imposed ramification then cuts out a closed subscheme, and makes $\rho$ negative. Thus, the general strategy of the proof is to find families of curves with a marked point in which the desired ramification of the canonical linear series occurs in exactly the expected codimension, which is $-\rho$. In this case, we can use limit linear series techniques, and in particular Corollary 4.4.12, to inductively construct more such families, producing Weierstrass points with more and more ramification.

We begin with the following rather trivial observation:

**Proposition 4.5.3.** Let $C$ be a nonsingular projective curve of genus 2. Then for any point $P \in C$, if $\alpha_0, \alpha_1$ denotes the ramification sequence of the canonical linear series at $P$, then either $\alpha_0 = \alpha_1 = 0$ (i.e., $P$ is not a Weierstrass point), or $\alpha_0 = 0$ and $\alpha_1 = 1$.

**Proof.** Since $\alpha_i \leq d - r - 1$, it is enough to rule out the case $\alpha_0 = 1$, which is Example 2.2.2. $\square$

The statement we will need for the induction step is a variant of the study we have already carried out for curves of genus 1 with ramification imposed at two points.

**Proposition 4.5.4.** Let $C$ be a projective nonsingular curve of genus 1, and $P_1, P_2 \in C$ distinct points. Given $r, d$, fix $\alpha^1, \alpha^2$ Schubert indices of type $(r, d)$ satisfying

\[ d - r - 1 \leq \alpha_j^1 + \alpha_{r-j}^2 \leq d - r \text{ for } j = 0, \ldots, r, \]

and suppose that equality is achieved on the right for at least one value of $j$. Then $G_{r}(C, (P_1, \alpha^1), (P_2, \alpha^2))$ consists of at most a single point. It is nonempty with its unique point corresponding to a $g_d^r$ on $C$ having ramification sequences exactly $\alpha^i$ at $P_i$ for $i = 1, 2$ if and only if the following two conditions are satisfied:
(I) for all \( i < j \) with
\[
a_i + a_{r-i} = a_j + a_{r-j} = d,
\]
we have \( P_1 - P_2 \) an \( (a_j - a_i) \)-torsion point;

(II) for all \( i \neq j \) with
\[
(a_i + 1)P_1 + a_{r-i}P_2 \sim a_jP_1 + a_{r-j}P_2,
\]
we have \( i < r \) and \( a_{i+1} = a_i + 1 \), and similarly for all \( i \neq j \) with
\[
a_i^2P_1 + (a_{r-i} + 1)P_2 \sim a_jP_1 + a_{r-j}P_2,
\]
we have \( i > 0 \) and \( a_{r-i+1} = a_{r-i} + 1 \).

Here, as usual we write \( a_j^i := a_j + j \).

Note that in condition (II), we necessarily have \( a_i^1 + a_{r-i}^1 = d - r - 1 \) and \( a_j^1 + a_{r-j}^1 = d - r \).

PROOF. The first observation is as follows: let \( \mathcal{L} \) be a line bundle of degree \( d \). Then for each \( i \), a nonzero section \( s_i \in \Gamma(C, \mathcal{L}) \) vanishing to order at least \( a_i^1 \) at \( P_1 \) and \( a_{r-i}^2 \) at \( P_2 \) is unique up to scalar, if it exists, since \( \deg \mathcal{L}(-a_i^1P_1 - a_{r-i}^2P_2) \leq 1 \). Moreover, if it exists, we must have that \( s_i \) vanishes to order \( a_i^1 + \epsilon_i^1 \) at \( P_1 \) and \( a_{r-i}^2 + \epsilon_i^2 \) at \( P_2 \), where each \( \epsilon_i^1 \) is 0 or 1 and \( \epsilon_i^2 \leq 1 \).

Now, as in the argument for Proposition 2.8.2, if \( \mathcal{L}, V \) is a \( g^d_a \) having at least the required vanishing, then for every \( i \) we must have a section \( s_i \in V \) vanishing to order at least \( a_i^1 \) at \( P_1 \) and \( a_{r-i}^2 \) at \( P_2 \). This immediately yields the necessity of (I), since then we must have
\[
\mathcal{O}_C(a_i^1P_1 + a_{r-i}^2P_2) \cong \mathcal{L} \cong \mathcal{O}_C(a_j^1P_1 + a_{r-j}^2P_2).
\]
In particular, the underlying line bundle \( \mathcal{L} \) of the desired \( g^d_a \) is determined by any pair of \( a_i^1, a_{r-i}^2 \) adding up to \( d \). Now suppose \( (\mathcal{L}, V) \) has exactly the required vanishing. Then, if \( \epsilon_i^1 = 1 \), we see that \( a_i^1 + 1 \) occurs in the vanishing sequence at \( P_1 \), so must be equal to \( a_{i+1}^1 \). On the other hand, we have \( \epsilon_i^1 = 1 \) if and only if \( \mathcal{L} \cong \mathcal{O}((a_i^1 + 1)P_1 + a_{r-i}^2P_2) \), so we conclude the necessity of the first part of (II). The second part follows similarly.

We now wish to show that if (I) and (II) are satisfied, a unique \( g^d_a \) with exactly the required vanishing exists. Certainly, the underlying line bundle \( \mathcal{L} \) is uniquely determined as \( \mathcal{O}_C(a_i^1P_1 + a_{r-i}^2P_2) \) for any \( i \) with \( a_i^1 + a_{r-i}^2 = d \). We will build up the space \( V \) piece by piece. Condition (I) implies that for each \( i \), there exists a section \( s_i \) of \( \Gamma(C, \mathcal{L}) \) vanishing to order at least \( a_i^1P_1 \) and \( a_{r-i}^2P_2 \), which we then know to be unique up to scalar. There are three cases to consider. If \( \epsilon_i^1 = \epsilon_i^2 = 0 \), then we set \( V_i \) to be the span of \( s_i \). If \( \epsilon_i^1 = 1 \) (so that necessarily \( \epsilon_i^2 = 0 \)), then we must have \( a_i^1 + a_{r-i}^2 = d - 1 \), and according to condition (II) we have \( a_{i+1}^1 = a_i^1 + 1 \). It then follows that \( a_{r-i-1}^2 = a_{r-i}^2 - 1 \). We can define \( V_i \) to be \( \Gamma(C, \mathcal{L})(-a_i^1P_1 - a_{r-i}^2P_2) \); since in this case \( \mathcal{L}(-a_i^1P_1 - a_{r-i}^2P_2) \) has degree 2, we see that \( V_1 \) has no base points, so a general section vanishes to order exactly \( a_i^1 \) at \( P_1 \) and \( a_{r-i}^2 \) at \( P_2 \). We observe that we have \( s_i = s_{i+1} \), which then vanishes to order \( a_{i+1}^1 \) at \( P_1 \) and \( a_{r-i}^2 \) at \( P_2 \). Finally, if \( \epsilon_i^2 = 1 \), we similarly define \( V_i \) to be \( \Gamma(C, \mathcal{L})(-a_i^1P_1 - a_{r-i}^2P_2) \), which is again 2-dimensional. From the above, we see that \( \epsilon_i^1 = 1 \) if and only if \( \epsilon_{i+1}^2 = 1 \), so the 2-dimensional spaces \( V_i \) are always paired with \( V_i = V_{i+1} \). Thus, if we set \( V = \sum_i V_i \), we find that \( \dim V \leq r + 1 \). By
observed that a sequence must contain a section vanishing to order at least $P$ in the 2-dimensional spaces $V$. Using the dimension-counting argument that showed that every $V$ is one-dimensional, we have also already seen this. On the other hand, the same dimension-counting argument that showed that every $V$ with the given vanishing sequence must contain a section vanishing to order at least $a_1^2$ at $P_1$ and $a_{r-1}^2$ at $P_2$ also shows that $V$ must contain at least a 2-dimensional space vanishing to order at least $a_1^2$ at $P_1$ and $a_{r-1}^2$ at $P_2$. But this immediately implies that $V$ must contain the 2-dimensional spaces $V_i$ as well, proving uniqueness. $\square$

We now rephrase Proposition 4.5.4 in the particular case we will use it.

**Corollary 4.5.5.** Let $C$ be a projective nonsingular curve of genus 1, and $P_1, P_2 \in C$ distinct points. Given $g \geq 2$, fix $\alpha^2$ a Schubert index of type $(g-1,2g-2)$, with $\alpha_{g-1}^2 = g-1$ maximal. Then:

1. If we set
   \[
   \alpha_1^1 = \begin{cases} 
   0 : & i = 0 \\
   g - 2 - \alpha_{g-1-j}^2 : & i = 1, \ldots, g-1,
   \end{cases}
   \]
   there exists a unique $g_{2g-2}$ on $C$ with ramification sequence exactly $\alpha_1^1$ at $P_1$ and $\alpha^2$ at $P_2$ whenever $P_1 - P_2$ is not $(\alpha_1^1 + i + 1)$-torsion or $(\alpha_1^1 + i)$-torsion for any $i > 0$.

2. If we fix a $j \in \{1, \ldots, g-1\}$ with either $j = g - 1$ or $\alpha_{g-1-j}^2 \neq g-2-j$, and set
   \[
   \alpha_1^1 = \begin{cases} 
   g - 1 - \alpha_{g-1-j}^2 : & i = 0, j \\
   g - 2 - \alpha_{g-1-j}^2 : & i \neq 0, j.
   \end{cases}
   \]
   there is at most one $g_{2g-2}$ on $C$ with ramification sequence exactly $\alpha_1^1$ at $P_1$ and $\alpha^2$ at $P_2$, and such a $g_{2g-2}$ exists if and only if the set of integer values $m$ for which $P_1 - P_2$ is $m$-torsion contains $\alpha_1^1 + j$, but does not contain $\alpha_1^1 + i + 1$ for any $i \neq 0, j$ with either $i = g - 1$ or $\alpha_{i+1}^1 > \alpha_1^1$, and does not contain $\alpha_1^1 + i$ for any $i \neq 0, j$ with $\alpha_{g-1-j}^2 > \alpha_1^1 + i$.

In particular, in (i) a $g_{2g-2}$ exists for a general choice of $P_2$ for any fixed choice of $P_1$, and in (ii) a $g_{2g-2}$ does not exist for general choice of $P_2$ for any fixed choice of $P_1$, but does exist if $2(\alpha_1^1 + j) > \alpha_{g-1-j}^2 + g$ and we choose $P_1, P_2$ so that $P_1 - P_2$ has order precisely $\alpha_1^1 + j$.

We will need a generalization of Winters’ theorem (Theorem 3.3.4) to higher-dimensional families of curves. Rather than try to state the most general possible theorem, we will state what we actually need for the argument:

**Theorem 4.5.6.** Let $B_0$ be regular and affine, and let $\pi_0 : X_0 \to B_0$ be a flat projective family of nodal curves, with each component of the nonsmooth locus of $\pi_0$ mapping isomorphically to $B_0$, and let $P_0$ be a section of $\pi_0$ with image contained in the smooth locus of $\pi_0$. Then there exists a regular scheme $B$ containing $B_0$
as a divisor, and a morphism $\pi : X \to B$ with section $P$ satisfying the following conditions:

(i) $\pi$ is smooth away from the preimage of $B_0$;
(ii) The restriction of $\pi$ to $B_0$ is isomorphic to $\pi_0$, and the restriction of $P$ recovers $P_0$;
(iii) $X$ is regular.

Moreover, if $B_0$ is of finite type over a field $k$, we can also choose $B$ to be of finite type over $k$.

Now we return to the proof of Theorem 4.5.2. In fact, Eisenbud and Harris prove a stronger version of the theorem, which requires an additional definition.

**Definition 4.5.7.** A Schubert index $\alpha$ of type $(g-1,2g-2)$ is **primitive** if it satisfies the following property: if $i_0$ is minimal with $\alpha_{i_0} \neq 0$, then $2(i_0 + 1) > \alpha_{g-1} + g$.

By convention, we also say $\alpha$ is primitive if $\alpha_i = 0$ for all $i$.

What Eisenbud and Harris proved was the following:

**Theorem 4.5.8.** Let $\alpha := \alpha_0, \ldots, \alpha_{g-1}$ be a primitive Schubert index of type $(g-1,2g-2)$, and suppose that $\sum_{i=0}^{g} \alpha_i < g-1$, or $\alpha_{g-1} = g-1$ with $\alpha_i = 0$ for $i < g-1$. Then there exists a projective nonsingular curve $C$ of genus $g$ and a point $P \in C$ such that the ramification sequence of the canonical linear series at $P$ is equal to $\alpha$.

This implies Theorem 4.5.2:

**Proposition 4.5.9.** Let $\alpha$ be a Schubert index of type $(g-1,2g-2)$, and suppose $\sum_{i=0}^{g-1} \alpha_i \leq g/2$. Then $\alpha$ is primitive.

**Proof.** We claim that if $\alpha_j > 0$ for some $j$, then

$$\sum_{i=0}^{g-1} \alpha_i \geq \alpha_{g-1} + g - j - 1.$$  

Indeed, if $j = g - 1$ this is immediate, while if $j < g - 1$ we have

$$\sum_{i=0}^{g-1} \alpha_i \geq \sum_{i=j}^{g-2} \alpha_i + \alpha_{g-1} \geq (g - 1 - j)\alpha_j + \alpha_{g-1},$$

so the desired inequality follows.

Now, the statement is true by convention if all $\alpha_i$ are 0, or equivalently, if $\alpha_{g-1} = 0$, so assume $\alpha_{g-1} > 0$. Then if $i_0$ is minimal with $\alpha_{i_0} > 0$ and $\sum_{i=0}^{g-1} \alpha_i \leq g/2$, we have $\alpha_{g-1} + g - i_0 - 1 \leq g/2$, so

$$2(i_0 + 1) \geq 2\alpha_{g-1} + g > \alpha_{g-1} + g,$$

as desired.

**Example 4.5.10.** The primitivity condition on $\alpha$ is actually substantially more general than the condition that $\sum_i \alpha_i \leq g/2$. For instance, if $\alpha_0 = \cdots = \alpha_{g-2} = 0$, then we see that any value of $\alpha_{g-1}$ up to its maximum possible value of $g - 1$ still gives primitivity. Theorem 4.5.8 handles all of these cases as well.
It is easy to see that a primitive Schubert index satisfies the semigroup condition. In fact, we can refine this relationship to give a more intuitive definition of primitivity as follows.

**Exercise 4.5.11.** Show that a Schubert index \( \alpha \) of type \((g - 1, 2g - 2)\) is primitive if and only if for every Schubert index \( \alpha' \) of type \((g - 1, 2g - 2)\) with \( \alpha'_j \leq \alpha_i \) for all \( i \), we have that \( \alpha' \) satisfies the semigroup condition.

We are now ready to prove Theorem 4.5.8.

**Proof.** The proof is by induction on \( g \), although in fact we have to induct on a slightly stronger statement: namely, we will show that for each \( \alpha \) as in the theorem statement, there is a smooth projective family \( \pi: X \rightarrow B \) of curves of genus \( g \) together with a section \( P_{\alpha}: B \rightarrow X \) with the property that if \( Z_\alpha \) denotes the locus of \( b \in B \) such that the fiber \((X)_b\) has a Weierstrass point at \((P_{\alpha})_b\) with ramification exactly \( \alpha \), then \( \text{codim}_{B_1} Z_\alpha = \sum_{i=0}^r \alpha_i \).

The base case is \( g = 2 \), where we necessarily have \( \alpha = 0,1 \). We can let \( C \) be any smooth projective curve of genus 2 over \( k \), and then let \( B_2 = C, X_2 = C \times_k C \), with \( \pi_2 \) given by the first projection, and the section \( P_2 \) given by the diagonal morphism. By Proposition 4.5.3, the locus \( Z_g \) consists precisely of the Weierstrass points of \( C \), which has codimension 1, as desired.

We now consider the induction step, supposing \( g \geq 3 \). Fix a genus-1 curve \( C \) over \( k \), and let \( B_1 \) be the complement of the diagonal in \( C \times_k C \). Let \( X_1 = B_1 \times_k C \), and let \( Q_1, Q_2: B_1 \rightarrow X_1 \) be the two disjoint sections coming from the inclusion of \( B_1 \) into \( C \times_k C \). Now, suppose we are given \( \alpha \) as in the theorem statement. Define \( \alpha' \) of type \((g - 2, 2g - 4)\) as follows: if \( \alpha_i = 0 \) for all \( i \), set \( \alpha'_i = 0 \) for all \( i \), too. Otherwise, if \( i \) is minimal with \( \alpha_i > 0 \), note that the primitivity condition implies that \( i > 0 \), and set

\[
\alpha'_i = \begin{cases} 
\alpha_{i+1} : j \neq i - 1 \\
\alpha_{i+1} - 1 : \text{otherwise.}
\end{cases}
\]

Then \( \alpha' \) clearly satisfies the condition that \( \sum_j \alpha'_j \leq g - 2 \), with equality only if \( \alpha'_j = 0 \) for all \( j < g - 2 \). We claim that \( \alpha' \) is also primitive. This is clear if \( \alpha_j = 0 \) for all \( j \), or if \( i = g - 1 \). Otherwise, primitivity of \( \alpha \) says precisely that \( 2i + 2 > \alpha_{g-1} + g \); letting \( i' \) be minimal with \( \alpha'_{i'} > 0 \), we want to show that 

\[
2i' + 2 > \alpha'_{i'} + g - 1 = \alpha_{g-1} + g - 1.
\]

If \( \alpha_i = 1 \), then \( i' \geq i \), so the desired statement follows immediately in this case. Finally, if \( \alpha_i \geq 2 \), then \( i' = i - 1 \). In this case, we calculate

\[
g - 1 > \sum_{j=i}^{g-2} \alpha_j = \sum_{j=i}^{g-2} \alpha_j + \alpha_{g-1} \geq 2(g - 1 - i) + \alpha_{g-1},
\]

from which we conclude \( 2i > \alpha_{g-1} + g - 1 \), as desired.

Thus, \( \alpha' \) satisfies the hypotheses of the theorem for \( g - 1 \), and applying the induction hypothesis, we have a family \( \pi_{g-1}: X_{g-1} \rightarrow B_{g-1} \) with a section \( P_{g-1} \) such that the locus on which \( P_{g-1} \) is a Weierstrass point of \( X_{g-1} \) with ramification sequence \( \alpha'_j \) has the expected codimension \( \sum_j \alpha'_j \). Let \( B_0 = B_1 \times_k B_{g-1} \), and \( X_0 \) the curve obtained from gluing the section \( Q_2 \) of \( X_1 \) to the section \( P_{g-1} \) of \( X_{g-1} \). Applying Theorem 4.5.6, we obtain a smoothing family \( \pi: X \rightarrow B \) with a section \( P \) extending the given family \( \pi_0 \) and section \( Q_1 \). Consider \( G'_d(X/B, (P, \alpha)) \). Observe that we obtain an open subset of \( G'_d(X/B, (P, \alpha)) \) by restricting to the limit linear series which have ramification sequence exactly \( \alpha \) at \( P \). Furthermore, if we set
\[ \beta = 0, \alpha'_0 + 1, \ldots, \alpha'_{g-2} + 1, \text{ and } \gamma = g - 1 - \beta_{g-1}, g - 1 - \beta_{g-2}, \ldots, g - 1 - \beta_0, \]

we can further restrict to those having ramification sequence exactly \( \beta \) along \( P_{g-1} \) on \( X_{g-1} \), and exactly \( \gamma \) along \( Q_2 \) on \( X_1 \), and this again gives an open subset, which we denote by \( U \). Now, note that if we take the canonical linear series on \( X_{g-1} \) and twist up by \( 2P_{g-1} \), we get a \( g_{2g-2}^{-1} \), and if the vanishing sequence of the canonical series at \( P_{g-1} \) was equal to \( \alpha' \) over a given point of \( B_{g-1} \), then the vanishing sequence of the \( g_{2g-2}^{-1} \) at \( P_{g-1} \) will be equal to \( \beta \). Then by the construction of \( \pi_{g-1} \) together with Corollary 4.5.5, \( U \) has codimension \( -\rho \) over \( B_0 \), so by Corollary 4.4.12 we conclude that \( U \) has codimension \( -\rho \) over all of \( B \). Restricting to the complement of \( B_0 \) then gives the desired inductive statement. \( \square \)
Representable functors and moduli spaces

In this appendix, we discuss how to define moduli spaces rigorously in terms of functors of points, and how to prove representability, and we investigate the examples of the relative Grassmannian and relative Picard scheme.

A.1. The functor of points and Yoneda’s lemma

Our discussion is motivated by two questions. The first arises as a matter of aesthetics. Recall that the fibered product $X \times_Z Y$ of two schemes (or objects in any category) over another is a scheme together projection morphisms to $X, Y$ which compose to give the same morphisms to $Z$, and is characterized by the universal property that for any $T$, and any morphisms $T \to X, T \to Y$ which give the same morphisms to $Z$, there is a unique morphisms $T \to X \times_Z Y$ recovering the original morphisms upon composing with the projections. Proving existence of fibered products can be a bit of a hassle, but the fact that they are unique up to unique isomorphism is just a brief, purely formal argument. Indeed, whenever we define an object by a universal property, we find that if there exists an object with the described property, it is unique up to unique isomorphism, and the argument is always purely formal nonsense. We therefore ask:

**Question A.1.1.** How can one formulate a rigorous framework for a “universal property” such that one can prove that any object satisfying a universal property is unique up to unique isomorphism?

The second question is far more substantive, and deals with our main interest: moduli spaces. These are algebraic varieties (or schemes, or more general objects) which are supposed to naturally parametrize certain algebrogemetric objects, as the Grassmannian $G(r, d)$ parametrizes $r$-dimensional subspaces of a fixed $d$-dimensional space. The problem is that usually such a description only describes the points as a set, and doesn’t explain what the geometric structure should be. Frequently (as with the case of the Grassmannian), there is a clear “natural” geometric structure, but it’s not clear how to formalize what the “right” structure is.

**Question A.1.2.** How does one formalize a notion of moduli space so that the space is uniquely determined as a scheme (or variety), and not simply as a set?

These questions are both addressed by the machinery of the functor of points, and Yoneda’s lemma. To give further motivation in the context of Question A.1.2, we observe that for topological spaces, or for varieties over an algebraically closed field, the set of points underlying a space/variety $X$ is the same as the set of morphisms from a single point to $X$. This is enough only to recover the underlying set of $X$, but if we want to start to understand the geometry, we could also study
morphisms from other geometric objects \( T \) to \( X \). For instance, consider the curve in \( \mathbb{A}^d_k \) parametrized by \((t, t^2, \ldots, t^d)\). This is an algebraic curve in \( \mathbb{A}^d_k \), but it’s not described as such. At its most elementary, this is a collection of points in \( \mathbb{A}^d_k \) parametrized by \( t \). We can also think of it as a morphism \( \mathbb{A}^1_k \rightarrow \mathbb{A}^d_k \) where the coordinate on \( \mathbb{A}^1_k \) is \( t \), since it is visibly described by polynomials in \( t \). The fact that the points come from a morphism guarantee that they are not some random collection of points in \( \mathbb{A}^d_k \) corresponding to the points in \( \mathbb{A}^1_k \), but must be the points of an algebraic curve in \( \mathbb{A}^d_k \). More generally, we can picture a morphism \( T \rightarrow X \) as a collection of points of \( X \), which are parametrized nicely by \( T \). With this in mind, we define a \textbf{T-valued point} of \( X \) to be simply a morphism \( T \rightarrow X \).

In the case of a moduli space such as the Grassmannian, a map \( T \rightarrow G(r, d) \) will give in particular an \( r \)-dimensional subspace for every point of \( T \), which is to say, a family of \( r \)-dimensional subspaces parametrized by \( T \). Since the conditions imposed on morphisms are more restrictive than for an arbitrary map of points, this family will have some sort of conditions which impose, roughly speaking, that the subspaces corresponding to points of \( T \) vary nicely as the points of \( T \) do. The idea is then that instead of just saying that the points of the Grassmannian correspond to subspaces, we will describe what we want to allow for families of subspaces over any given \( T \), and this will specify what the morphisms from \( T \) to \( G(r, d) \). This is turn becomes a universal property which will uniquely determine \( G(r, d) \), once we prove that it exists.

We briefly review the most basic terminology of category theory. Recall that a \textbf{category} \( \mathcal{C} \) consists of a collection of \textit{objects} \( \text{Obj}(\mathcal{C}) \) and \textit{morphisms} between objects: i.e., for each ordered pair \((a, b)\) of (not necessarily distinct) objects of \( \mathcal{C} \), we have a set \( \text{Mor}(a, b) \) of morphisms from \( a \) to \( b \). We are also given the data of \textit{composition} of morphisms: for any \((a, b, c)\) of objects on \( \mathcal{C} \), a map of sets \( \text{Mor}(a, b) \times \text{Mor}(b, c) \rightarrow \text{Mor}(a, c) \); given \( f \in \text{Mor}(a, b) \) and \( g \in \text{Mor}(b, c) \), we write the resulting element of \( \text{Mor}(a, c) \) with the usual notation \( g \circ f \).

The two conditions for such a collection of data to form a category are:

I) that composition is associative: i.e. that for all \((a, b, c, d)\) and all \( f, g, h \) in \( \text{Mor}(a, b), \text{Mor}(b, c) \) and \( \text{Mor}(c, d) \) respectively, that \( h \circ (g \circ f) = (h \circ g) \circ f \);

II) that for any object \( a \in \text{Obj}(\mathcal{C}) \), there is an identity element \( 1_a \in \text{Mor}(a, a) \) such that for any \( b \) and any morphism \( f \in \text{Mor}(a, b) \), we have \( f \circ 1_a = f \), and for any \( f \in \text{Mor}(b, a) \) we have \( 1_a \circ f = 1_a \).

We also have the notion of a \textbf{functor}. A functor can be either \textbf{covariant} or \textbf{contravariant}. A covariant functor is a mapping between categories; specifically, \( F : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) associates to each object \( a \) of \( \mathcal{C}_1 \) an object \( F(a) \) of \( \mathcal{C}_2 \), and associates, for any pair \((a, b)\) of objects of \( \mathcal{C}_1 \) and any morphism \( f \in \text{Mor}(a, b) \), a morphism \( F(f) \in \text{Mor}(F(a), F(b)) \). A functor must have the properties that \( F(1_a) = 1_{F(a)} \) for any \( a \in \text{Obj}(\mathcal{C}_1) \), and that for any \((a, b, c)\) and \( f \in \text{Mor}(a, b), g \in \text{Mor}(b, c) \), we have \( F(g \circ f) = F(g) \circ F(f) \). A contravariant functor is the same, except that it reverses directions of morphisms, so that if \( f \in \text{Mor}(a, b) \), then \( F(f) \in \text{Mor}(F(b), F(a)) \).

Now, let \( \mathcal{C} \) be a category, and \( X \in \text{Obj}(\mathcal{C}) \). We observe that we have associated to \( X \) a natural contravariant functor \( h_X : \mathcal{C} \rightarrow \text{Set} \) defined by \( h_X(T) = \text{Mor}(T, X) \) (with \( h_X \) acting on morphisms by composition). Motivated by geometric categories in a manner we shall explain later, we call \( h_X \) the \textbf{functor of points} of \( X \).

Yoneda’s lemma says that \( h_X \) determines \( X \) uniquely. To state it more precisely, let’s suppose we have two contravariant functors \( F_1, F_2 : \mathcal{C} \rightarrow \text{Set} \). Then a
morphism \( \varphi : F_1 \to F_2 \) consists of the data, for all objects \( X \in \mathcal{C} \), of a function \( \varphi(X) : F_1(X) \to F_2(X) \), such that for any morphism \( f : Y \to X \) in \( \mathcal{C} \), we have that the diagram

\[
\begin{array}{ccc}
F_1(X) & \xrightarrow{\varphi(X)} & F_2(X) \\
\downarrow_{F_1(f)} & & \downarrow_{F_2(f)} \\
F_1(Y) & \xrightarrow{\varphi(Y)} & F_2(Y)
\end{array}
\]

commutes. \( \varphi \) is an isomorphism if it has an inverse, or as one can check is equivalent, if every \( \varphi(X) \) is a bijection.

Note that if we have a morphism \( X \to Y \) in \( \mathcal{C} \), we obtain a morphism

\[
h_X \to h_Y
\]

by composition, and the same holds for isomorphisms.

**Lemma A.1.3.** (Yoneda’s lemma) Given objects \( X, Y \in \mathcal{C} \), suppose we have a morphism \( \varphi : h_X \to h_Y \) of the associated functor of points. Then \( \varphi \) is induced by a unique morphism \( f : X \to Y \). In particular, any isomorphism \( h_X \to h_Y \) is induced by an isomorphism \( X \to Y \).

**Proof.** We consider the map \( \varphi(X) : h_X(X) \to h_Y(X) \). In \( h_X(X) \) we have the identity element, and we set \( f = (\varphi(X))(\text{id}) \in \text{Mor}(X,Y) \). It is then a simple exercise to check that this induces a bijection between morphisms \( h_X \to h_Y \) and morphisms \( X \to Y \).

Let’s notice that if we have a contravariant functor \( F : \mathcal{C} \to \text{Set} \), then an object \( X \in \text{Obj}(\mathcal{C}) \) together with an \( \eta \in F(X) \) induces a morphism \( h_X \to F \). Indeed, given a morphism \( \varphi : Y \to X \), we obtain a map \( F(\varphi) : F(X) \to F(Y) \), and declaring \( (F(\varphi))(\eta) \) to be the image of \( \varphi \) defines the morphism \( h_X \to F \).

**Definition A.1.4.** Given a functor \( F : \mathcal{C} \to \text{Set} \), we say that a pair \( X \in \text{Obj}(\mathcal{C}), \eta \in F(X) \) represents \( F \) if the induced morphism \( h_X \to F \) is an isomorphism. In this case, \( \eta \) is called the universal object (or universal family) for \( F \). We say that \( F \) is representable if there exists a pair \( (X, \eta) \) representing \( F \).

Yoneda’s lemma immediately implies:

**Corollary A.1.5.** If \( (X, \eta) \) represents a functor \( F \), then the pair is unique up to unique isomorphism; that is, if \( (Y, \eta') \) also represents \( F \), there is a unique isomorphism \( \varphi : X \cong Y \) sending \( \eta' \) to \( \eta \).

We remark that the object \( \eta \) is an important part of the data for representing a functor. For instance, in the case of fibered products, it is precisely the pair of projection morphisms needed to state the universal property. However, we will frequently omit it from notation.

We immediately see that any universal property described by characterizing the maps into a given object (as is the case, for instance, with fibered products) is in fact requiring the object in question to represent a functor, and from Yoneda’s lemma we immediately conclude that such universal properties determine an object up to unique isomorphism. Covariant functors have a parallel theory (which arises, for instance, in the universal property of the symmetric product of Proposition 2.4.3), and between the contravariant and covariant formulations, we see that Yoneda’s lemma gives a very satisfactory answer to Question A.1.1.
Similarly, we see that once we have specified for a given moduli problem not only what objects we want to parametrize, but what we allow as families of such objects parametrized by any given \( T \), we have described a functor for the moduli space to represent, and if it exists, it will be unique. We see that although universal properties arise naturally in both the covariant and contravariant context, moduli problems are most naturally stated in terms of contravariant functors, which is why we have focused our exposition in that direction.

As another illustration of the usefulness of Yoneda’s lemma, we apply it to constructing morphisms of objects in terms of their functors of points. We can directly rephrase Yoneda’s lemma as follows.

**Corollary A.1.6.** Given \( X, Y \in \text{Obj}(\mathcal{C}) \), a morphism \( f : X \to Y \) is equivalent to the following:

For each \( T \in \text{Obj}(\mathcal{C}) \), a function \( f_T \) from the \( T \)-valued points of \( X \) to the \( T \)-valued points of \( Y \), such that for any \( g : T' \to T \) in \( \mathcal{C} \), we have \( f_T \circ g_X = g_Y \circ f_T \), where \( g_X : \text{Mor}_\mathcal{C}(T, X) \to \text{Mor}_\mathcal{C}(T', X) \) is induced by composition with \( g \), and similarly for \( g_Y \).

Indeed, the collection of \( f_T \) is equivalent to the data of a morphism \( h_X \to h_Y \), so this follows immediately from Yoneda’s lemma. However, this corollary is very useful in dealing with schemes representing functors, as it means that we can construct morphisms between them solely in terms of the functors, even if we don’t have the slightest idea how to understand the geometry of the schemes!

**A.2. A criterion for representability**

We now move on to see what more we can say about schemes representing functors, and how to prove representability. The key idea is that, with the appropriate language, we can prove representability locally, by gluing together representing schemes. The magic will be that gluing in this context is actually much easier than gluing schemes (or varieties) which do not arise from representing functors. We will illustrate the technique by proving the existence of fibered products.

This section is rather technical, and can probably be skipped on a first reading.

We begin by deriving a certain property that a contravariant functor \( F : \text{Sch} \to \text{Set} \) must have in order to be representable: the property of being a “Zariski sheaf”.

In fact, we will fix a base scheme \( S \), and work with the category \( \text{Sch}_S \) of schemes over \( S \). There is no loss of generality here, since if we want to work with the category \( \text{Sch} \) of all schemes, we can simply set \( S = \text{Spec} \mathbb{Z} \).

The basic observation is the following:

**Proposition A.2.1.** (Gluing of morphisms of schemes) Let \( X \) and \( Y \) be schemes over \( S \), and \( \{ U_i \} \) an open covering of \( X \). Then restriction to the \( U_i \) induces a bijection between morphisms \( f : X \to Y \) over \( S \), and collections of morphisms \( f_i : U_i \to Y \) over \( S \), such that for all \( i, j \) we have \( f_{ij} = f_i \) as morphisms over \( S \).

We leave the proof as an exercise.

This is useful in various contexts, including the following.

**Exercise A.2.2.** Show that for any ring \( R \) and \( n \geq 1 \), the functor \( X \mapsto \Gamma(X, \mathcal{O}_X)^n \) on the category of schemes over \( R \) is represented by \( \text{Spec} R[x_1, \ldots, x_n] \) together with the universal object \( (x_1, \ldots, x_n) \).
For us, the main point of Proposition A.2.1 is that in order for a functor \( F : \text{Sch}_S \to \text{Set} \) to be representable, it needs to satisfy a certain property, analogous to the condition for a presheaf to be a sheaf:

**Definition A.2.3.** A contravariant functor \( F : \text{Sch}_S \to \text{Set} \) is a *Zariski sheaf* if it satisfies the following condition:

For every \( X \in \text{Obj}(\text{Sch}_S) \), and every open cover \( \{U_i\} \) of \( X \), the natural map

\[
\{\eta \in F(X)\} \to \{\eta_i \in F(U_i) \} : \forall i,j, \quad \eta_i|_{U_i \cap U_j} = \eta_j|_{U_i \cap U_j}
\]

is a bijection.

It is simply a matter of definition-chasing to see that Proposition A.2.1 can be rephrased to say the following:

**Corollary A.2.4.** Let \( F : \text{Sch}_S \to \text{Set} \) be a contravariant functor. In order for \( F \) to be representable, it is necessary that \( F \) be a Zariski sheaf.

This is by no means a sufficient condition, but we will see that if \( F \) is a Zariski sheaf, and we want to construct an \( X \) representing \( F \), then it is enough to carry out the construction locally. This yields a very simple construction of fibered products of schemes, and we will also apply the same technique to Grassmannians. In order to formulate this precisely, we need to develop the notion of “open subfunctor,” which is elementary but somewhat technical. Our definition is motivated by considering the functors of points of an open subscheme \( U \) of a scheme \( X \). The first condition is quite simple: for any scheme \( T \) over \( S \), the morphisms \( T \to U \) are a subset of the morphisms \( T \to X \). The more subtle idea is as follows: for any morphism \( f : T \to X \) over \( S \), taking the preimage of \( U \) gives us an open subscheme \( T_U \) of \( T \), and a map \( f_U : T_U \to U \) (still over \( S \)). Moreover, we see that if \( T' \to T \) is any morphism such that the composition \( T' \to X \) has image contained in \( U \), then \( T' \to T \) factors uniquely through the inclusion \( T_U \to T \). More formally, for any element \( f \in h_X(T) \), we have an open subscheme \( \iota : T_U \to T \) such that: for all \( T' \), and \( g \in h_T(T') \), if \( f \circ g \) lies in the subset \( h_U(T') \subseteq h_X(T') \) then there is a unique \( h \in h_{T_U}(T') \) such that \( g = \iota \circ h \).

We can now formalize this situation.

**Definition A.2.5.** Let \( F, G \) be contravariant functors \( \text{Sch}_S \to \text{Set} \), and \( G \to F \) a morphism of functors. We say that \( G \) is an open subfunctor of \( F \) if:

(I) for every \( T \in \text{Ob}(\text{Sch}_S) \), the map \( G(T) \to F(T) \) is injective.

(II) for every \( T \in \text{Ob}(\text{Sch}_S) \), and every \( \eta \in F(T) \), there exists an open subscheme \( U \) of \( T \) and an \( \eta' \in G(U) \) such that \( \eta \) and \( \eta' \) agree under the induced maps to \( F(U) \), and furthermore, for any \( f : T' \to T \), if \((F(f))(\eta) \in G(T') \subseteq F(T') \), then there exists a unique \( g : T' \to U \) such that \( f = \iota \circ g \), where \( \iota : U \to T \) is the inclusion map.

Note that the condition on the \((U, \eta')\) means that it represents a functor (specifically, the functor \( \text{Sch}/T \to \text{Set} \) which assigns to \( f : T' \to T \) the one-point set if \( F(f)(\eta) \in G(T') \), and the empty set otherwise), so is unique if it exists. We can then easily talk about a cover of a functor by open subfunctors:

**Definition A.2.6.** A collection \( F_i \to F \) of open subfunctors of \( F \) are said to cover \( F \) if for every \( T \) and \( \eta \in F(T) \), if \( U_i \) is the open subscheme of \( T \) associated to \( \eta \) by \( F_i \), then the \( U_i \) cover \( T \).
We observe that if \( U \subseteq X \) is an open subscheme, then \( h_U \) is an open subfunctor of \( h_X \), and if \( \{ U_i \} \) is an open cover of \( X \), then \( h_{U_i} \) is a cover by open subfunctors of \( h_X \).

Now, suppose we have a cover of \( F \) by open subfunctors \( F_i \), and each \( F_i \) is represented by \((U_i, \eta_i)\). We want to glue the \((U_i, \eta_i)\) together to give an \((X, \eta)\) representing \( F \). We will see that in order to do this, we will need that \( F \) is a Zariski sheaf. However, the statement itself is surprisingly simple.

**Theorem A.2.7.** Suppose that \( F : \text{Sch}_S \to \text{Set} \) is a Zariski sheaf, and that \( F \) has a cover by open subfunctors \( F_i \to F \). Suppose further that each \( F_i \) is represented by \((U_i, \eta_i)\), with \( U_i \in \text{Obj}(\text{Sch}_S) \) and \( \eta_i \in F_i(U_i) \).

Then there exists \((X, \eta)\) representing \( F \), with maps \( \psi_i : U_i \to X \) such that:

(i) each \( \psi_i \) is an open immersion;
(ii) the \( \psi_i(U_i) \) cover \( X \);
(iii) \( (F(\psi_i))(\eta) = \eta_i \) for each \( i \).

Before we give the proof, we need to understand what is necessary in general to glue together schemes, without worrying about functors.

**Proposition A.2.8.** Let \( \{ U_i \} \) be a collection of schemes, and for each \( i \neq j \), suppose we have an open subscheme \( U_{i,j} \subseteq U_i \). Suppose we also have for every \( i \neq j \) an isomorphism \( \varphi_{i,j} : U_{i,j} \cong U_{j,i} \), satisfying:

(I) for each \( i \neq j \), we have \( \varphi_{i,j} = \varphi_{j,i}^{-1} \);
(II) for each \( i,j,k \) pairwise distinct, \( \varphi_{i,j}(U_{i,j} \cap U_{i,k}) = U_{j,i} \cap U_{j,k} \), and \( \varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j} \) on \( U_{i,j} \cap U_{i,k} \).

Then we can glue together the \( U_i \) along the \( (U_{i,j}, \varphi_{i,j}) \) to obtain a scheme \( X \). More precisely, there is a scheme \( X \), and morphisms \( \psi_i : U_i \to X \), such that:

(i) each \( \psi_i \) is an open immersion;
(ii) the \( \psi_i(U_i) \) cover \( X \);
(iii) for each \( i \neq j \), we have \( \psi_i(U_{i,j}) = \psi_i(U_i) \cap \psi_j(U_j) \);
(iv) for each \( i \neq j \), we have \( \psi_i = \psi_j \circ \varphi_{i,j} \) on \( U_{i,j} \).

The proof is left as an exercise (specifically, Exercise II.2.12 of [Har77]).

Notice (comparing the statement of Theorem A.2.7 to that of Proposition A.2.8) that gluing together schemes representing a functor is drastically simpler than gluing together arbitrary schemes we don’t have to worry about keeping track of isomorphisms on intersections of open subsets, or what happens on triple intersections. The fact that everything comes from a single functor deal with all of that transparently.

Before giving the proof of Theorem A.2.7, we observe that we can “intersect” two open subfunctors to obtain a new open subfunctor, in the obvious way: given \( G, G' \) two open subfunctors of \( F \), we consider \( G(T), G'(T) \subseteq F(T) \) under the maps provided, and get a new functor by intersecting inside \( F(T) \) for every \( T \). If \( U, U' \subseteq T \) are the open subsets provided by (ii) of the definition, we check that \( U \cap U' \) works for the intersected functor. Also, to reduce the amount of notation necessary, if we have a functor \( F \), object \( T \), and \( \eta \in F(T) \), together with a morphism \( T' \to T \), we will write \( \eta|_{T'} \) to denote the element of \( F(T') \) obtained as the image of \( \eta \) under the induced map \( F(T) \to F(T') \).

**Proof of Theorem A.2.7.** We first use the \( \eta_i \) to glue together the \( U_i \) into a single scheme \( X \). According to the previous proposition, we need open subschemes
$U_{i,j} \subseteq U_i$ and isomorphisms $\varphi_{i,j} : U_{i,j} \cong U_{j,i}$ satisfying the stated compatibility conditions. The idea is simple: by the definition of an open subfunctor, if we are given $i \neq j$, then associated to $\eta_i \in F(U_i)$, the open subfunctor $F_j$ gives us an open subscheme, which we denote $U_{i,j}$, of $U_i$, and an element $\eta_{i,j} \in F_j(U_{i,j})$ such that $\eta_i$ agrees with $\eta_{i,j}$ in $F(U_{i,j})$, and such that $(U_{i,j},\eta_{i,j})$ represents the functor of morphisms $T' \to U_i$ such that $\eta_i|_{T'} \in F_j(T') \subseteq F(T')$.

One checks that in fact $(U_{i,j},\eta_{i,j})$ represents $F_i \cap F_j$. It follows that $(U_{i,j},\eta_{i,j})$ and $(U_{j,i},\eta_{j,i})$ represent the same functor, so by Yoneda’s lemma we obtain a unique isomorphism $\varphi_{i,j} : U_{i,j} \cong U_{j,i}$ sending $\eta_{i,j}$ to $\eta_{j,i}$. We then claim that the $U_{i,j}$ and $\varphi_{i,j}$ satisfy the conditions of Proposition A.2.8. Indeed, the first condition is immediate, while the second follows from writing everything in terms of the functor $F_i \cap F_j \cap F_k$, and following through the definitions, again using Yoneda’s lemma.

It thus follows that we obtain a scheme $X$ and maps $\psi_i : U_i \to X$ as in the proposition. It is then enough to show that we have an $\eta \in F(X)$ such that $(X,\eta)$ represents $F$, and $(F(\psi_i))(\eta) = \eta_i$ for each $i$. Here we finally use that $F$ is a Zariski sheaf: for each $i$, we can consider $\eta_i \in F_i(U_i)$ as an element of $F(U_i)$, and considering the $U_i$ as open subsets of $X$, following through the definitions we find that $\eta_i|_{U_i \cap U_j} = \eta_j|_{U_i \cap U_j}$ for every $i,j$. Thus, by the Zariski sheaf condition, the $\eta_i$ glue uniquely to give an element $\eta \in F(X)$ with $\eta|_{U_i} = \eta_i$ for all $i$. We claim that $(X,\eta)$ represents $F$. Indeed, given $T$ and $\zeta \in F(T)$, the $F_i$ give us an open cover $T_{U,i}$ of $T$, and because $(U_i,\eta_i)$ represent $F_i$, we get morphisms $T_{U,i} \to U_i$, with $\zeta|_{T_{U,i}} = \eta_i|_{T_{U,i}}$, and which agree on the intersections. We can therefore glue to obtain a morphism $T \to X$, with $\zeta = \eta|_T$, and it follows by definition that $(X,\eta)$ represent $F$.

As an initial illustration of the utility of Theorem A.2.7, we will apply it to prove the existence of fibered products of schemes.

**Theorem A.2.9.** Fibered products exist in the category of schemes, and therefore also in the category of schemes over $S$ for any fixed scheme $S$.

**Proof.** Given $X,Y$ mapping to $Z$, let $F_{X \times_Z Y}$ be the functor described by the universal property of $X \times_Z Y$: that is, for all schemes $T$, we set $F_{X \times_Z Y}(T)$ to be the set of pairs of morphisms $T \to X$, $T \to Y$ which compose to give the same morphism $T \to Z$. Thus, the theorem states precisely that $F_{X \times_Z Y}$ is representable.

It is straightforward to check from Proposition A.2.1 that $F$ is a Zariski sheaf. By Theorem A.2.7, it therefore suffices to produce a cover by open subfunctors which are each representable. The next observation is that given $U,V,W$ open subschemes of $X,Y,Z$ respectively, with the images of $U$ and $V$ contained in $W$, then $F_{U \times_W V}$ is an open subfunctor of $F_{X \times_Z Y}$. Moreover, if we have a family $\{U_i,V_i, W_i\}$ of such open subschemes, the subfunctors $F_{U_i \times_W V_i}$ cover $F_{X \times_Z Y}$ if, for all points $x \in X$, $y \in Y$ mapping to the same point of $Z$, we have some $i$ such that $x \in U_i$ and $y \in V_i$. We choose our family $\{U_i,V_i, W_i\}$ to consist of all triples of affine open subschemes with $U_i$ and $V_i$ mapping into $W_i$. This satisfies the above condition, since if $z$ is the common image of $x$ and $y$, we can first choose $W_i$ an affine neighborhood of $z$, and then choose $U_i$ and $V_i$ to be affine neighborhoods of $x$ and $y$ respectively contained in the preimages of $W_i$. Thus, in this way, we obtain a cover of $F_{X \times_Z Y}$ by open subfunctors $F_{U_i \times_W V_i}$, with all $U_i, V_i, W_i$ affine, and we see that we have reduced to the case that all three schemes are affine, say $U_i = \text{Spec} A_i$, $V_i = \text{Spec} B_i$, and $W_i = \text{Spec} C_i$. 


Finally, we claim that $F_{U_i \times W_i V_i}$ is represented by $\text{Spec} A_i \otimes_{C_i} B_i$, together with the maps to $U_i$ and $V_i$ induced by the natural homomorphisms $A_i \to A_i \otimes_{C_i} B_i$ and $B_i \to A_i \otimes_{C_i} B_i$. The fact that this gives the correct functor on affine schemes is immediate from the universal property of tensor product of rings, and the general statement follows similarly making use of the fact that if $T$ is any scheme and $R$ any ring, then morphisms $T \to \text{Spec} A$ correspond to ring homomorphisms $A \to \Gamma(T, \mathcal{O}_T)$ (see Exercise II.1.18 of [Har77]).

\[\square\]

Exercise A.2.10. Fill in the details of the proof of Theorem A.2.9.

### A.3. Grassmannians

We now formally define and prove the existence of Grassmannians, and more generally, relative Grassmannians.

In the discussion of §A.1, we concluded that the key to defining a moduli functor for the Grassmannian would be to identify what it should mean to have a nicely varying collection of subspaces of a fixed vector space $V$, parametrized by points of some $T$. The idea is simple enough. We already have a way to define a nice family of vector spaces parametrized by points of $T$: namely, this should be a vector bundle on $T$. Thus, if we want a family of subspaces, this naturally leads us to consider a notion of subbundle. There is a slight subtlety in getting the right definition, especially from the sheaf context.

**Definition A.3.1.** Given a scheme $T$, and a vector bundle (i.e., locally free sheaf) $\mathcal{E}$ on $T$, a subbundle of $\mathcal{E}$ is a subsheaf $\mathcal{F}$ of $\mathcal{E}$ which is locally free, and for which the quotient $\mathcal{E}/\mathcal{F}$ is also locally free.

**Remark A.3.2.** In fact, if $\mathcal{E}/\mathcal{F}$ is locally free, it follows that $\mathcal{F}$ is locally free, but not conversely. We state the definition in this manner to emphasize the motivating geometric context. The reason for the necessity of assuming that $\mathcal{E}/\mathcal{F}$ is locally free is that an injective map of locally free sheaves does not necessarily correspond to an injective map of the corresponding geometric vector bundles: we can have an injective map of sheaves where the rank of the map drops at a point of $T$, and then the map of geometric vector bundles is not. It turns out that the condition that $\mathcal{E}/\mathcal{F}$ be locally free is equivalent to injectivity of the associated map of geometric vector bundles, which is what should be required of a subbundle.

Another statement equivalent to $\mathcal{E}/\mathcal{F}$ being locally free, and more relevant to our immediate purpose, is that the map $\mathcal{F} \to \mathcal{E}$ remains injective when we look at the fiber at any point of $T$. Thus, this is precisely what is necessary to ensure that $\mathcal{F}$ is really giving us a subspace (of fixed dimension) for each point of $T$.

Another important consequence of the condition that $\mathcal{E}/\mathcal{F}$ be locally free is that it ensures that the definition is invariant under pullback, so that if $\mathcal{F}$ is a subbundle of $\mathcal{E}$ on $T$, and $f : T' \to T$ any morphism, then $f^* \mathcal{F}$ is a subbundle of $f^* \mathcal{E}$.

Thus, given a $k$-vector space $V$, we can define the Grassmannian $G(r, V)$ to represent the functor which associates to a scheme $T$ over $\text{Spec} k$ the rank-$r$ subbundles of the trivial bundle on $T$ corresponding to $V$, which as a sheaf is just $V \otimes_k \mathcal{O}_T$. If we think of $V$ as determining a vector bundle on $\text{Spec} k$, we can also describe the trivial bundle on $T$ equivalently as the pullback from $\text{Spec} k$ to $T$ of the vector bundle corresponding to $V$. At this point, the generalization to the relative
Grassmannian is clear: if we replace \( \text{Spec } k \) by a more general scheme \( S \), we replace \( V \) by a vector bundle \( \mathcal{E} \) on \( S \), and then for any scheme \( T \) over \( S \), we associate the set of rank-\( r \) subbundles of the pullback to \( T \) of \( \mathcal{E} \).

**Definition A.3.3.** Given \( d \geq r > 0 \), a scheme \( S \), and a vector bundle \( \mathcal{E} \) of rank \( d \) on \( S \), we define the associated relative Grassmannian functor

\[
\mathcal{G}(r, \mathcal{E}) : \text{Sch}/S \to \text{Set}
\]

given by

\[
\mathcal{G}(r, \mathcal{E})(T) = \{ \mathcal{F} \subseteq \pi_T^* \mathcal{E} : \mathcal{F} \text{ a subbundle of rank } r \},
\]

where \( \pi_T \) is the structure map \( T \to S \) that comes with an object of \( \text{Sch}/S \).

We will now show that this functor is representable, and in the process we will obtain a great deal more information about it. The theorem we will prove is the following.

**Theorem A.3.4.** For any scheme \( S \), and vector bundle \( \mathcal{E} \) on \( S \) of rank \( d \), given nonnegative \( r \leq d \), the functor \( \mathcal{G}(r, \mathcal{E}) \) is represented by a scheme \( G(r, \mathcal{E}) \) over \( S \), together with a universal subbundle \( \mathcal{F} \to \pi^* \mathcal{E} \) of rank \( r \), where \( \pi : G(r, \mathcal{E}) \to S \) is the structure map.

Concretely, for any \( S \)-scheme \( T \), with structure morphism \( \pi_T \), and any rank-\( r \) subbundle \( \mathcal{F}_T \) of \( \pi_T^* \mathcal{E} \), there is a unique morphism \( \varphi : T \to G(r, \mathcal{E}) \) over \( S \) such that \( \mathcal{F}_T = \varphi^* \mathcal{F} \) as a subbundle of \( \pi_T^* \mathcal{E} \).

We also have that \( \pi \) is smooth of relative dimension \( r(d - r) \) and proper, and there exists an open cover of \( G(r, \mathcal{E}) \) by open subsets each isomorphic to \( \mathbb{H}^r(d - r) \) for some \( U \subseteq S \) open.

As per our criterion for representability, the first step is the following lemma.

**Lemma A.3.5.** \( \mathcal{G}(r, \mathcal{E}) \) is a Zariski sheaf.

**Proof.** This follows essentially immediately from the definitions, as a subsheaf is uniquely determined by a collection of subsheaves on an open cover which agree on the intersections. \( \square \)

To prove representability, it therefore suffices to produce an open cover of \( \mathcal{G}(r, \mathcal{E}) \) by subfunctors which are each representable. The idea is quite simple, at least in case of the classical Grassmannian \( G(r, d) \) of subspaces of the vector space \( k^d \): if we fix a \((d - r)\)-dimensional subspace \( W \) of \( k^d \), then the subset \( U_W \) of \( G(r, d) \) corresponding to \( r \)-dimensional spaces \( V \) with \( V \cap W = (0) \) should be open in \( G(r, d) \), and it should be isomorphic to \( \mathbb{H}^r(d - r) \). We see the last part by choosing a basis of \( k^d \) so that (in terms of this basis) a vector \( (c_1, \ldots, c_d) \in W \) if and only if \( c_1 = \cdots = c_r = 0 \). Then every subspace in \( U_W \) can be uniquely represented as the row space of a \( r \times d \) matrix whose first \( r \times r \) submatrix is equal to the identity. Thus, \( U_W \) is isomorphic to \( \mathbb{H}^r(d - r) \) by considering the remaining \( r(d - r) \) entries of the matrix.

To make this more precise, we will need the classical concept of determinantal loci. The first definition is as follows.

**Definition A.3.6.** Let \( S \) be a scheme, and \( \mathcal{E}, \mathcal{F} \) vector bundles on \( S \), with a morphism \( \varphi : \mathcal{E} \to \mathcal{F} \). We say that \( \varphi \) has rank less than or equal to \( r \) if the associated morphism

\[
\wedge^{r+1} \varphi : \wedge^{r+1} \mathcal{E} \to \wedge^{r+1} \mathcal{F}
\]
is equal to 0.

The philosophy behind this definition is that while rank is a good notion for maps of vector spaces, we should not try to assign a rank to a map of modules. Geometrically, this corresponds to the idea that a map of vector bundles can give different ranks when restricted to the fibers at different points, so it doesn’t make sense to try to assign a single number. We do however obtain good theories of loci where the rank is at most (or at least) a given number.

The following proposition is standard linear algebra, translated to the module setting.

**Proposition A.3.7.** In the situation of Definition A.3.6, suppose also that \( r \) is strictly less than the ranks of both \( \mathcal{E} \) and \( \mathcal{F} \). Let \( \{ U_i \} \) be an affine open cover of \( S \) such that for each \( i \), we have \( \mathcal{E}|_{U_i} \) and \( \mathcal{F}|_{U_i} \) both free, and fix trivializations \( \mathcal{E}|_{U_i} \cong \mathcal{O}^{d_1}_{U_i} \) and \( \mathcal{F}|_{U_i} \cong \mathcal{O}^{d_2}_{U_i} \). Then \( \varphi \) has rank less than or equal to \( r \) if and only if for each \( i \), the matrix representing \( \varphi \) on \( U_i \) under the chosen trivializations has every \((r+1) \times (r+1)\) minor equal to 0 on \( U_i \).

We then have:

**Corollary A.3.8.** Let \( S \) be a scheme, and \( \mathcal{E}, \mathcal{F} \) vector bundles on \( S \), with a morphism \( \varphi : \mathcal{E} \to \mathcal{F} \). Then there is a unique closed subscheme \( Z \subseteq S \) such that a morphism \( \pi : T \to S \) factors through \( Z \) if and only if \( \pi^* \varphi \) has rank less than or equal to \( r \).

**Proof.** Define the functor \( F : \text{Sch}/S \to \text{Set} \) to associate to \( \pi : T \to S \) the one-point set if \( \pi^* \varphi \) has rank less than or equal to \( r \), and the empty set otherwise. Then the corollary simply says that \( F \) is represented by a closed subscheme of \( S \). If \( r \) is at least as big as either of the ranks of \( \mathcal{E} \) or \( \mathcal{F} \), we see immediately that we can set \( Z = S \). Otherwise, since formation of wedge products commutes with pullback, Proposition A.3.7 implies that the functor is in fact represented by the closed subscheme determined locally on \( S \) by the vanishing of the appropriate \((r+1) \times (r+1)\) minor equal to 0 on \( U_i \). □

Due to the construction via minors, we have the following terminology:

**Definition A.3.9.** In the situation of Corollary A.3.8, the closed subscheme \( Z \) is called the \( r \)th **determinantal subscheme** associated to \( \varphi \).

Determinantal subschemes are an important subject in their own right, and we will study them and their generalizations in a later appendix.

Returning to our earlier motivating discussion of the classical Grassmannian \( G(r, d) \), let \( W \) be a \((d-r)\)-dimensional subspace of \( k^d \). Given \( \pi : T \to \text{Spec} k \), we want to formally express the condition that a subbundle \( \mathcal{F} \subseteq \mathcal{O}^d_T \) is disjoint from \( \pi^* W \). It turns out that simply asking for \( \mathcal{F} \cap \pi^* W \) to be the zero subsheaf is not good enough – we could easily have that the subspace given by \( \mathcal{F} \) at some point of \( T \) has non-zero intersection with \( W \), but there is no open subset of \( T \) on which \( \mathcal{F} \) has non-zero intersection with \( \pi^* W \). On the other hand, for a vector space \( V \) to have \( V \cap W = (0) \) is equivalent to saying that the map \( V \to k^d/W \) has full rank \( r \). We can thus impose the correct condition by requiring that the \((r-1)\)st determinantal locus of the map \( \mathcal{F} \to \pi^*(k^d/W) \) is the empty subscheme of \( T \). In the general case, we make the following definition:
Definition A.3.10. In the situation of Definition A.3.3, suppose that \( U \) is an open subscheme of \( S \), and \( \mathcal{W} \subseteq \mathcal{E} \) a subbundle of rank \( d - r \). Then we define the subfunctor \( \mathcal{U}_{U, \mathcal{W}} \subseteq \mathcal{G}(r, d) \) as follows: given \( \pi : T \to S \), a subbundle \( \mathcal{F} \subseteq \pi^* \mathcal{E} \) is in \( \mathcal{U}_{U, \mathcal{W}} \) if and only if \( T \) maps into \( U \subseteq S \), and \( (r - 1) \text{st} \) determinantal subscheme for the map \( \mathcal{F} \to \pi^*(\mathcal{E}/\mathcal{W}) \) has rank uniformly less than or equal to \( r - 1 \) is the empty subscheme.

We have now produced the desired family of subfunctors, so the next step is to prove that they form a cover by open subfunctors, and are representable.

Lemma A.3.11. In the situation of Definition A.3.10, the subfunctor \( \mathcal{U}_{U, \mathcal{W}} \) of \( \mathcal{G}(r, d) \) is open. As the \( (U, \mathcal{W}) \) vary, the functors \( \mathcal{U}_{U, \mathcal{W}} \) form an open cover of \( \mathcal{G}(r, \mathcal{E}) \), and in fact, in order to produce a cover of \( \mathcal{G}(r, \mathcal{E}) \), we may take any affine open cover \( \{ U_i \} \) of \( S \) on which \( \mathcal{E} \) is trivialized, and only need finitely many choices of \( \mathcal{W} \) for each \( U_i \).

Proof. It is clear from Corollary A.3.8 that \( \mathcal{U}_{U, \mathcal{W}} \) is an open subfunctor: indeed, given any \( \pi : T \to S \) and \( \mathcal{F} \subseteq \pi^* \mathcal{E} \) a \( T \)-valued point of \( \mathcal{G}(r, \mathcal{W}) \), we can take the open subscheme of \( T \) required in the definition of an open subfunctor to be the complement of the \( (r - 1) \text{st} \) determinantal subscheme of \( \mathcal{F}_{T(U)} \to \pi^*(\mathcal{E}/\mathcal{W}) \) inside of \( \pi^{-1}(U) \). Denote this open subscheme by \( V_{\pi, U, \mathcal{W}} \).

Next, for a given \( U \) on which \( \mathcal{E} \) is trivialized, we claim that we can choose a finite collection of \( \mathcal{W} \) such that the \( V_{\pi, U, \mathcal{W}} \) cover \( \pi^{-1}(U) \). This will prove the desired statement, since if \( \{ U_i \} \) is an open cover of \( S \), the \( \pi^{-1}(U_i) \) cover \( T \). Fixing an isomorphism \( \mathcal{E}|_T \cong O_T^{\oplus d} \), we claim that it is enough to consider subbundles \( \mathcal{W}_I \subseteq O_T^{\oplus d} \) generated by a choice of \( d - r \) standard basis vectors corresponding to a choice of subset \( I \subseteq \{ 1, \ldots, d \} \). Indeed, given a point \( t \in \pi^{-1}(U) \), we will have \( t \in V_{\pi, U, \mathcal{W}_I} \) if and only if \( \mathcal{F}|_t \) injects into \( (O_T^{\oplus d}/\mathcal{W}_I)|_t \), so our claim amounts to the standard fact that given a field \( k \) and an \( r \)-dimensional subspace \( V \subseteq k^d \), there is some collection \( I \) of \( r - d \) standard basis vectors \( e_i \) of \( k^d \) such that \( V \cap \text{span}(e_i : i \in I) = (0) \).

Lemma A.3.12. For any \( U, \mathcal{W} \) as in Definition A.3.10, assume further that \( U \) is affine. Then the functor \( \mathcal{U}_{U, \mathcal{W}} \) is represented by a vector bundle over \( U \) of rank \( r(d - r) \).

Proof. Since \( U \) is affine, the map \( \mathcal{W} \to \mathcal{E}|_U \) splits by Theorem III.3.5, Proposition III.6.3 (c), Proposition III.6.7, and Exercise III.6.1 of [Har77], so if we let \( \mathcal{W}' \subseteq \mathcal{E}|_U \) be the complementary subbundle arising from such a splitting, we have \( \mathcal{E} = \mathcal{W}' \oplus \mathcal{W} \), and \( \mathcal{W}' \cong \mathcal{E}|_U/\mathcal{W} \). The condition that the \( (r - 1) \text{st} \) determinantal subscheme of \( \mathcal{F} \to \pi^*(\mathcal{E}|_U/\mathcal{W}) \) is empty is equivalent to saying that the map is an isomorphism, since both the source and target are locally free of rank \( r \). Thus, we necessarily have \( \mathcal{F} \cong \pi^* \mathcal{W}' \), and the subbundles we wish to classify can be thought of simply as the images under different morphisms \( \pi^* \mathcal{W}' \to \pi^* \mathcal{E} \), subject to the condition that the composition \( \pi^* \mathcal{W}' \to \pi^* \mathcal{E} \to \pi^* \mathcal{W}' \) is the identity. Such a morphism is uniquely determined by the induced morphism \( \pi^* \mathcal{W}' \to \pi^* \mathcal{W} \), and since these are locally free sheaves of ranks \( r \) and \( d - r \) respectively, such morphisms are parametrized by the module

\[
\text{Hom}(\pi^* \mathcal{W}', \pi^* \mathcal{W}) = \Gamma(T, \mathcal{H}\text{om}(\pi^* \mathcal{W}', \pi^* \mathcal{W})) = \Gamma(T, \pi^* \mathcal{H}\text{om}(\mathcal{W}', \mathcal{W})).
\]

Now, \( \mathcal{H}\text{om}(\mathcal{W}', \mathcal{W}) \) is locally free of rank \( r(d - r) \), and we thus see that \( \mathcal{U}_{U, \mathcal{W}} \) is represented by the associated geometric vector bundle.
The following proposition will imply that the relative Grassmannian is proper. The idea is to prove directly that if we have a one-parameter family of subspaces, a limit subspace always exists. The concern one might have a priori is that a basis can become linearly dependent over a particular point, and the main content of the proposition is explaining that in this case, one can always replace it with a basis which remains independent.

**Proposition A.3.13.** Let $R$ be a discrete valuation ring, with fraction field $K$. Let $W \subseteq K^d$ be an $r$-dimensional subspace. Then $W \cap R^d$ is a rank-$r$ free submodule of $R^d$, with free quotient, and it is the only such submodule of $R^d$ which generates $W$ in $K^d$.

**Proof.** It is enough to show that $W \cap R^d$ is a rank-$r$ free submodule of $R^d$, with free quotient, since if $W'$ is any other such submodule of $R^d$, generating $W$ in $K^d$, we must have $W' \subseteq W$ and then they must be equal.

We next claim that if $v_1, \ldots, v_r$ is a basis of $W$ which lies in $R^d$, and has the property that its elements remain linearly independent in $(R/m)^d$, then $v_1, \ldots, v_r$ is also a basis for $W \cap R^d$. Indeed, given a linear combination

$$v = \sum_{i=1}^r a_i v_i$$

with the $a_i \in K$ and $v \in R^d$, it is enough to show that the $a_i$ must in fact be in $R$. Let $i_0$ be an index so that $a_{i_0}$ has minimal valuation; if we consider $v' = \sum_{i=1}^r \frac{a_i}{a_{i_0}} v_i$, the new coefficients are all in $R$, with at least one of them a unit, and we conclude from linear independence of the $v_i$ modulo $m$ that at least one coordinate of $v'$ is likewise a unit. Since the coordinates of $v'$ are $1/a_{i_0}$ times the coordinates of $v$, we conclude that the valuation of $a_{i_0}$ must have been nonnegative, which proves the claim. We note moreover that the linear independence of the $v_i$ modulo $m$ also implies, by Nakayama’s lemma, that the quotient $R^d/(W \cap R^d)$ is free, as desired.

It thus remains to produce a basis $v_1, \ldots, v_r$ as above. Start with any basis $v_1, \ldots, v_r$, scaled so that the $v_i$ are in $R^d$. Let $A$ be the $d \times r$ matrix with columns given by the $v_i$; then the $v_i$ are linearly dependent modulo $m$ if and only if all the $r \times r$ minors of $A$ vanish modulo $m$. Since the $v_i$ are linearly independent in $K$, at least one such minor is not equal to 0 in $R$, so we obtain a well-defined number $n \in \mathbb{Z}_{\geq 0}$ as the minimal valuation of the nonzero minors $r \times r$ minors of $A$, and we wish to show that by modifying our choice of the $v_i$, we can obtain $n = 0$. Since our valuation is discrete, it is enough to show that if $n > 0$, we can modify the choice of the $v_i$ to reduce $n$. Now, if $n > 0$, then a linear dependence modulo $m$ consists of $a_i \in R$, not all in $m$, such that

$$v = a_1 v_1 + \cdots + a_r v_r \in mR^d.$$  

If $a_i \not\in m$, then we can replace $v_i$ by $\frac{1}{a_i} v$, where $a$ is a coordinate of $v$ with minimal (necessarily positive) valuation. This has the effect of dividing all the minors of $A$ by $a$, so we strictly decrease $n$, as desired. □

We can now put everything together to prove Theorem A.3.4.
Proof of Theorem A.3.4. Given Lemmas A.3.5, A.3.11, and A.3.12, we conclude representability of the Grassmannian from Theorem A.2.7. Using also that vector bundles are locally isomorphic to affine space, we find that $G(r, E)$ is smooth and of finite type over $S$, with the asserted local description. Finally, in the case that $S$ is Noetherian, properness follows from Proposition A.3.13 by the valuative criterion for properness\(^1\) (Theorem II.4.7 of [Har77]). The general case follows from the observation that properness is local on the base, and if we choose $U$ in $S$ on which $E$ is trivialized, then $G(r, E)|_U$ is simply the base change to $U$ of $G(r, \mathcal{O}_Z^{\oplus d})$. Since properness is closed under base change, the general case reduces to the Noetherian case, and we conclude the desired statement. \(\square\)

Remark A.3.14. In fact, it is not too difficult to prove directly using the Plucker imbedding that relative Grassmannians are projective, which also renders unnecessary the argument of Proposition A.3.13. However, we have elected the present approach partly because the projectivity will not be relevant to us, and partly for the philosophical reason that while producing a closed imbedding in a projective space proves properness, it does not explain properness. In contrast, the argument of Proposition A.3.13 explains precisely why a limit of a family of subspaces always exists.

Remark A.3.15. In the case that $S$ is a variety, we find in particular that $G(r, E)$ is also a variety, nonsingular if $S$ is. The reason for working in the context of schemes rather than varieties is that we obtain more information this way: the fact that a functor of schemes is represented by a variety is a stronger statement than if the corresponding functor of varieties is representable. Moreover, by working with schemes we now have explicit information about morphisms from non-reduced schemes to Grassmannians, and in particular we obtain a description of the tangent space for free.

A.4. Picard schemes

Our next topic is moduli of line bundles on (families of) curves. The proofs in this case are harder and for the most part we will omit them, but we will at least give formal definitions and statements of everything we will need. The book [BLR91] includes a comprehensive survey of this material, with sketches and citations to the literature for the relevant proofs, and for simplicity we will cite it rather than the various original papers. However, we mention that [Kle05] also gives a survey of Picard schemes, with proofs as well as a discussion of the history. It is less useful to us because its presentation focuses on the case of irreducible fibers, but it is nonetheless a good introduction to the subject.

This section is rather lengthy, but the main results are summarized in Corollary A.4.15 below.

Although we were able to give in Section 2.4 heuristic arguments on the geometry of Picard varieties of curves over a field, to make these rigorous requires a definition which determines Picard varieties not just as sets, but as varieties. Thus, as with the Grassmannian, the first order of business is to define a moduli functor associated to a curve, or family of curves. Given $X$ a projective curve over

\(^1\)More general statements of the valuative criterion do not require a Noetherian base, but still require a quasiseparated hypothesis. This is why we find it more convenient to reduce to the Noetherian case.
Spec \kappa, one might guess that the functor \( \mathcal{P}ic(X/k) \) should associate to a scheme \( T \) over Spec \( k \) the set of isomorphism classes of line bundles on \( X \times_k T \). This is reasonable, and indeed, leads to the correct definition in the context of stacks, but we see that it cannot be representable, since it is not a Zariski sheaf. The reason is that isomorphisms which exist over an open cover of \( T \) need not patch together to give an isomorphism over all of \( T \). Concretely, given a line bundle \( \mathcal{L} \) on \( X \times_k T \), if \( T \) has a nontrivial line bundle \( \mathcal{M} \), then \( \mathcal{L} \times p_2^* \mathcal{M} \) is not isomorphic to \( \mathcal{L} \). On the other hand, there is an open cover \( \{ U_i \} \) of \( T \) on which \( \mathcal{M} \) is trivialized, so we do have \( \mathcal{L}|_{X \times_k U_i} \cong \mathcal{L} \times p_2^* \mathcal{M}|_{X \times_k U_i} \) for each \( i \), which violates the Zariski sheaf condition. This looks bad, but it turns out that under reasonable circumstances, if we simply mod out by pullbacks of line bundles from Pic(\( T \)), we get good behavior. For reasons of later convenience, we start by consider the degree 0 case.

**Definition A.4.1.** Let \( \pi : X \to B \) be a flat, proper family of nodal curves,\(^2\) and suppose that \( \pi \) has a section \( B \to X \). Then the degree-\( 0 \) **Picard functor** \( \mathcal{P}ic^0(X/B) : \text{Sch}/B \to \text{Set} \) is given by

\[
\mathcal{P}ic^0(X/B)(T) = \text{Pic}^0_T(X \times_B T)/p_2^* \text{Pic}(T),
\]

where \( \text{Pic}^0_T(X \times_B T) \) denotes the subgroup of \( \text{Pic}(X \times_B T) \) consisting of line bundles which have degree 0 on every component of every fiber of \( p_2 \).

The definition of course makes sense more generally, but will not be the right definition in general; see Remark A.4.16 below.

**Theorem A.4.2.** In the situation of Definition A.4.1, the functor \( \mathcal{P}ic^0(X/B) \) is represented by a smooth, separated scheme \( \mathcal{P}ic^0(X/B) \) of finite type and relative dimension \( g \) over \( B \), where \( g \) is the genus of the fibers of \( \pi \). Moreover, \( \text{Pic}^0(X/B) \) has geometrically connected fibers, and a \( B \)-ample line bundle.

Explicitly, there is a **Poincare line bundle** \( \mathcal{L} \) on \( X \times_B \text{Pic}^0(X/B) \), unique up to tensoring by \( p_2^* \text{Pic}(\text{Pic}^0(X/B)) \), such that \( \mathcal{L} \) has degree 0 on every component of every fiber of \( p_2 \), and for any \( T \to B \), and any line bundle \( \mathcal{L} \) on \( X \times_B T \) having degree 0 on every component of every fiber of \( p_2 \), there is a unique morphism \( \phi : T \to \text{Pic}^0(X/B) \) over \( B \) such that \( \mathcal{L} \cong \phi^* \mathcal{L} \otimes p_2^* \mathcal{M} \) for some \( \mathcal{M} \in \text{Pic}(T) \).

**Proof.** First, by the existence of a section together with Proposition 8.1.4, the definition preceding Theorem 8.4.4, and Corollary 9.2.13 of loc. cit., we find that our Picard functor agrees with that of loc. cit., and that (assuming representability) its fibers are geometrically connected. Then the statement of the theorem is Theorem 9.4.1 of [BLR91], except that the assertion that the relative dimension is \( g \) can be deduced from Lemma 9.3.5 of loc. cit., since the \( W \) in the latter statement has dimension \( g \).

**Remark A.4.3.** The smoothness and dimension of \( \text{Pic}^0(X/B) \) can be easily understood with a little deformation theory. Indeed, one shows that the tangent space of a fiber is given by \( H^1(X_b, \mathcal{O}_{X_b}) \), and that \( H^2(X_b, \mathcal{O}_{X_b}) \) is an obstruction space for the deformation problem. Smoothness follows immediately, since \( X_b \) being one-dimensional implies that \( H^2(X_b, \mathcal{O}_{X_b}) = 0 \). We also conclude that the (relative) dimension is \( g \), since \( h^1(X_b, \mathcal{O}_{X_b}) = g \) (this is Serre duality in the case \( X_b \) is nonsingular, and can be taken as the definition of \( g \) in the general case).

\(^2\)Recall that for us, a curve is always assumed geometrically reduced and connected.
In the case that $\pi$ is smooth, we write $\text{Pic}^0(X/B)$ instead of $\text{Pic}^0(X/B)$, since there is only one component for each fiber. We now observe that in this situation, we can easily use the valuative criterion for properness to prove that $\text{Pic}^0(X/B)$ is proper, and hence projective.

**Proposition A.4.4.** In the situation of Definition A.4.1, suppose further that $\pi$ is smooth. Then $\text{Pic}^0(X/B)$ is proper over $B$, and hence projective.

**Proof.** The existence of a $B$-ample line bundle is essentially the same as quasi-projectivity,\(^3\) so projectivity follows from properness, by Theorem 5.5.3 of [GD61]. We check properness via the valuative criterion. Let $R$ be a discrete valuation ring, with fraction field $K$, and suppose we have a morphism $\text{Spec} R \to B$, and a lift of this on $\text{Spec} K$ to a morphism to $\text{Pic}^0(X/B)$; that is, a line bundle $\mathcal{L}_\eta$ on $X$ of degree 0 on $X_\eta := X|_{\text{Spec} K}$. We want to show that there is an extension of $\mathcal{L}_\eta$ to a line bundle $\mathcal{L}$ on all of $X|_{\text{Spec} R}$ (note that we don’t need to show uniqueness, because we already have that $\text{Pic}^0(X/B)$ is separated). But there is some divisor $D$ on $X_\eta$ such that $\mathcal{L} = \mathcal{O}_{X_\eta}(D)$, and because $\pi$ is smooth and $\text{Spec} R$ is regular, we have that $X|_{\text{Spec} R}$ is regular, so we can just let $D'$ be the closure of $D$ in $X|_{\text{Spec} R}$, which is still a divisor. Thus, if we set $\mathcal{L} = \mathcal{O}_{X|_{\text{Spec} R}}(D')$, we obtain the desired extension of $\mathcal{L}_\eta$.

We now explore the situation in more detail in the case that the fibers of $\pi$ are all curves of compact type, beginning in the case of a (nodal) curve over a field. The following proposition is a somewhat technical formulation which describes how a line bundle on a (base change of) a nodal curve is determined by line bundles on each component of its normalization, together with gluing information.

**Proposition A.4.5.** Let $X$ be a projective nodal curve over a field $k$, and let $Y_1, \ldots, Y_n$ be the connected components of the normalization $\overline{X}$ of $X$. Let $Q_1, \ldots, Q_m$ be the nodes of $X$, and $P_1, P'_1, \ldots, P_n, P'_n$ their preimages in $\overline{X}$.

Given $T$ over $\text{Spec} k$, use subscript $T$ to denote base change to $T$. Given a line bundle $\mathcal{L}$ on $X_T$, pullback to each $(Y_i)_T$ gives a tuple $(\mathcal{L}_{1,T}, \ldots, \mathcal{L}_{n,T}, \varphi_1, \ldots, \varphi_m)$ consisting of a line bundle $\mathcal{L}_i$ on each $(Y_i)_T$, and isomorphisms

$$\varphi_j : \mathcal{L}_{j,T}(p_{j,T}) \simeq \mathcal{L}'_{j,T}(p'_{j,T})$$

for each $j = 1, \ldots, m$, where $P_j$ lies on $Y_j$, and $P'_j$ lies on $Y'_j$.

Then given also $\mathcal{L}'$ on $X$ inducing $(\mathcal{L}'_1, \ldots, \mathcal{L}'_n, \varphi'_1, \ldots, \varphi'_m)$, isomorphisms $\psi : \mathcal{L} \to \mathcal{L}'$ are in bijection with tuples of isomorphisms $\psi_i : \mathcal{L}_i \to \mathcal{L}'_i$ commuting with the $\varphi_j$ and $\varphi'_j$. Moreover, every tuple $(\mathcal{L}_1, \ldots, \mathcal{L}_n, \varphi_1, \ldots, \varphi_m)$ as above is isomorphic to one arising from a line bundle $\mathcal{L}$ on $X_T$.

**Proof.** The basic observation is that because sections of $\mathcal{O}_X$ correspond to tuples of sections of the $\mathcal{O}_{Y_i}$ which agree above the nodes, the same holds for $\mathcal{L}$, because it is isomorphic to $\mathcal{O}_X$ in a neighborhood of each node. It is clear that an isomorphism $\mathcal{L} \to \mathcal{L}'$ induces a tuple of isomorphisms as described, and that two distinct isomorphisms give rise to two distinct tuples. Given a tuple of isomorphisms $\psi_i$, we want to produce an isomorphism $\psi : \mathcal{L} \to \mathcal{L}'$, but we can do this using our explicit description of the sections of $\mathcal{L}$ and $\mathcal{L}'$, together with the fact that the

\(^3\)Depending on the precise definition used.
\[\psi_i\] are required to commute with the \(\varphi_j\) and \(\varphi'_j\), which means precisely that the morphisms \(\psi_i\) agree above each node of \(X\).

Finally, we show that a tuple can always be obtained from a line bundle \(L\) on \(X\). We do this by explicit construction: denote by \(\nu: X \to X\) the normalization map, and let \(\tilde{L}\) be the line bundle on \(X\) induced by the \(L_i\). For an open subset \(U \subseteq X\), define \(L(U)\) to be the subset of \(\tilde{L}(U)\) of sections whose values at \(P_j\) and \(P'_j\) agree under \(\varphi_j\) for each \(j\) with \(Q_j \in U\). This is clearly an invertible sheaf on the complement of the \(Q_j\). To check that it is also invertible at each \(Q_j\), we may choose an open neighborhood \(U_j\) of \(Q_j\) sufficiently small so that \(\tilde{L}\) may be trivialized on \(\tilde{U}_j := \nu^{-1}(U_j)\), and furthermore, we may choose the trivialization \(\tilde{L}|_{\tilde{U}_j} \sim O_{\tilde{U}_j}\) to be compatible with the map \(\varphi_j\), in the sense that it commutes with the canonical identification \(O_{\tilde{U}_j}|_{P_j} = O_{\tilde{U}_j}|_{P'_j}\). In this case, we see that our trivialization identifies \(L|_{U_j}\) with the sections of \(O_{\tilde{U}_j}\) whose values at \(P_j\) and \(P'_j\) agree, which is exactly our description of \(O_{U_j}\). Thus, \(L\) is invertible even at the nodes, and it is straightforward from the construction that it recovers the initial data of the \(L_i\) and \(\varphi_j\) under restriction, so we conclude the desired statement. \(\square\)

**Remark A.4.6.** From a stack-theoretic point of view, one can rephrase Proposition A.4.5 as saying that the stack of line bundles on \(X\) is simply the (2-) fibered product of the stacks of line bundles on each \(Y_i\) over the stacks of line bundles on the nodes.

The importance of considering not only objects, but also isomorphisms in the statement of Proposition A.4.5 is illustrated by the following corollary. One might think that different gluing maps \(\varphi_j\) would always lead to distinct line bundles on \(X\), but in fact we see that in the compact type case, the \(\varphi_j\) are completely irrelevant.

**Corollary A.4.7.** Let \(X\) be a curve of compact type over a field \(k\), and let \(Y_1, \ldots, Y_n\) be the irreducible components of \(X\). Then the natural pullback morphisms from \(X\) to each \(Y_i\) induce an isomorphism

\[
\text{Pic}^\delta(X/k) \sim \text{Pic}^0(Y_1/k) \times_k \cdots \times_k \text{Pic}^0(Y_n/k).
\]

**Proof.** Proposition A.4.5 lets us describe the functor \(\text{Pic}^\delta(X/k)\) in terms of line bundles of degree 0 on the \(Y_i\), together with appropriate gluing maps. To show injectivity of the natural map on \(T\)-valued points, we need only show that in fact the gluing maps are irrelevant, in the following sense: given tuples \((L_1, \ldots, L_n, \varphi_1, \ldots, \varphi_m)\) and \((L_1, \ldots, L_n, \varphi'_1, \ldots, \varphi'_m)\) as in Proposition A.4.5, there exist automorphisms \(\psi_j : L_i \sim L'_i\) which commute with the \(\varphi_j\) and \(\varphi'_j\).

Because the dual graph of \(X\) is a tree, without loss of generality we may assume that the \(Y_i\) are ordered so that for each \(i > 1\), the component \(Y_i\) intersects \(Z_{i-1} := Y_1 \cup \cdots \cup Y_{i-1}\) in exactly one point, where we have the gluing maps \(\varphi_j\) and \(\varphi'_j\). Since \((P_j)_T\) and \((P'_j)_T\) are sections, the gluing maps differ by an element of \(O^*_T\), and we may choose \(\psi_2\) to be the automorphism of \(L_2\) determined by the same element of \(O^*_T\), which gives the desired behavior. We then do the same for each successive \(Y_i\), using that it meets \(Z_{i-1}\) in exactly one node, so there is always exactly one condition that needs to be satisfied. Every node will occur exactly once, and we ultimately uniquely determine a choice of the \(\psi_1\) with the desired property (subject to the choice of setting \(\psi_1\) to be the identity). This proves injectivity.
Surjectivity is somewhat similar: we need to show that given any choice \((L_1, \ldots, L_n)\), there exists some choice of gluing maps \(\varphi_1, \ldots, \varphi_m\). The difficulty here occurs when \(T\) itself has nontrivial line bundles, since in this case we could have \(\mathcal{L}_{i}^{\ell} \mid (P_j^\ell)\) not isomorphic to \(\mathcal{L}_{i}^{\ell} \mid (P_j^\ell)\). However, because we are free to twist each \(L_i\) by (the pullback of) any element of \(\text{Pic}(T)\), we can proceed inductively just as in the proof of injectivity, twisting on one component at a time to produce isomorphisms one node at a time. \(\square\)

We take a moment to extend Corollary A.4.7 to the case of nodal curves not of compact type, giving rigorous proofs for the geometric assertions in Section 2.4. As a preliminary step, we have the following exercise, describing the functor of points of the punctured affine line.

**Exercise A.4.8.** Given a scheme \(S\), consider the functor \(F : \text{Sch}/S \to \text{Set}\) which sends \(T\) to \(\Gamma(T, O^*_{T})\). Show that when \(S = \text{Spec } \mathbb{Z}\), this functor is represented by \(\mathbb{G}_{m} := \text{Spec } \mathbb{Z}[t, t^{-1}]\), together with \(t \in \Gamma(\mathbb{G}_{m}, O^*_{\mathbb{G}_{m}})\). Show that in the general case, the functor is represented by \((\mathbb{G}_{m})_S := \mathbb{G}_{m} \times S \text{Spec } \mathbb{Z}\), together with the pullback of \(t\).

The following exercise addresses the general situation.

**Exercise A.4.9.** Let \(X\) be a nodal curve over \(k\), with dual graph \(\Gamma\). Choose an orientation of \(\Gamma\), and consider the morphism

\[
\prod_{v \in V(\Gamma)} (\mathbb{G}_{m})_k \to \prod_{e \in E(\Gamma)} (\mathbb{G}_{m})_k
\]

obtained by mapping \((g_v)_{v \in V(\Gamma)}\) to \((g'_e)_{e \in E(\Gamma)}\), where \(g'_e = g_{h(e)}g_{t(e)}^{-1}\). Show the following:

(a) The kernel of (A.4.1) is isomorphic to \((\mathbb{G}_{m})_k\), and the cokernel \(K\) is (non-canonically) the product of \(#E(\Gamma) - #V(\Gamma) + 1\) copies of \((\mathbb{G}_{m})_k\).

(b) Let \(Y_v\) for \(v \in V(\Gamma)\) be the components of the normalization of \(X\). Show that under the restriction morphism

\[
\text{Pic}^0(X/k) \to \prod_{v \in V(\Gamma)} \text{Pic}^0(Y_v/k),
\]

the preimage of the point \((\Theta_{Y_v})_{v \in V(\Gamma)}\) is isomorphic to \(K\), and more generally, the fibers of the restriction morphism are each (non-canonically) identified with \(K\).

(c) Conclude that \(X\) is of compact type if and only if \(\text{Pic}^0(X/k)\) is proper.

We now move on to considering the case of curves in families. The first result is the following properness statement.

**Corollary A.4.10.** In the situation of Theorem A.4.2, suppose further that the fibers of \(\pi\) are all curves of compact type. Then \(\text{Pic}^0(X/B)\) is proper, and in fact projective.

**Proof.** As in the proof of Proposition A.4.4, projectivity follows from properness. From Proposition A.4.4 and Corollary A.4.7, we see that because the fibers of \(\pi\) are of compact type, the fibers of \(\text{Pic}^0(X/B)\) are proper. We also observe that \(\text{Pic}^0(X/B)\) has a section, corresponding to the trivial line bundle, and it follows that \(\text{Pic}^0(X/B)\) is proper by Corollary 15.7.11 of [GD66]. \(\square\)
So far, we have restricted to the case of line bundles of degree 0 on all com-
ponents. However, we are ultimately interested in line bundles of higher degree
$d$, and – in the reducible case – varying multidegree. We conclude with an explanation of
how to deduce the arbitrary multidegree case from the case of $\text{Pic}^d(X/B)$, under
some mild additional hypotheses on $X$ over $B$.

We begin with the case that $\pi$ is smooth.

**Definition A.4.11.** In the situation of Definition A.4.1, suppose further that
$\pi$ is smooth. Then for any $d \in \mathbb{Z}$, define the degree-$d$ **Picard functor** by

$$\mathcal{P}ic^d(X/B)(T) = \text{Pic}^d_T(X \times_B T)/\mathbb{P}_2^d \text{Pic}(T),$$

where $\text{Pic}^d_T(X \times_B T)$ denotes the subgroup of $\text{Pic}(X \times_B T)$ consisting of line bundles
which have degree $d$ on every fiber of $p_2$.

**Proposition A.4.12.** In the situation of Definition A.4.11, we have that $\mathcal{P}ic^d(X/B)$
is represented by a scheme $\text{Pic}^d(X/B)$ together with a Poincare line bundle $\mathcal{L}$
on $X \times_B \text{Pic}^d(X/B)$. Moreover, $\text{Pic}^d(X/B)$ is (noncanonically) isomorphic to
$\text{Pic}^d(\mathcal{L})$ as a $B$-scheme.

**Proof.** Let $\sigma : B \to X$ be a section of $\pi$, which we have assumed to exist
in Definition A.4.1. Then the image of $\sigma$ is a divisor, which moreover has degree
1 in every fiber, so we have $\mathcal{O}_X(\sigma)$ an element of $\text{Pic}^d(X,B)$ over $B$. Then for
any $f : T \to B$, tensoring by $f^*(\mathcal{O}_X(\sigma))$ induces a map from $\mathcal{P}ic^d(X/B)(T)$ to
$\mathcal{P}ic^d(X/B)(T)$. This is inverted by tensoring with $f^*(\mathcal{O}_X(-\sigma))$, so we conclude
that the functors $\mathcal{P}ic^d(X/B)$ and $\mathcal{P}ic^d(\mathcal{L})$ are isomorphic. Thus, representabil-
ity of $\mathcal{P}ic^d(X/B)$ follows from Theorem A.4.2. \qed

It remains to discuss Picard schemes of fixed multidegree for families of nodal
curves. For this, we will make use of the terminology introduced in Section 4.2.

**Definition A.4.13.** Given an almost-local smoothing family $\pi : X \to B$, let
$\Gamma$ and $\text{cl}_b$ be as in Proposition 4.2.5. Given also a $B$-scheme $S$, a line bundle $\mathcal{E}$
on $X \times_B S$ has **multidegree** $\tilde{d} = \sum_{v \in V(\Gamma)} d_v$ if, for every $s \in S$ and every component
$Y$ of the fiber $(X \times_B S)_s$, we have

$$\text{deg } \mathcal{L}|_Y = \sum_{v \in V(\Gamma), \text{cl}_b(v) = v_Y} d_v,$$

where $b$ is the image of $s$ in $B$, and $v_Y$ is the vertex of $\Gamma_b$ corresponding to $Y$.

Given $\tilde{d}$, the degree-$\tilde{d}$ **Picard functor** $\mathcal{P}ic^{\tilde{d}}(X/B) : \text{Sch}/B \to \text{Set}$ is given by

$$\mathcal{P}ic^{\tilde{d}}(X/B)(T) = \text{Pic}^{\tilde{d}}_T(X \times_B T)/\mathbb{P}_2^{\tilde{d}} \text{Pic}(T),$$

where $\text{Pic}^{\tilde{d}}_T(X \times_B T)$ denotes the subset of $\text{Pic}(X \times_B T)$ consisting of line bundles
which have multidegree $\tilde{d}$.

Note that by hypothesis, the irreducible components of fibers of $\pi$ are geomet-
rically irreducible, so in the above definition, the dual graphs of $(X \times_B S)_s$ and $X_b$
are canonically identified. The basic result is then the following.

**Proposition A.4.14.** In the situation of Definition A.4.13, we have that $\mathcal{P}ic^{\tilde{d}}(X/B)$
is represented by a scheme $\text{Pic}^{\tilde{d}}(X/B)$ together with a Poincare line bundle $\mathcal{L}$.
Moreover, $\text{Pic}^\vec{d}(X/B)$ is (noncanonically) isomorphic to $\text{Pic}^\vec{d}(X/B)$ as a $B$-scheme.

**Proof.** Given $\vec{d}$, it is enough to prove that $\mathcal{P}ic^\vec{d}(X/B)(B) \neq \emptyset$, since if $\mathcal{L}$ is a line bundle of multidegree $\vec{d}$ on $X$, we can tensor by it to define an isomorphism

$$\mathcal{P}ic^\vec{d}(X/B) \xrightarrow{\sim} \text{Pic}^\vec{d}(X/B)$$

as in the proof of Proposition A.4.12. Let $d = \sum_e d_e$, and let $\sigma : B \to X$ be a section. Then $\mathcal{O}_X(d\sigma)$ defines a line bundle on $X$ having degree $d$ on fibers, and we claim that we can twist by combinations of $\mathcal{O}_{(e,v)}$ (see Notation 4.2.8) to obtain a line bundle of multidegree $\vec{d}$. Indeed, twisting by $\mathcal{O}_{(e,v)}$ has the effect of lowering the multidegree by 1 in index $v$ and raising it by 1 in index $v'$, where $v'$ is the other vertex adjacent to $e$. One easily checks by induction on $\# V(\Gamma)$ that, because $\Gamma$ is a tree, such operations act transitively on multidegrees of any fixed total degree, giving the desired statement.

We thus obtain a line bundle on $X$ of multidegree $\vec{d}$, and we can tensor by this to obtain the desired isomorphism $\text{Pic}^\vec{d}(X/B) \xrightarrow{\sim} \text{Pic}^\vec{d}(X/B)$. □

Putting together Theorem A.4.2, Corollary A.4.10 and Proposition A.4.14, we can summarize the main results of this section as follows.

**Corollary A.4.15.** In the situation of Definition A.4.13, there is a scheme $\text{Pic}^\vec{d}(X/B)$ together with a Poincare line bundle $\tilde{\mathcal{L}}$ on $X \times_B \text{Pic}^\vec{d}(X/B)$ having multidegree $\vec{d}$ and satisfying the universal property that for any $T \to B$, and any line bundle $\mathcal{L}$ on $X \times_B T$ having multidegree $\vec{d}$, there is a unique morphism $\varphi : T \to \text{Pic}^\vec{d}(X/B)$ over $B$ such that $\mathcal{L} \cong \varphi^* \tilde{\mathcal{L}} \otimes p_2^* \mathcal{M}$ for some $\mathcal{M} \in \text{Pic}(T)$.

Moreover, we have that $\text{Pic}^\vec{d}(X/B)$ is smooth and projective over $B$, with geometrically connected fibers of dimension $g$, where $g$ is the genus of the fibers of $\pi$.

**Remark A.4.16.** While it is usually a bad idea to try to avoid using stacks when confronted with a moduli functor which fails to satisfy the Zariski sheaf condition due to the presence of automorphisms, the case of $\mathcal{P}ic(X/B)$ is somewhat exceptional. Morally speaking, the fact that the automorphism groups of any two line bundles are the same means that the sheafification of the original functor will be representable, but possibly by an algebraic space.\footnote{In which case we should technically say that it is an algebraic space, rather than that it is represented by one.} The Picard functor is exceptional in that, under the mild additional hypotheses we have imposed, we have representability by a scheme, sheafification need only be carried out in the Zariski topology, and the process only involves modding out, rather than adding new elements from ones which exist locally on $T$ but can’t be patched together on all of $T$.

The reason for the hypothesis that $\pi$ has a section is precisely to ensure that this last statement holds – otherwise, we may have to add elements when sheafifying, and in particular not every element of $\mathcal{P}ic(X/B)(T)$ would come from an actual line bundle on $X \times_B T$. This in turn would mean that the universal object would not come from an actual line bundle on $X \times_B \text{Pic}(X/B)$, which would cause sufficient difficulties that it would probably be easier simply to work with the stack.
On the other hand, the reason (or at least one reason) for the hypothesis in the definition of smoothing family that the irreducible components of fibers be geometrically irreducible is that it ensures that our definition of multidegree makes sense. In fact, if we define a functor $\mathcal{P}ic(X/B)$ simply by omitting any degree restrictions, without the hypothesis on fibers it will in general not be representable by a scheme, but rather will only be an algebraic space. This issue is caused precisely by the poor behavior of multidegree in such families.
Dimension theory for schemes and morphisms

In this appendix, we discuss various aspects of the theory of dimension of schemes, focusing our attention on how to phrase results to avoid the various pathologies that arise. We begin with a brief survey of pathologies, useful hypotheses, and positive results. We then describe a notion a morphism having at least a given relative dimension; this will behave well quite generally, and will be the most natural language in which to state the foundational dimension results of limit linear series theory.

We will work throughout with locally Noetherian schemes, to maintain some degree of good behavior.

B.1. Dimension and schemes: a concise survey

We begin by recalling basic definitions:

**Definition B.1.1.** Let $X$ be a locally Noetherian scheme. Then the *dimension* $\dim X$ of $X$ is the supremum over $n$ such that there exists a chain

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

of irreducible closed subsets of $X$. If the supremum does not exist, we say that $X$ is infinite-dimensional.

Given also a closed subscheme $Z$ of $X$, the *codimension* $\text{codim}_X Z$ of $Z$ in $X$ is the infimum over irreducible components $Z'$ of $Z$ of the supremum over $n$ such that these exists a chain

$$Z' \subsetneq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

of irreducible closed subsets of $X$.

See Remark B.1.16 below for a discussion of the infimum in the definition of codimension.

The model for a well-behaved dimension theory is that of varieties over a field. In this case, we have a number of desirable properties:

- Every variety $X$ has a finite dimension.
- If $X$ is a variety and $U \subseteq X$ an open subvariety, then $\dim X = \dim U$.
- If $X$ is a variety and $Z$ a closed subvariety, then $\dim X = \dim Z + \text{codim}_X Z$.
- If $X$ is a variety, and $Z_1 \subseteq Z_2$ closed subvarieties, then $\text{codim}_X Z_1 = \text{codim}_X Z_2 + \text{codim}_{Z_2} Z_1$.\(^1\)

\(^1\)Here the irreducibility hypotheses are critical; it is easy to construct counterexamples when either of $Z_2$ or $X$ are not assumed irreducible.
In the above, we assume that all varieties and subvarieties are nonempty and irreducible.

As soon as we move away from varieties, most of the above properties immediately begin to fail as stated. We begin with an example which illustrates how even the most innocuous deviation from being of finite type over a field can create problems.

**Example B.1.2.** Let \( R = k[t] \) be the local ring of \( \mathbb{A}^1_k \) at the origin, and consider \( X = \mathbb{A}^1_k \times_k \text{Spec } R \) where \( \text{Spec } R = \mathbb{A}^1_R = \text{Spec } k[x, t] \). Then \( \dim X = 2 \). However, the complement of the zero set of \( t \) is an open subset \( U \) which is equal to \( \mathbb{A}^1_K \), where \( K \) is the fraction field of \( R \), so we see that \( U \) has dimension 1, which is strictly smaller than \( \dim X \).

Next consider the zero set \( Z \) of \( xt - 1 \). This has codimension 1 in \( X \), but it consists of a single point, supported in \( U \), and therefore \( \dim Z = 0 \). Thus \( \dim X > \dim Z + \text{codim}_X Z \).

In addition, an example of Nagata (see Exercise 9.6, p. 230 of [Eis95]) gives a Noetherian (indeed, regular) affine scheme which has infinite dimension (however, we will see in Corollary B.1.5 below that we will never have infinite codimension). Putting these examples together, we conclude that dimension is simply poorly behaved for even mild generalizations of varieties. The problem in these examples is that we may get different lengths for maximal chains of irreducible closed subsets, depending on where the chain starts.

At the same time, there are Noetherian examples where the last condition fails, meaning that even if we fix where a given chain starts we may have maximal chains of different lengths; However, it turns out that such examples are rarer, and can be easily avoided. This is addressed by the catenary condition, discussed below.

We now turn towards positive statements and results. First, because irreducible closed subsets of schemes have unique generic points, we observe that dimension and codimension can also be phrased in terms of chains of specializations of points. We then conclude the following:

**Proposition B.1.3.** If \( Z \) is a closed subscheme of a locally Noetherian scheme \( X \), then

\[
\text{codim}_X Z = \inf_{\eta} \dim \mathcal{O}_{X, \eta},
\]

where \( \eta \) ranges over generic points of irreducible components of \( Z \). In addition, if \( U \subseteq X \) is an open subscheme meeting every irreducible component of \( Z \), then

\[
\text{codim}_X Z = \text{codim}_U (Z \cap U).
\]

Note that because of the infimum in the definition of codimension, we could equally well have considered all closed irreducible subsets of \( Z \) in the definition, and correspondingly all points of \( Z \) in Proposition B.1.3, as is done in §II.3 of [Har77]. Thus, unlike dimension, codimension can be considered a local property.

The foundational result in dimension theory is the Krull principal ideal theorem, which holds for any Noetherian ring. We include a converse statement as well.

**Theorem B.1.4 (Krull).** Let \( R \) be a Noetherian ring, and \( P \) a prime ideal. If \( P \) is minimal over an ideal generated by some \( f_1, \ldots, f_c \in R \), then \( P \) has codimension at most \( c \). Conversely, if \( P \) has codimension \( c \), then there exist \( f_1, \ldots, f_c \in R \) such that \( P \) is minimal over the ideal generated by the \( f_i \).
The proof has some substance, but can be done in a couple of pages; see Theorem 10.2 and Corollary 10.5 of [Eis95]. In terms of schemes, it says that if a closed subscheme $Z$ of a locally Noetherian scheme $X$ can (locally) be cut out by $c$ equations, then every component of $Z$ has codimension at most $c$. We observe the following immediate consequence.

**Corollary B.1.5.** Noetherian local schemes are finite-dimensional. Any closed subscheme of a locally Noetherian scheme has finite codimension.

**Proof.** In the local case, the dimension of the local ring is the codimension of the maximal ideal, which by Theorem B.1.4 is bounded by the number of generators. In the locally Noetherian case, Proposition B.1.3 implies that the codimension is an infimum over dimension of Noetherian local schemes, so is likewise finite. □

These statements suggest that in moving from varieties to schemes, one often gets better behavior by considering codimension than by considering dimension. To fully explore this, we introduce the following technical condition.

**Definition B.1.6.** We say that a scheme $X$ is catenary if, given any $Z_1 \subset Z_2$ integral closed subschemes of $X$, any two maximal chains of integral closed subschemes of $X$ containing $Z_1$ and contained in $Z_2$ have the same length. We say that $X$ is universally catenary if $X$ is locally Noetherian and every scheme $Y$ locally of finite type over $X$ is catenary.

It is easy to check the following:

**Proposition B.1.7.** Let $X$ be catenary and irreducible. Then for any irreducible closed subschemes $Z_1 \subset Z_2$ of $X$, we have

$$\text{codim}_X Z_2 + \text{codim}_{Z_2} Z_1 = \text{codim}_X Z_1.$$  

Thus, the catenary condition ensures that codimension behaves as well as could be hoped. Importantly, the catenary condition is satisfied quite generally. Indeed, we have the following:

**Theorem B.1.8.** Let $X$ be locally of finite type over a (locally) Cohen-Macaulay scheme. Then $X$ is universally catenary.

**Proof.** See Theorem 17.9 of [Mat86]. □

Thus, being catenary is a very mild condition. In particular, any scheme of finite type over a field or any other regular scheme is catenary. Noting that the catenary condition can be checked on the level of local rings, we also have that any scheme whose local rings can be imbedded in regular local rings is catenary. It follows that the spectrum of a complete local ring is always catenary, by Theorem 29.4 of [Mat86].

Finally, we recall some results on dimension in relation to morphisms, beginning with statements relating to fibers. Chevalley proved the following theorem on semicontinuity of fiber dimension.

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$^2$Note that, using Nakayama’s lemma to see that generators of the maximal ideal are equivalent to generators of the Zariski cotangent space, this argument also implies the fact that the dimension of the Zariski cotangent space of a Noetherian local ring is always at least the dimension of the ring.
Theorem B.1.9. Let \( f : X \to Y \) be a morphism locally of finite type. For any \( n \geq 0 \), the locus \( F_n(X) := \{ x \in X : \dim_x(f^{-1}(f(x))) \geq n \} \) is closed in \( X \), where \( \dim_x \) denotes the dimension of any components meeting \( x \).

In particular, if also \( f \) is closed, the locus \( F_n(Y) := \{ y \in Y : \dim f^{-1}(y) \geq n \} \) is closed.

See Theorem 13.1.3 and Corollary 13.1.5 of [GD67].

We also have the following statement relating dimensions to fiber dimension.

Theorem B.1.10. Let \( f : X \to Y \) be a morphism and \( x \in X \) any point. Then

\[
\dim \mathcal{O}_{X,x} \leq \dim \mathcal{O}_{Y,f(x)} + \dim \mathcal{O}_{f^{-1}(f(x)),x},
\]

with equality holding if \( f \) is open.

See Theorem 10.10 of [Eis95] for the inequality, and equality in the case that \( f \) is flat; the more general statement is Theorem 14.2.1 of [GD66].

The following result is sometimes called the “dimension formula.”

Proposition B.1.11. Let \( Y \) be a universally catenary scheme, and \( f : X \to Y \) a morphism locally of finite type. Let \( Z' \subseteq Z \) be irreducible closed subschemes of \( X \), and let \( \eta' \) and \( \eta \) be their respective generic points. Then

\[
\operatorname{codim}_X Z - \operatorname{codim}_{f^{-1}(f(x))} f^{-1}(Z) = \dim Z_{f(x)} - \dim Z'_{f(\eta')}.
\]

See Proposition 5.6.5 of [GD65].

We conclude with a pair of statements bounding how codimension can change under intersection and preimage. The first theorem is a difficult result of Serre, using his theory of intersection multiplicities.

Theorem B.1.12. Let \( Z_1, Z_2 \) be irreducible subschemes of an irreducible regular scheme \( X \), and \( Z \) any irreducible component of \( Z_1 \cap Z_2 \). Then \( \operatorname{codim} Z \leq \operatorname{codim} Z_1 + \operatorname{codim} Z_2 \).

See Theorem V.3 of [Ser65] for the statement in terms of local rings.

Hochster was then able to reduce the following more general statement to Serre’s theorem.

Theorem B.1.13. Let \( f : X \to Y \) be a morphism of irreducible schemes, with \( Y \) regular, and \( Z \) an irreducible closed subscheme of \( Y \). Then any irreducible component \( Z' \) of \( f^{-1}(Z) \) has

\[
\operatorname{codim}_X Z' \leq \operatorname{codim}_Y Z.
\]

See Theorem 7.1 of [Hoc75].

Note that, since \( Y \) being regular implies that it is catenary, Theorem B.1.12 is the special case of Theorem B.1.13 in which \( f \) is a closed immersion.

Remark B.1.14. The above results are much easier to prove in the case that the base scheme is smooth over a field. Indeed, if \( Y \) is smooth over a field, then the diagonal \( \Delta(Y) \subseteq Y \times Y \) is locally cut out by \( \dim Y \) equations. We can then realize \( f^{-1}Z \) as the preimage of the diagonal under the morphism \( X \times Z \to Y \times Y \) induced by \( f \), and the desired inequality follows from Theorem B.1.4.

Example B.1.15. Note that the regularity is a vital hypothesis Theorems B.1.12 and B.1.13 to be true: if \( X \) is a cone over a quadric surface, and \( Z_1, Z_2 \) are cones corresponding to two distinct lines in one ruling of the surface, then
each has codimension 1, but their intersection is only at the cone point, which has codimension 3.

**Remark B.1.16.** The infimum in the definition of codimension may seem mysterious, but there are two basic reasons for using it. The simplest is that, if $X$ is a variety over a field (in particular, irreducible), and $Z$ is a closed algebraic subset, then this definition renders the identity $\dim Z + \operatorname{codim}_X Z = \dim X$ valid. If we had replaced the infimum in the definition with a supremum, this identity would fail whenever $Z$ had components of distinct dimensions.

Another reason is that one is frequently interested in lower bounds on codimension, for instance in a hypothesis that the singular locus of a variety have codimension at least 2, or in the statement that the undefined locus of a rational map between nonsingular projective varieties has codimension at least 2. In both these cases, we mean that there is no component of codimension 1, and because of our choice of definition, the blanket statement on codimension implies that the bound holds for every component.

As it happens, in this book we will be interested almost exclusively in upper bounds on codimension, so to obtain optimal statements we will frequently make statements in terms of every component of a given algebraic subset or subscheme; the Krull principal ideal theorem is the first case of this, and Serre’s theorem (along with Hochster’s generalization) is the second.

### B.2. Relative dimension for morphisms

We now introduce a notion of a morphism having at least a certain relative dimension. The reason for doing so is as follows: in the context of limit linear series, we want to prove “smoothing theorems” via dimension counts. That is, we will show that if we have a suitable family $\pi: X \to B$ of curves, then the relative (limit) linear series moduli space $G^r_d(X/B)$ has dimension at least $\rho$ greater than $B$. Then, if a special fiber corresponding to limit linear series on a curve $X_0$ has dimension exactly $\rho$, we conclude that no component of this fiber can be a component of $G^r_d(X/B)$, so that every limit linear series on $X_0$ must be in the closure of linear series on other (smooth) curves in the family. This is made precise in Theorem 4.4.10 and its corollaries. There is no difficulty in making the above precise when $B$ is of finite type over a field, but as soon as we generalize even slightly – for instance, to letting $B$ be the spectrum of a discrete valuation ring – standard language is lacking. Examples such as Example B.1.2 show that we cannot be certain that we will have $\dim G^r_d(X/B) \geq \dim B + \rho$. We can work instead with codimension, but this is unnatural because the intuition is fundamentally about dimension rather than codimension, and while the space $G^r_d(X/B)$ is canonical, the ambient space in which it is constructed is not. We could also work with dimensions local rings, as in Theorem 5.3 of [Oss06], but this is also rather awkward.\(^3\) We therefore elect instead to develop a robust notion of what it should mean for a scheme to have at least a certain relative dimension over another.

These ideas are taken from [Oss13], and are therefore not yet standard. Because the main application for us will be to smoothing theorems for limit linear series, after making the definition and some preliminary observations, we explain

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\(^3\)Additionally, this approach does not generalize to stacks, which becomes important in the higher-rank case.
in Propositions B.2.7 and B.2.8 how one deduces smoothing statements from relative dimensions.

The definition is the following:

**Definition B.2.1.** Let \( f : X \to Y \) be a morphism locally of finite type of universally catenary schemes. We say that \( f \) has **relative dimension at least** \( n \) if for any irreducible closed subscheme \( Y' \) of \( Y \), and any irreducible component \( X' \) of \( X_{|Y'} \), with generic point \( \eta \), we have

\[
\dim X'_{f(\eta)} - \operatorname{codim}_{Y'} f(X') \geq n.
\]

We say that \( f \) has **universal relative dimension at least** \( n \) if for all universally catenary \( Y \)-schemes \( S \), the base change \( X \times Y S \to S \) has relative dimension at least \( n \).

**Remarks B.2.2.**

(i) The terminology is justified by the observation that if \( Y \) is of finite type over a field, then (B.2.1) is equivalent to \( \dim X'_{f(\eta)} - \operatorname{dim} Y' \geq n \). Although the latter expression is independent of \( f \), the property of having relative dimension at least \( n \) may still depend on \( f \) in general; see Remark B.3.8.

(ii) If \( f : X \to Y \) has relative dimension at least \( n \), then we see immediately from the definition that every irreducible component of every fiber of \( f \) must have dimension at least \( n \); see Proposition B.2.5 below for a stronger version of this statement.

(iii) Note that here we allow \( n \) to be negative, and even then the condition is not vacuous; see Example B.2.4.

(iv) Universally catenary schemes are by definition closed under morphisms which are locally of finite type, so we have in particular that if a morphism has universal relative dimension at least \( n \), then every locally finite type base change has relative dimension at least \( n \).

(v) Although fiber dimension is upper semicontinuous, the same is not true for our definition of relative dimension. See Example B.3.6 below.

We begin with two classes of examples of morphisms satisfying Definition B.2.1.

**Example B.2.3.** If \( f \) is universally open with every generic fiber having every component of dimension at least \( n \) (in particular, if \( f \) is smooth of relative dimension \( n \) or larger), then \( f \) has universal relative dimension at least \( n \). Indeed, because of semicontinuity of fiber dimension, the condition is preserved under base change (including under restriction to irreducible subschemes of the base), so this follows immediately from the fact that open morphisms map generic points to generic points.

**Example B.2.4.** For closed immersions, the first difference between relative dimension and codimension is that codimension is defined to be \( c \) if the minimum codimension over all irreducible components of the source is equal to \( c \), while the relative dimension is at least \( -c \) only if the maximum codimension over all irreducible components of the course is (at most) \( c \). More generally, for closed immersions relative dimension measures intersection codimension behavior, and universal relative dimension measures “superheight,” introduced by Hochster in §7 of [Hoc75].

In particular, if \( f : X \to Y \) is a closed immersion, \( Y \) is regular, and every component of \( X \) has codimension at most \( c \) in \( Y \), it is a theorem of Serre (Theorem V.3 of [Ser65]) that \( f \) has relative dimension at least \( -c \), and building on Serre’s
theorem Hochster proved in Theorem 7.1 of [Hoc75] that \( f \) has universal relative dimension at least \(-c\).

In a different direction, if \( Y \) is universally catenary, and everywhere locally \( X \) is cut out by at most \( c \) equations in \( Y \), then \( f \) has universal relative dimension at least \(-c\) by Krull’s principal ideal theorem. More generally, if \( X \) is everywhere locally cut out by the vanishing of the \((k + 1) \times (k + 1)\) minors of an \( n \times m \) matrix, then \( f \) has universal relative dimension at least \(-(n - k)(m - k)\) (see Exercise 10.4 of [Eis95]). Further, because \( Y \) is catenary, and such descriptions are preserved under restriction, we see that the same also holds for intersections of closed subschemes of the above forms, if we sum the corresponding relative dimensions.

Finally, we note that since our definitions are blind to non-reduced structure, it is in fact enough for the above descriptions of \( X \) to hold up to taking reduced structures. In particular, if \( Y \) has pure dimension \( c \) at a closed point \( x \), then setting \( X = \{x\} \) we find that \( X \hookrightarrow Y \) has universal relative dimension at least \(-c\).

We next make some preliminary observations:

**Proposition B.2.5.** If \( f : X \to Y \) has relative dimension at least \( n \), then for any irreducible closed subset \( Y' \) of \( Y \), and any irreducible component \( X' \) of \( f^{-1}(Y') \), for every \( x \in X' \) we have

\[
\dim_x X'_{f(x)} - \text{codim}_{Y'} f(X') \geq n.
\]

**Proof.** Observing that in Definition B.2.1, we have \( X'_{f(\eta)} \) irreducible and hence equidimensional, the statement follows immediately from semicontinuity of fiber dimension.

**Proposition B.2.6.** If \( f : X \to Y \) has relative dimension at least \( n \), and \( U \subseteq X \) is a nonempty open subset, then the induced morphism \( U \to Y \) has relative dimension at least \( n \). Conversely, if \( \{U_i\} \) is an open cover of \( X \) and each morphism \( U_i \to Y \) has relative dimension at least \( n \), then \( X \to Y \) has relative dimension at least \( n \).

The same statements hold for universal relative dimension.

**Proof.** The first statement is clear from the definition, since \( X'_{f(\eta)} \) is irreducible and of finite type over a field, so its dimension doesn’t change when restricting to open subsets.

The second statement follows similarly: if \( \{U_i\} \) is an open cover of \( X \), then some \( U_i \) contains the generic point \( \eta \) of \( X' \), and both the dimension and codimension in (B.2.1) are unchanged by restriction to \( U_i \).

The universal case follows immediately, since open subsets (respectively, open covers) are preserved under base change.

We conclude this section with two propositions describing the main applications of Definition B.2.1; in the context of moduli spaces, they constitute “smoothing theorems” based on dimension counts and analysis of special fibers. See Corollaries 4.4.11 and 4.4.12 for an example of this. The strongest statements occur in the case of nonnegative relative dimension.

**Proposition B.2.7.** Given \( f : X \to Y \), suppose that \( f \) has relative dimension at least \( n \), and there exists \( x \in X \) such that the fiber \( X_{f(x)} \) has dimension \( n \) at \( x \).
Then there exists a neighborhood $U$ of $x$ on which $f$ pure fiber dimension $n$, and on any such neighborhood $f$ is open.

If further $f$ has universal relative dimension at least $n$, then $f$ is universally open on $U$.

If further $Y$ is reduced and the fiber of $f$ is geometrically reduced at $x$, then $f$ is flat at $x$.

Note that this proposition can also be viewed as a complement to the standard criterion for flatness in terms of fiber dimension in the case of a Cohen-Macaulay scheme over a regular scheme, or alternatively, as a complement to Chevalley’s criterion (Corollary 14.4.4 of [GD66]), which asserts that if $Y$ is geometrically unibranch, then equidimensionality implies universal openness.

**Proof.** Applying Proposition B.2.5 and semicontinuity of fiber dimension, there exists an open neighborhood $U$ of $x$ on which $f$ also has pure fiber dimension $n$. We claim that $f$ is necessarily open on $U$. Let $U'$ be any non-empty open subset of $U$; by Proposition B.2.6, we have that $f$ has relative dimension at least $n$ and pure fiber dimension $n$, so it is enough to show that the image of such a morphism is open. Since the morphism is locally of finite type, by Corollary 1.10.2 of [GD64] it is enough to show that the image is closed under generization. Let $y$ specialize to $y'$ in $Y$, with $y'$ in the image of $f$, say $y' = f(x')$. Set $Y'$ to be the closure of $y$, and $X'$ any irreducible component of $f^{-1}(Y')$ containing $x'$. Then from the definition of having relative dimension at least $n$, together with the hypothesis that the fibers have dimension $n$, we conclude that $X'$ dominates $Y'$, and in particular $y$ is in the image of $f$, as desired.

Next, if $f$ has universal relative dimension at least $n$, we conclude that $f$ is universally open on $U$ because according to Corollary 8.10.2 of [GD66] we may check that a morphism is universally open after finite type base change.

Finally, if the fiber of $f$ is geometrically reduced at $x$, and $Y$ is reduced, flatness of $f$ follows from Theorem 15.2.2 of [GD66]. □

It is clear that Proposition B.2.7 will never apply in the case of negative relative dimension. Because it is sometimes important for applications, we also state a weaker version which works even when the relative dimension is negative.

**Proposition B.2.8.** Given $f : X \to Y$, suppose that $Y$ is irreducible, and there exist $x \in X$ and $Y' \subseteq Y$ closed and irreducible containing $f(x)$ and with support strictly smaller than $Y$, such that

(I) $f$ has relative dimension at least $n$;

(II) every irreducible component $X'$ of $f^{-1}(Y')$ containing $x$ has

$$\dim X'_{f(\eta)} - \text{codim}_{Y'} f(X') = n,$$

where $\eta$ is the generic point of $X'$.

Then for every irreducible component $X''$ of $X$ containing $x$, we have

$$f(X'') \not\subseteq Y'.$$

If further we have

(III) $Y'$ has codimension $c$ in $Y$, and the inclusion $Y' \hookrightarrow Y$ has universal relative dimension at least $-c$,
then we have
\[ \dim X''(\eta) - \codim_Y \overline{f(X'')} = n, \]
where \( \eta' \) is the generic point of \( X'' \).

**Proof.** Given \( X'' \) an irreducible component of \( X \) containing \( x \), let \( \eta' \) be its generic point. By (I), we have
\[ \dim X''(\eta) - \codim_Y \overline{f(X'')} \geq n. \]
If we had \( f(X'') \subseteq Y' \), then \( X'' \) would also be an irreducible component of \( f^{-1}(Y') \), so by (II) we would have
\[ \dim X''(\eta') - \codim_{Y'} \overline{f(X'')} = n. \]
But (in light of our running catenary hypotheses) this contradicts the hypothesis that \( \codim_Y Y' > 0 \). Thus the first assertion holds.

Next suppose that (III) is also satisfied, and let \( X' \) be an irreducible component of \( f^{-1}(Y') \) containing \( x \) and contained in \( X'' \), and \( \eta \) its generic point. Then, since we already have the opposite inequality, we want to show that
\[ \dim X''(\eta) - \codim_Y \overline{f(X'')} \leq n = \dim X'(\eta) - \codim_{Y'} \overline{f(X')} \]
But applying the catenary hypothesis together with the hypothesis on the universal relative dimension of \( Y' \hookrightarrow Y \), we have
\[ \codim_{Y'} \overline{f(X')} - \codim_Y \overline{f(X'')} = \codim_Y \overline{f(X')} - \codim_Y \overline{f(X'')} \leq \codim_Y \overline{f(X')} - \codim_{X''} X' - \codim_{Y'} \overline{f(X'')} = \codim_{Y''} \overline{f(X')} - \codim_{X''} X'. \]
But the catenary hypothesis also allows us to use the dimension formula, Proposition B.1.11 to conclude that the righthand side above is equal to \( \dim X'(\eta) - \dim X''(\eta') \), yielding the desired inequality.

**Remark B.2.9.** We elaborate slightly on the hypotheses and conclusions of Proposition B.2.8. First, we observe that under the hypotheses of the proposition, although \( \dim X'(\eta) - \codim_{Y'} \overline{f(X')} \) remains unchanged when we replace \( X' \) by \( X'' \), the separate terms need not. Indeed, the map \( \mathbb{A}^2 \rightarrow \mathbb{A}^2 \) given by \((x, y) \mapsto (x, xy)\) has dimension dimension at least 0, and if we restrict to the line \( x = 0 \) in the target, we get that the \( \dim X'(\eta) = \dim X''(\eta') = 1 \), so the proposition applies. However, without restricting (and letting \( \eta' \) be the generic point of \( X \)), we have \( \dim X'(\eta) = \dim X''(\eta') = 0 \). This behavior is discussed further in Remark B.3.9.

The final conclusion of Proposition B.2.8 may seem superfluous in the context of smoothing arguments, but in fact it plays an important role in inductive arguments. See for instance the proof of Theorem 4.5.8. We see from the second part of Example B.3.6 that for this final conclusion, hypothesis (III) is indeed necessary. In the notation of that example, consider the map
\[ \overline{C} \times \overline{C} \setminus \Delta \rightarrow C \times \overline{C}. \]
This has relative dimension at least \(-1\), and if we restrict to \( Z \), we find that the conditions of Proposition B.2.8 are satisfied with \( n = -1 \), except that \( Z \) does not.
have relative dimension at least $-1$ in $C \times \tilde{C}$. However, in this case (again setting $\eta'$ to be the generic point of $X$) we have $\dim X_{f(\eta)} - \text{codim}_Y \overline{f(X)} = 0$.

### B.3. Formal properties of relative dimension

In this section, we investigate the formal behavior of relative dimension, including behavior with respect to compositions in general, and with respect to composition with smooth morphisms.

Our first observation is the following.

**Proposition B.3.1.** Suppose that $f : X \to Y$ is smooth and surjective of relative dimension $m$, and $g : Y \to Z$ a morphism with $Z$ universally catenary. Then $g$ has relative dimension at least $n$ if and only if $g \circ f$ has relative dimension at least $n + m$. Additionally, the same statement holds for universal relative dimension.

**Proof.** Because smoothness is preserved under base change, and composition commutes with base change, the universal statement follows immediately from the non-universal statement. Next, observe that if $g \circ f$ is locally of finite type then $g$ is locally of finite type by Lemma 17.7.5 of [GD67].

Now, if $Z'$ is an irreducible closed subscheme of $Z$, then by smoothness, every component $X'$ of $X|_{Z'}$ dominates a component $Y'$ of $Y|_{Z'}$, and by surjectivity of $f$, every component $Y''$ of $Y|_{Z'}$ is dominated by a component $X'$ of $X|_{Z'}$. Thus, given such $X'$ and $Y'$, having generic points $\eta$ and $\xi$ respectively, we have $g(f(\eta)) = g(\xi)$, and by smoothness $\dim X'_{g(f(\eta))} = \dim Y'' + m$; the desired statement follows. □

Now we come to the more substantial statement that Definition B.2.1 behaves well with respect to composition.

**Lemma B.3.2.** Suppose that $f : X \to Y$ has relative dimension at least $m$, and $g : Y \to Z$ has relative dimension at least $n$. Then $g \circ f$ has relative dimension at least $m + n$.

The same holds for universal relative dimension.

**Proof.** The universal statement follows immediately from the non-universal statement, since composition commutes with base change.

For the non-universal statement, let $Z'$ be an irreducible component of $Z$, and $X'$ an irreducible component of $X|_{Z'}$, with generic point $\eta$; then we wish to show that

$$\dim X'_{g(f(\eta))} - \text{codim}_{Z'} g(f(X')) \geq m + n.$$  

Let $Y'$ be an irreducible component of $Y|_{Z'}$ containing $f(\eta)$, with generic point $\xi$.

$$\eta \in X' \quad \xrightarrow{f} \quad X$$  

$$\xi \in Y' \quad \xrightarrow{g} \quad Y$$  

Then $X'$ is an irreducible component of $X|_{Y'}$, so we have

$$\dim X'_{f(\eta)} - \text{codim}_{Y'} \overline{f(X')} \geq m, \quad \text{and} \quad \dim Y'_{g(\xi)} - \text{codim}_{Z'} g(Y') \geq n.$$
Now, if $\zeta$ is any generic point of an irreducible component $Y''$ of the fiber $Y'_{g(\eta)}$ with $f(\eta) \in Y''$, note that $g(\zeta) = g(f(\eta))$.

\[
\begin{array}{c}
\xymatrix{ X'_{f(\eta)} & X'_{g(\eta)} \\
\zeta & Y'' \\
\downarrow & \\
\zeta & Y'_{g(\eta)} }
\end{array}
\]

The relationship between $\dim X'_{g(\eta)}$ and $\dim X'_{f(\eta)}$ is given by

\[
\dim X'_{g(\eta)} = \dim X'_{f(\eta)} + \dim Y'_{g(\eta)} - \dim Y'_{f(\eta)} + \dim X'_{g(\eta)}.
\]

Adding the two inequalities and using the above equation, we find that

\[
\dim X'_{g(\eta)} - \dim Y'_{g(\eta)} + \dim Y'_{f(\eta)}\quad \text{and} \quad \dim Y'_{g(\eta)} - \dim Y'_{f(\eta)} + \dim X'_{g(\eta)} \geq m + n.
\]

We now apply our catenary hypothesis, first to conclude that

\[
\dim Y'_{g(\eta)} = \dim Y'_{f(\eta)} + \dim \pi_{Y'_{g(\eta)}} - \dim \pi_{Y'_{f(\eta)}},
\]

and second, to apply the dimension formula, Proposition B.1.11, to the morphism $Y' \to \overline{g(Y')}$ at the point $\zeta$, concluding that

\[
\dim Y'' = \dim Y'_{g(\eta)} + \dim Y'_{f(\eta)} - \dim Y'_{g(\eta)}.
\]

Putting these three equations together with the previous inequality gives the desired inequality. □

Using Lemma B.3.2 and the standard Grothendieck six conditions argument (see for instance Remark 5.5.12 of [GD60]), we see that Definition B.2.1 also behaves well under post-composition by a smooth morphism.

**Corollary B.3.3.** Given morphisms $f : X \to Y$ and $g : Y \to Z$, suppose $g$ is smooth of relative dimension $n$, with $Z$ universally catenary. Then $f$ has universal relative dimension at least $m$ if and only if $g \circ f$ has universal relative dimension at least $m + n$.

**Proof.** The “only if” direction is immediate from Lemma B.3.2. For the converse, first observe that $Y$ is universally catenary, and the smoothness of $g$ implies the diagonal $\Delta_g : Y \to Y \times_Z Y$ has universal relative dimension at least $-n$, because if we factor $\Delta_g$ as a closed immersion followed by an open immersion, the closed immersion is locally cut out by $n$ equations (See Proposition 2.2.7 of [BLR91]). Now, if $g \circ f$ has universal relative dimension at least $m + n$, then the second projection $p_2 : X \times_Z Y \to Y$ is the base change of $g \circ f$ to $Y$, so likewise has universal relative dimension at least $m + n$. On the other hand, the graph morphism $\Gamma_f : X \to X \times_Z Y$ is the base change of $\Delta_g$ under $f \times \text{id}$, so it has universal relative dimension at least $-n$. Since $f = p_2 \circ \Gamma_f$, we conclude that $f$ has universal relative dimension at least $m$ from Lemma B.3.2. □

As another consequence of Lemma B.3.2, we obtain the following wide classes of examples.
Corollary B.3.4. Suppose that \( f : X \to Y \) is a closed immersion, and \( g : Y \to Z \) is smooth of relative dimension \( n \), with \( Z \) universally catenary. If either \( Z \) is regular and every component of \( X \) has codimension at most \( c \) in \( Y \), or \( X \) may be expressed locally as an intersection of determinantal conditions with expected codimensions adding up to \( c \), then \( g \circ f \) has universal relative dimension at least \( n - c \).

Alternatively, suppose that \( f : X \to Y \) is a morphism of smooth \( S \)-schemes, with \( S \) universally catenary, and \( m \) and \( n \) the relative dimensions of \( X \) and \( Y \) over \( S \), respectively. Then \( f \) has universal relative dimension at least \( m - n \).

Again, we mention that universal relative dimension is insensitive to non-reduced structures, so in fact the descriptions of the corollary only need to hold up to taking reduced structures in order to conclude the desired statements.

Proof. The first statement is a direct consequence of Lemma B.3.2, together with Examples B.2.3 and B.2.4. For the second, we observe that \( f \) is necessarily locally of finite type, so \( X \) can locally be written as a closed subscheme of \( \mathbb{A}^N_Y \). Then, because \( X \) is smooth over \( S \) of relative dimension \( m \), and \( \mathbb{A}^N_Y \) is smooth over \( S \) of relative dimension \( N + n \), we have that \( X \) is everywhere locally cut out by \( N + n - m \) equations inside \( \mathbb{A}^N_Y \) by Proposition 2.2.7 of [BLR91], so we have reduced the second statement to the first.

We conclude our examination of relative dimension with positive and negative examples, and a number of remarks on various aspects of the subject.

Example B.3.5. If \( Y \) is a smooth variety over a field, and \( f : X \to Y \) is a blowup with smooth center, the second part of Corollary B.3.4 implies that \( f \) has strong relative dimension at least 0.

We next provide some negative examples.

Example B.3.6. As usual, the normalization \( \widetilde{C} \) of an irreducible nodal curve \( C \) provides an interesting example to consider. It has relative dimension at least 0, but not universal relative dimension at least 0. Indeed, if we take \( \widetilde{C} \times_C \widetilde{C} \), we obtain \( \widetilde{C} \) together with two isolated points, each of which maps to one of the preimages of the node under projection to \( \widetilde{C} \).

Another base change which does not have relative dimension at least 0 is obtained by taking the product with \( \overline{C} \), considered over the base field. In this case, if we let \( \Delta \subseteq \overline{C} \times C \) be the diagonal, and \( Z \subseteq C \times \overline{C} \) its image, then the restriction of \( \overline{C} \times C \) to (the preimage of) \( Z \) likewise consists of a copy of \( \Delta \cong \overline{C} \) together with two isolated points.

Notice that this means that relative dimension does not have a semicontinuity property: the map \( \overline{C} \times \overline{C} \to C \times \overline{C} \) has relative dimension at least 0 over the nonsingular locus of \( C \times \overline{C} \), but not over all of \( C \times \overline{C} \).

Example B.3.7. An example of a closed immersion of codimension \( c \) which does not have relative dimension at least \( -c \) is given by the standard example of failure of subadditivity of codimension for intersections: let \( X \) be a cone over a quadric surface, and \( Z \) the cone over a line in the surface. Then \( Z \) has codimension 1, but we claim that the inclusion \( Z \to X \) does not have relative dimension at least \( -1 \). Indeed, if \( Z' \) is the cone over any other line in the same ruling, then \( Z' \cap Z \) is equal to the cone point, which has codimension 2 in \( Z' \).
Remark B.3.8. According to Corollary B.3.4, relative dimension does not depend on the map \( f \) when \( X \) and \( Y \) are both smooth over a common base. However, in the general case, we see from Example B.3.6 that there is a nontrivial dependence on \( f \). Indeed, the morphism \( \tilde{C} \times \tilde{C} \to C \times \tilde{C} \) considered in that example does not have relative dimension at least 0, but there are many other morphisms with the same source and target which do. For instance, if we choose a point \( P \in \tilde{C} \) and consider the composition

\[
\tilde{C} \times \tilde{C} \overset{p_2}{\to} \tilde{C} \to C \times \{P\} \subseteq C \times \tilde{C},
\]

where the second morphism is the normalization, then we have a composition of morphisms of relative dimension at least 1, 0 and \(-1\) respectively, so obtain relative dimension at least 0, as asserted.

Remark B.3.9. Relative dimension of morphisms can also shed light on the behavior of constructibility of images, in the sense of understanding when a dominant morphism fails to have open image. The first example of this is always the map \( \mathbb{A}^2 \to \mathbb{A}^2 \) given by \((x, y) \mapsto (x, xy)\), and it is natural to notice that the failure of openness occurs at a jump in fiber dimension, and to wonder whether this phenomenon is general.

In fact, Example B.3.6 yields an example of a dominant morphism of varieties for which the image is not open and the fiber dimension does not jump. Namely, in the notation of that example, consider the map

\[
\tilde{C} \times \tilde{C} \setminus \Delta \to C \times \tilde{C}.
\]

This is a dominant morphism of varieties, with all fibers 0-dimensional, but its image is \((C \times \tilde{C} \setminus Z) \cup \{P_1, P_2\}\), where \(P_1\) and \(P_2\) are the intersection of \(Z\) with the singular locus of \(C \times \tilde{C}\).

However, we see immediately from Proposition B.2.7 that if \( f : X \to Y \) has relative dimension at least \(n\), and there is no jumping of fiber dimension (i.e., all fibers have dimension \(n\)), then \( f \) necessarily has open image. By Corollary B.3.4, the relative dimension hypothesis applies in particular whenever \(X\) and \(Y\) are smooth and \(\dim X - \dim Y = n\). Thus, the behavior of the first example is in fact rather general.

Remark B.3.10. Suppose that we have \( f : X \to Y \) and \( g : Y \to Z \), where \( g \) is a closed immersion, and \( g \circ f \) has relative dimension at least \(n\). Then it is immediate from the definition that \( f \) has relative dimension at least \(n\) as well, but it is natural to wonder, if \( g(Y) \) has codimension \(c\) in \(Z\), whether in fact \( f \) has relative dimension at least \(n + c\). However, this is not the case. For instance, if \( f \) is the inclusion of the cone over a line into the cone over a quadric surface, as in Example B.3.7, then we know that \( f \) does not have relative dimension at least \(-1\). However, if we compose with the inclusion of \(Y\) into \(\mathbb{A}^3\), then because \(\mathbb{A}^3\) is regular, we have that the composition has relative dimension at least \(-2\), in fact universally.

Remark B.3.11. Example B.3.6 also demonstrates that the statement of Corollary B.3.3 fails if we replace universal relative dimension with relative dimension. Indeed, in the notation of the example, the composed morphism

\[
\tilde{C} \times \tilde{C} \to C \times \tilde{C} \to C
\]

has relative dimension at least 1, and \(C \times \tilde{C} \to C\) is smooth of relative dimension 1, but \(\tilde{C} \times \tilde{C} \to C \times \tilde{C}\) does not have relative dimension at least 0.
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