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Vector bundles on reducible curves and applications

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1. Introduction

Let C be a projective, irreducible, non-singular curve of given genus $g \geq 2$ defined over an algebraically closed field. Let E be a vector bundle on C. The slope of E is defined as

$$\mu(E) = \frac{\deg E}{\text{rank} E}$$

A vector bundle E is said to be semistable (resp. stable) if and only if for every subbundle F of E, $\mu(F) \leq$ (resp. $<$)$\mu(E)$. The set of all stable bundles of a given rank r and degree d on C form a moduli space $U(r,d)$. If r and d are coprime, the condition for semistability is the same as the condition for stability and $U(r,d)$ is projective. If r and d are not coprime, this is no longer the case. A projective moduli space can be obtained, though, by considering semistable vector bundles modulo a suitable equivalence relation.

If C is a singular curve, the results above are not true. The moduli space of stable bundles $U(r,d)$ is no longer projective even when r and d are relatively prime. Newstead (see [**N**]) gave a compactification for a fixed irreducible curve by considering torsion-free sheaves instead of just vector bundles on the curve. This was later generalized by Seshadri ([**Se**]) to include reducible curves and by Pandariphande ([**P**]) when the curve moves in the moduli space of a fixed genus.

A different approach was taken by Gieseker in the case of an irreducible nodal curve for $r = 2$ by considering vector bundles on various semistable models of the given curve. His methods were extended in [**X**] to the case of a reducible curve (and $r = 2$) and then by Schmitt [**Sc2**] to any rank and curves moving in the moduli space.

In the case of rank one, many results about non-singular curves and their linear series have been proved by degenerating to singular curves. This requires knowing what is the equivalent in the singular case of a linear series. For a reducible curve of compact type, the theory of limit linear series from [**EH1**] gave an excellent solution. In order to do something similar in higher rank, one needs to extend the concept of limit linear series to rank greater than one. At the same time, one must deal with the stability condition.

In this paper, we present some scattered results about the moduli spaces of (semi) stable vector bundles on reducible nodal curves, the concept of generalized limit linear series and some of their applications. On neither of these themes does the paper try to be comprehensive. Its main purpose is to present a few examples of some techniques that have proved useful in the resolution of problems related to vector bundles with the hope that they will become more widely used.

2. Moduli spaces of slope-stable torsion-free sheaves

Seshadri in [**Se**] gave the following compactification(s) of a moduli space of vector bundles on a reducible curve. While his construction is more general, we shall concentrate on the nodal case.

Let C be a nodal curve with components C_i. When the curve is reducible, the notion of torsion-free sheaf mentioned in the introduction for irreducible curves can be generalized to the concept of depth one sheaves. A sheaf E on C is said to be of depth one if any torsion section vanishes identically on some component of C.

For an irreducible curve, the concept of stability could have been formulated equivalently by the condition that for every subbundle F of E,

$$\frac{\chi(F)}{\mathrm{rank}(F)} \leq (<) \frac{\chi(E)}{\mathrm{rank}(E)}$$

where χ denotes the Euler-Poincaré characteristic. On a reducible curve, even if E has constant rank, there will be subsheaves of E with different ranks on the various components. Therefore, in order to generalize the definition above, one needs to make clear how to count the relative rank of every component. This is done in the following way:

A polarization of C is the choice of rational weights

$$w_i, \quad 0 < w_i < 1, \quad \sum w_i = 1.$$

A depth one sheaf E of rank n on C is said to be (semi)stable for the given polarization if for every subsheaf F of E with rank r_i on the component C_i,

$$\frac{\chi(G)}{\sum w_i r_i} (\leq) < \frac{\chi(E)}{r}.$$

We will refer to the number $\sum w_i r_i$ as the w-rank of F. There is then a moduli space parameterizing (equivalence classes of semi)stable torsion-free sheaves on C.

The moduli space of torsion-free sheaves on a nodal reducible curve is itself reducible. The description of its components was given in [**T2**] for curves of compact type (that is, curves whose Jacobian is compact or whose dual graph has no non-trivial cycles) and in [**T3**] for arbitrary nodal curves. If each component of the curve has genus at least one, the number of components of the moduli space depends only on the rank and the dual graph of the curve rather than the genus of the various components of C. For example, if the curve is of compact type with M components all of genus at least one, then the moduli space of vector bundles of rank r on C has r^{M-1} components.

More generally, consider any nodal curve C. Assume that C is the fiber of a one-dimensional family $\pi : \mathcal{C} \to S$. Fix a vector bundle $\mathcal{E}$ on $\mathcal{C}$ so that the restriction to every fiber of π is of degree d. The multidegree (d_i) is the collection of degrees of the restriction of this vector bundle to the components C_i of C. If we tensor $\mathcal{E}$ by a line bundle $\mathcal{E} = \mathcal{O}_{\mathcal{C}}(\sum a_i C_i)$, the numbers d_i will be modified by multiples of r and the total degree d will remain invariant. For example, if C has only two components glued at one point, the degree on one of the components can be increased by an arbitrary multiple of r while the degree on the other component will decrease by subtracting the same multiple of r. Let us say that two multidegrees are equivalent if they can be obtained from one another by such a transformation. Then, the set of components of the moduli space corresponds one to one with the set of equivalence classes of multidegrees by this equivalence relation.

We provide a sketch of proof of this fact in the case of a curve C with two components C_1, C_2 and a single node obtained by identifying $Q \in C_1$ with $P \in C_2$ (for the more general statements and proofs see [**T3**]):

2.1. **Proposition** *Let C be the nodal curve of genus g obtained by identifying $Q \in C_1$ with $P \in C_2$, where C_1, C_2 are two non-singular curves of genus at least one. Then the moduli space of torsion-free sheaves on C semistable with respect to a generic polarization is connected and has r irreducible components each of dimension $r^2(g-1)+1$.*

Proof. Assume that E is a vector bundle stable by a generic polarization w_1, w_2. Consider the subsheaf F of E consisting of the sections of E that vanish identically on the second component C_2. Then, $\chi(F) = \chi(E_{|C_1}) - r$ while the w-rank of F is $w_1 r$. The stability condition gives

$$\frac{\chi(E_{|C_1}) - r}{w_1 r} = \frac{\chi(F)}{w_1 r} \leq \frac{\chi(E)}{r}$$

Hence, $\chi(E_{|C_1}) \leq w_1 \chi(E) + r$. Reversing the role of C_1, C_2 and using that $\chi(E) = \chi(E_{|C_1}) + \chi(E_{|C_2}) - r, w_1 + w_2 = 1$, one obtains

$$(*) \qquad w_1 \chi(E) \leq \chi(E_{|C_1}) \leq w_1 \chi(E) + r.$$

If w_1 is generic, $w_1 \chi(E)$ is not an integer and there are therefore r possible values of $\chi(E_{|C_1})$ that satisfy $(*)$. Moreover (see (cf. [**T2**], [**T3**]) if $(*)$ is satisfied and $E_{|C_1}, E_{|C_2}$ are semistable then the vector bundle E is also semistable. If in addition, at least one of the vector bundles $E_{|C_1}, E_{|C_2}$ is stable, then so is E. In fact it suffices if they are both semistable and no destabilizing subsheaf of $E_{|C_1}$ glues with a destabilizing subsheaf of $E_{|C_2}$ for E to be stable.

The moduli space of semistable vector bundles of a given rank and degree on each of the curves C_1, C_2 is irreducible of dimension $r^2(g_1-1)+1, r^2(g_2-1)+1$ with $g_1 \geq 1,\ g_2 \geq 1$ the genera of the corresponding curves. Once $E_{|C_1}, E_{|C_2}$ have been fixed, a vector bundle on C is determined by giving a projective isomorphism of the fibers $(E_{|C_1})_Q \cong (E_{|C_2})_P$. Fix a degree for E_1 satisfying (*). As the degree of E is fixed, this determines the degree for $E_{|C_2}$. One gets a variety of dimension

$$r^2(g_1-1)+1+r^2(g_2-1)+1+r^2-1 = r^2(g-1)+1.$$

This is the dimension of the moduli space of vector bundles on C. We obtain in this way an open set of one of the components of the moduli space of vector bundles on C. The closure of this component may contain some points corresponding to vector bundles that are stable on C but whose restriction to each of the components is not necessarily semistable. Moreover, the different components of the moduli space are not disjoint, the points of intersection correspond to torsion-free sheaves that are not locally free at the nodes. For example, in the situation above, the components corresponding to $\deg(E_{|C_1}) = d_1, \deg(E_{|C_2}) = d - d_1$ and $\deg(E'_{|C_1}) =$

$d_1 - 1, \deg(E'_{|C_2}) = d - d_1 + 1$ have an intersection whose generic point is a torsion-free sheaf F such that $\deg(F_{|C_1}) = d_1 - 1, \deg(F_{|C_2}) = d - d_1$ and the fiber at the node N is of the form $(\mathcal{O}_N)^{r-1} \oplus \mathcal{M}_N$ with $\mathcal{M}_N$ the maximal ideal at the node.

On an elliptic curve, there are no stable vector bundles of rank r and degree d when r, d are not coprime. Assume that C_1 is an elliptic curve and that $d_1 = \deg(E_{|C_1})$ gives a solution to $(*)$ (or a suitable set of relations deduced from the graph of the nodal curve). Write h for the greatest common divisor of r, d_1. Then, the generic point in the component corresponding to this value d_1 is a vector bundle whose restriction to C_1 is a direct sum of h vector bundles of rank $\frac{r}{h}$ and degree $\frac{d_1}{h}$. Note that this still gives the right dimension to the component of the moduli space of vector bundles on C: while there is an h-dimensional space of these restrictions of vector bundles to C_1 there is also an h-dimensional set of automorphisms of these restrictions acting on the set that identify isomorphic bundles. □

As it was mentioned before, the stability of a vector bundle for a given polarization on a reducible curve does not imply that the restriction of this bundle to each component is stable. But these restrictions cannot be too unstable either. As an example we present the following:

2.2. **Proposition** *Let C be a nodal curve with an irreducible component $\bar{C}$ intersecting the rest of the curve in α nodes. Let E be a vector bundle on C that is stable by a generic polarization. Assume that the restriction of E to $\bar{C}$ is the direct sum of r line bundles $L_1 \oplus \cdots \oplus L_r$. Then, $|\deg L_i - \deg L_j| \leq \alpha - 1$.*

Proof. Consider the subsheaf F of E that vanishes on all components of $C - \bar{C}$ and that restricts to $\bar{C}$ to the sheaf of sections of L_1 that vanish at the nodes. Then $\chi(F) = \chi(L_1) - k$ while the weighted rank $\operatorname{rank}(F) = \bar{w}$. The stability condition then gives

$$\chi(L_1) - \alpha \leq \bar{w}\frac{\chi(E)}{r}.$$

Consider the subsheaf F'_i of E that restricts to $\bar{C}$ to the sheaf of sections of $L_2 \oplus \cdots \oplus L_r$ and on all components of $C - \bar{C}$ restricts to the sections of E that glue with the above on $\bar{C}$. Then $\chi(F') = \chi(E) - \chi(L_1)$ while the weighted rank $\operatorname{rank}(F') = (r-1)\bar{w} + r(1-\bar{w}) = r - \bar{w}$. The stability condition then gives

$$\frac{\chi(E) - \chi(L_1)}{r - \bar{w}} \leq \frac{\chi(E)}{r}.$$

One obtains

$$\bar{w}\frac{\chi(E)}{r} \leq \chi(L_1) \leq \bar{w}\frac{\chi(E)}{r} + \alpha$$

As $\bar{w}$ is generic, $\bar{w}\frac{\chi(E)}{r}$ is not an integer. As the above equation is valid for all L_i, not just L_1, the result follows. □

While the choice of a polarization can be arbitrary, there is a natural one that can be defined as follows: For a given semistable curve C, define d_i as the degree of the canonical sheaf of C restricted to the component C_i. The canonical polarization has weights

$$w_i = \frac{d_i}{2g-2}$$

Pandharipande showed in [**P**] that there is a moduli space of torsion-free sheaves over the moduli space of stable curves if one considers the canonical polarization.

3. Hilbert compactification of the moduli space and its relationship with the slope-stable moduli space

Gieseker took the following approach towards compactifying the moduli space of vector bundles on a nodal curve. There is an isomorphism between moduli spaces of vector bundles $U(r,d) \cong U(r,d+kr)$ given by tensoring with a line bundle of degree k. Hence, one can assume that d is sufficiently large. In this situation, on a non-singular curve, a stable vector bundle is globally generated, hence it induces a map from the curve to the Grassmannian.

Start now the other way around. Consider the Hilbert scheme of curves in $\mathrm{Gr}(H^0(E),r)$ of suitable Hilbert polynomial. Take the connected component of points corresponding to immersions of a given nodal curve and consider its closure. The restriction to the curve of the universal bundle on the Grassmannian gives a vector bundle on the curve. In order to obtain a moduli space for vector bundles, one should first restrict to the points that are stable under the action of the linear group and then take quotient by this action.

This provides a natural compactification of the locus of vector bundles. Its main advantage is that all the points of this space correspond to vector bundles on a curve, namely the restriction of the universal bundle on the Grassmannian. Its main drawback is that the curve is not fixed. Other curves may appear that differ from the given nodal curve in adding a few rational components between the nodes of the original nodal curve. As this is precisely what happens with limit liner series (see next section), this seems a suitable model to use in this case.

Several extensions of Gieseker's results to higher rank were given by Nagaraj-Seshadri [**NS1**], [**NS2**] and Kausz [**K**] using different methods. Caporaso ([**C**]) gave a compactification of the Hilbert stable set of curves over $\bar{\mathcal{M}}_g$ when $r=1$. Finally, Schmitt (see [**Sc2**]) extended Gieseker's construction not only to higher rank but also to the whole moduli space of semistable curves in $\bar{\mathcal{M}}_g$.

The two compactifications of Seshadri and Gieseker-Schmitt are in fact closely related: for a non-singular curve, slope stability and Hilbert stability are equivalent. This was first proved for rank 2 by Gieseker and Morrison in [**GM**] (see also [**T4**] for a different proof). For general rank, one of the implications was proved in [**T5**] and the equivalence in [**Sc1**].

On reducible curves, Hilbert stability implies that the distribution of degrees on the various components is the same as that allowed by slope stability in the case of the canonical weights (that is conditions similar to those in $(*)$ in the proof of 2.1 are satisfied). Similarly, the restriction to the various components cannot be too unstable, so conditions as those stated in 2.2 are also satisfied. For instance, rational components on Hilbert stable curves must have at least two nodes and in this case the restriction of the vector bundles to the components is of the form $\mathcal{O}^{\oplus j} \oplus \mathcal{O}(1)^{\oplus r-j}$.

4. Limit linear series

Assume that we have a family of curves $\pi : \mathcal{C} \to S$ such that $\mathcal{C}_t$ is non-singular for $t \neq t_0$ while the special fiber $\mathcal{C}_{t_0}$ is a curve of compact type (that is, a curve whose Jacobian is compact or whose dual graph has no cycles).

We want to consider a linear series for vector bundles on $\mathcal{C} - \mathcal{C}_{t_0}$ and we would like to describe the limit of this object on $\mathcal{C}_{t_0}$.

A linear series for vector bundles on $\mathcal{C}-\mathcal{C}_{t_0}$ will be given by a vector bundle $\mathcal{E}$ on $\mathcal{C}-\mathcal{C}_{t_0}$ such that the restriction to each fiber has a preassigned rank r and degree d together with a family of spaces of sections given by a locally free subsheaf $\mathcal{V}$ of rank k of $\pi_*(\mathcal{E})$.

Up to making a few base changes and blow-ups (that may add a few rational components to the central fiber), we can assume that the vector bundle and locally-free subsheaf can be extended to the whole family.

On the central fiber, one obtains then a limit linear series in the sense of [**T1**] section 2, that can be defined as follows:

4.1. **Definition. Limit linear series** *A limit linear series of rank r, degree d and dimension k on a curve of compact type with M components consists of data (I), (II) below for which data (III), (IV) exist satisfying conditions (a)-(c).*

(I) For every component C_i, a vector bundle E_i of rank r and degree d_i and a k-dimensional space of sections V_i of E_i.

(II) For every node obtained by gluing $Q \in C_i$ with $P \in C_j$ an isomorphism of the projectivisation of the fibers $(E_i)_Q$ and $(E_j)_P$

(III) A positive integer a

(IV) For every node obtained by gluing $Q \in C_i$ and $P \in C_j$, bases $s^t_{Q,i}, s^t_{P,j}, t = 1...k$ of the vector spaces V_i and V_j

Subject to the conditions

(a) $(\sum_{i=1}^M d_i) - r(M-1)a = d$

(b) At the nodes, the sections glue with each other through the isomorphism in II and the orders of vanishing at P_j, Q_i of corresponding sections of the chosen basis satisfy $ord_P s^t_{P,j} + ord_Q s^t_{Q,i} \geq a$

(c) Sections of the vector bundles $E_i(-aQ)$ are completely determined by their value at the nodes.

This is a generalization of the concept of limit linear series for line bundles. In the line bundle case, $a = d = \deg E_i$ for all i. Hence, (III) is irrelevant and so is the projective isomorphism in (II) as the fibers are one-dimensional. Conditions (a), (c) and the first part of (b) are automatically satisfied and one only needs to impose the second part of (b).

Let us check that the limit of a linear series on a non-singular curve is actually a limit linear series. Given a family of curves $\pi : \mathcal{C} \to S$ as above, fix a vector bundle on the generic fiber and extend it (after some base change and normalizations) to a vector bundle on the central fiber. This limit vector bundle on the central fiber is not unique. We could modify it by, for example, tensoring the bundle on the whole curve $\mathcal{C}$ with a line bundle with support on the central fiber. This would leave the vector bundle on the generic curve unchanged but would modify the vector bundle on the reducible curve (by adding to the restriction to each component a linear combination of the nodes). In this way, for each component C_i, one can choose (in many different ways) a version $\mathcal{E}_i$ of the limit vector bundle such that the sections of $\mathcal{E}_{i|C_j}$, $i \neq j$ are trivial. In particular, this implies that $\mathcal{V}_i$ restricts to a space of sections of dimension k on the component C_i. Taking $E_i = \mathcal{E}_{i|C_i}$, $V_i = \mathcal{V}_{i|C_i}$, one obtains data in (I) satisfying condition (c).

We want to see how to choose the data above in order to obtain the integer a in (III).

Order the components of C so that C_1 has only one node and $C_1 \cup \dots \cup C_i$ is connected. From the connectivity, C_i intersects some $C_{j(i)}$, $j(i) < i$ and from the fact that the curve is of compact type, this $j(i)$ is uniquely determined. Let C'_i be the connected component of $C - C_i$ containing $C_{j(i)}$. The restriction of $\mathcal{E}_{j(i)}(-bC'_i)$ to any component of C other than $C_i, C_{j(i)}$ is identical to the restriction of $\mathcal{E}_{j(i)}$ to that component. The restriction to C_i (resp $C_{j(i)}$) is changed by tensoring with $\mathcal{O}_{C_i}(bP_i)$ (resp $\mathcal{O}_{C_{j(i)}}(-bQ_{j(i)})$) if $P_i, Q_{j(i)}$ are the two points that get identified to form the node. Hence, in order to satisfy the conditions in the previous paragraph, we can take $\mathcal{E}_i = \mathcal{E}_{j(i)}(-b_iC'_i)$ for some integer b_i. As the curve has only a finite number of components, we may assume the b_i to be all identical. We denote this number by a.

It is easy to check then that condition (a) is satisfied.

As $\mathcal{E}_i = \mathcal{E}_j(-aC'_i)$ the gluing defining $\mathcal{E}_{i|C}$ determine those of $\mathcal{E}_{j|C}$. Hence, the isomorphisms in (II) are well determined and the first part of b) is satisfied.

Let us check the second part of (b). If $C_i, C_j,\ j < i$ intersect, then $j = j(i)$ (in the notations above). Hence $\mathcal{E}_i = \mathcal{E}_j(-aC'_i)$. Let s be a section of $\mathcal{V}_{j|C}$. Assume that s vanishes on C'_i with multiplicity α (and therefore vanishes at the point P of intersection with C'_i with order at least α). Let t_0 be a local equation for C. Then $(t_0^{a-\alpha})s$ is a non-trivial section of $\mathcal{E}_j(-aC'_i) = \mathcal{E}_i$. Hence it gives rise to a section on $\mathcal{V}_i$ that vanishes to order at least $a - \alpha$ on P.

Given a vector bundle $\mathcal{E}$ stable by some polarization, the vector bundles $\mathcal{E}_i$ that give rise to the limit linear series are no longer stable by this polarisation, as the distribution of degrees among the components gets changed. But conditions of "quasistability" are preserved, that is, conditions that say that the restriciton of the vector bundle to each component is not far from being stable (analogous to 2.2). This fact is very important in applications as it restricts the possibilities for these vector bundles.

5. Subbundles of a vector bundle

Let E be a vector bundle of rank r and degree d. Let E' be a subsheaf of E of fixed rank r'. Up to increasing the degree of E', we can assume that the quotient is again a vector bundle. Hence, we have an exact sequence

$$(**)\qquad 0 \to E' \to E \to E'' \to 0$$

Define

$$s_{r'}(E) = r'd - r\max\{\deg(E')\}$$

where E' varies among all subsheaves of E of rank r'. If E is stable, then $s_{r'}(E) \geq 0$.

If $E',\ E''$ are fixed, the set of vector bundles that fit in an exact sequence as $(**)$ is parameterized by $H^1(C, (E'')^* \otimes E')$. If there exists a stable E fitting in one of these exact sequences, then $H^0(C, (E'')^* \otimes E') = 0$, otherwise there would be a non-trivial map $E'' \to E'$ and therefore a non-trivial endomorphism of E contradicting stability. This implies that $h^1(C, (E'')^* \otimes E')$ is constant.

Let $E',\ E''$ vary

$$E' \in U(r', d'),\ E'' \in U(r - r', d - d').$$

Then, the set of stable E varies in a space of dimension

$$(r')^2(g-1)+1+(r-r')^2(g-1)+1+r'(d-d')-(r-r')d'+r'(r-r')(g-1)-1 =$$
$$= r^2(g-1) + 1 + r'd - rd' - r'(r-r')(g-1)$$

The following result was conjectured by Lange in [**L**]

5.1. **Theorem** *If $r'd - rd' - r'(r-r')(g-1) \geq 0$, the generic vector bundle in $U(r,d)$ has subbundles of rank r' and degree d'. For $0 \leq r'd - rd' \leq r'(r-r')(g-1)$, the set of vector bundles with such a subbundle is an irreducible set of the moduli space of codimension $r'(r-r')(g-1) - (r'd - rd')$.*

The rank two case was proved by Lange in [**L**] . Several special cases were obtained (see[**BL**] and the references there).

The result was proved for the generic curve in [**T7**] and then in its full generality in [**RT**].

The main point of any proof of the result is to show that there exist extensions as in $(*)$ with a stable E and for $r'd - rd' < r'(r-r')(g-1)$ that such an E has only a finite number of subbundles of the given rank and degree.

Reducible curves were used in [**T6**], [**T7**], [**T8**]. We will sketch below the main points of the proof when $r = 2$, $r' = 1, d = 1$. In order to prove the result for the generic curve, it suffices to show it for a special curve. Take g elliptic curves $C_1, ..., C_g$ with marked points P_i, Q_i on them and identify Q_i with P_{i+1} to form a nodal curve of arithmetic genus g. Consider the component of the moduli space of vector bundles on this curve with distribution of degrees one on the first component and zero on the remaining ones. For a generic point on this component, E_1 is an indecomposable vector bundle of degree one while the remaining E_i are direct sums of two line bundles.

The largest degree of a line subbundle of E_1 is zero. In fact, every line subbundle of degree zero on C_1 can be immersed in E_1. For a particular choice of such a subbundle, we can assume that it glues at Q_1 with a fixed direction. On the remaining components, the degree of the largest subbundle is zero and there are precisely two subbundles of this degree inside each E_i. On the other hand, every line bundle of degree -1 on C_i can be immersed in C_i by a two-dimensional family of maps. Given a fixed direction at both P_i and Q_i, one can find a subbundle of E_i of degree -1 that glues with these two directions at the nodes.

In order to obtain a line subbundle on the total curve, we must choose line subbundles of each of the components that glue with each other. If all the gluing are generic, the choice that gives the largest degree will be as follows: Take a subbundle of degree zero of E_1 that glues with one of the two subbundles of degree zero of E_2. Choose a subbundle of degree zero on each of the components C_i for even i. Choose a subbundle of degree -1 on the components C_i for odd i that glues with the chosen subbundles in adjacent components. The degree of the subbundle obtained in this way is $d' = -\frac{g-2}{2}$ if this number is an integer and $d' = -\frac{g-1}{2}$ otherwise.

In order to obtain subbundles of higher degree, take the gluing in the last k components so that line subbundles of degree zero glue with each other and glue with the chosen subbundle in the previous $g-k$ components. One then obtains a subbundle of degree $d' = -\frac{g-k-2}{2}$ (again if this number is an integer). In this case, the gluing depends on three rather than four parameters at the last k nodes. Hence, the codimension of the set is $k = g + 2d' - 2$ as expected.

6. Some applications to Brill-Noether Theory

A Brill-Noether subvariety $B^k_{r,d}$ of the stable set in $U(r,d)$ is a subset of $U(r,d)$ whose points correspond to stable bundles having at least k independent sections (often denoted by $W^{k-1}_{r,d}$). Brill-Noether varieties can be locally represented as determinantal varieties. This implies that, when non-empty, the dimension of $B^k_{r,d}$ at any point is at least the so called Brill-Noether number

$$\rho^k_{r,d} = dimU(r,d) - h^0(E)h^1(E) = r^2(g-1) + 1 - k(k-d+r(g-1))$$

with expected equality. Moreover, $B^{k+1}_{r,d}$ is contained in the singular locus, again with expected equality. It is not true that the above expectations actually occur for all meaningful values of r, d, k. For an account of the state of the art in Brill-Noether Theory see [**GT**].

Using limit linear series, one can prove the following:

6.1. **Theorem** *(see* [**T1**], [**T9**]*) Let C be a generic non-singular curve of genus $g \geq 2$. Let d, r, k be positive integers with $k > r$. Write*

$$d = rd_1 + d_2,\ k = rk_1 + k_2,\ d_2 < r,\ k_2 < r$$

and all $d_i,\ k_i$ non-negative integers. Assume that one of the following conditions is satisfied

$$(1) g - (k_1+1)(g - d_1 + k_1 - 1) \geq 1,\ d_2 \geq k_2 \neq 0$$
$$(2) g - k_1(g - d_1 + k_1 - 1) > 1,\ k_2 = 0$$
$$(3) g - (k_1+1)(g - d_1 + k_1) \geq 1,\ d_2 < k_2.$$

Then the set $B^k_{r,d}$ of rank r degree d and with k sections on C is non-empty and has (at least) one component of the expected dimension ρ.

Given a family of curves

$$\mathcal{C} \to T$$

construct the Brill-Noether locus for the family

$$\mathcal{B}^k_{r,d} = \{(t,E) | t \in T,\ E \in B^k_{r,d}(\mathcal{C}_t)\}.$$

The dimension of this scheme at any point is at least $\dim T + \rho^k_{r,d}$. Assume that we can find a point (t_0, E_0) such that the dimension at the point (t_0, E_0) of the fiber of $\mathcal{B}^k_{r,d}$ over t_0 $\dim(B^k_{r,d}(\mathcal{C}_{t_0}))_{E_0} = \rho^k_{n,d}$. Then $\dim \mathcal{B}^k_{r,d} \leq \rho^k_{r,d} + \dim T$ and therefore we have equality. The dimension of the generic fiber of the projection $\mathcal{B}^k_{r,d} \to T$ is at most the dimension of the fiber over t_0, namely $\rho^k_{r,d}$. But it cannot be any smaller, as the fiber over a point $t \in T$ is $B^k_{r,d}(\mathcal{C}_t)$ which has dimension at least $\rho^k_{n,d}$ at any point. Hence, there is equality which is the result we are looking for.

As the particular curve $\mathcal{C}_{t_0}$, we take g elliptic curves $C_1, ..., C_g$ with marked points P_i, Q_i on them and identify Q_i with P_{i+1} to form a node. Then, one needs to prove that there is a component of the set of limit linear series on the curve of dimension exactly ρ with the generic point corresponding to a stable bundle.

Let us sketch how one could proceed in the particular case $g = 5, k = 4, d = 9$ (and hence $\rho = 5$).

On the curve C_1, take the vector bundle to be a generic indecomposable vector bundle of degree nine. On the curve C_2 take $\mathcal{O}(P_2 + 3Q_2) \oplus \mathcal{O}(2P_2 + 2Q_2)$. On the curve C_3, take $\mathcal{O}(P_3 + 3Q_3)^{\oplus 2}$. On the curve C_4, take $\mathcal{O}(3P_4 + Q_4)^{\oplus 2}$. On the curve C_5, take the direct sum of two generic line bundles of degree four.

The sections and gluing at the nodes are taken as follows: there is a unique section s_1 of E_1 that vanishes at Q_1 with order four. Take as space of sections the space that contains s_1, two sections that vanish at Q_1 to order three and one section that vanishes to order two.

On C_2, glue $\mathcal{O}(2P_2+2Q_2)$ with the direction of s_1. Take a section that vanishes at P_2 to order three, the section that vanishes at P_2 to order one and at Q_2 to order three, one section that vanishes at P_2 to order one and at Q_2 to order two and the section that vanishes at P_2 to order two and at Q_2 to order two.

On C_3, take generic gluing. Take two sections that vanish at P_3 with order one and at Q_3 to order three and two sections that vanish at P_3 to order two and at Q_3 to order one.

On C_4, take generic gluing. Take two sections that vanish at P_4 to order one and at Q_4 to order two and two sections that vanish at P_4 to order three and at Q_4 to order one.

On C_5, take generic gluing. Take two sections that vanish at P_5 to order two and two sections that vanish at P_5 to order three.

In this way, each section on C_i glues at Q_i with a section on C_{i+1} so that the sum of the orders of vanishing is four. Note also that we have been using sections of our vector bundles that vanish as much as possible between the two nodes. So it is not possible to obtain a limit linear series of dimension larger than four with these vector bundles.

Let us count the dimension of the family so obtained. The vector bundle E_1 varies in a one-dimensional family and has a one-dimensional family of endomorphisms. The vector bundles $E_2,\ E_3,\ E_4$ are completely determined and have a family of endomorphisms of dimensions two, four and four respectively while E_5 varies in a two-dimensional family and has a two-dimensional family of endomorphisms. The resulting vector bundle is stable as the restriction to each component is semistable and the first one is actually stable (see section 2). Therefore it has a one-dimensional family of automorphisms. The gluings at each of the nodes vary in a four-dimensional family except for the first one that varies in a three-dimensional family. Therefore the total dimension of the family is

$$1+0+0+0+2-(1+2+4+4+4+2)+1+(3+4+4+4)=5=\rho.$$

If we try to deform the vector bundle by either taking more general restrictions to some component curves or more general gluing at some nodes, the limit linear series does not extend to this deformation. Hence, the point we describe is a general point in the set of limit linear series. Therefore, the result is proved.

One of the main questions for classical Brill-Noether Theory (that is in the case $r=1$) comes from the infinitesimal study of $B^k_{r,d}$. The tangent space to $U(r,d)$ can be identified with the set of infinitesimal deformations of the vector bundle E which is parameterized by $H^1(C,E\otimes E^*)\cong H^0(C,K\otimes E^*\otimes E)$. The tangent space to $B^k_{r,d}$ inside the tangent space to $U(r,d)$ can be identified with the orthogonal to the image of the Petri map

$$P_V: H^0(C,E)\otimes H^0(C,K\otimes E^*)\to H^0(C,K\otimes E\otimes E^*).$$

If this map is injective for a given E, then $B^k_{r,d}$ is non-singular of the right dimension at E. Hence, in order to prove that the expected results hold, it would be sufficient to prove the injectivity of this map for all possible E on say, a generic curve. This is true in rank one ([**G1**]). Unfortunately, for rank greater than one, the map is not injective in general. There is one case though in which one has an analogous result.

Assume that E is a vector bundle of rank two and canonical determinant. Let $U(2,K)$ be the moduli space of stable rank two vector bundles with determinant the canonical sheaf. Consider the set

$$B^k_{2,K} = \{E \in U(2,K) | h^0(C,E) \geq k\}$$

Then $B^k_{2,K}$ can be given a natural scheme structure. Its expected dimension is (see [**GT**], [**T10**])

$$\dim U(2,K) - \binom{k+1}{2}$$

The tangent space to $B^k_{2,K}$ at a point E is naturally identified with the orthogonal to the image of the symmetric Petri map

$$S^2(H^0(C,E)) \to H^0(C, S^2(E))$$

6.2. **Theorem** *(cf.* [**T10**]*) Let C be a generic curve of genus g defined over an algebraically closed field of characteristic different from two. Let E be a semistable vector bundle on C of rank two with canonical determinant. Then, the canonical Petri map*

$$S^2H^0(C,E) \to H^0(S^2E)$$

is injective.

The proof of this fact is in many ways similar to the proof of the injectivity of the classical Petri map in [**EH2**] with the (considerable) added complication brought in by higher rank.

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