LIFTING TROPICAL INTERSECTIONS

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Abstract. In general, one cannot expect that every point of the intersection of the tropicalizations of two varieties can be lifted to the intersection of the varieties themselves. A result of Bogart, Jensen, Speyer, Sturmfels and Thomas asserts that one always has such a lifting if the tropicalizations intersect suitably transversely. In joint work with Sam Payne, we generalize and strengthen their result in three directions: we weaken the transversality hypothesis, we work with intersections in more general ambient varieties, and we also treat intersection multiplicities. We hope that these results can be applied to moduli spaces to prove correspondence theorems relating enumerative algebraic geometry to enumerative tropical geometry.

1. Correspondence theorems and tropical intersections

If one wishes to use tropical geometry to prove results in enumerative algebraic geometry – for instance, problems relating to Schubert calculus – the natural approach is in two steps: understand the tropicalized version of the problem, and then prove a correspondence theorem relating the tropical version to the version in algebraic geometry. One approach to the latter is to prove directly that the tropicalized objects can be lifted to algebrogeometric objects. We propose an alternate approach to this sort of question: one could instead work with tropical moduli spaces. If the tropical moduli spaces are the tropicalization of the moduli spaces in algebraic geometry (as is the case for instance for the Grassmannian), a correspondence theorem becomes a question on lifting intersections inside tropicalized moduli spaces to intersections in the original moduli spaces. This provides additional motivation for the following question, which is in any case quite natural and fundamental:

Question 1.1. What is the relationship between intersection and tropicalization? In particular, under what conditions do they commute?

If we have $X, X' \subseteq (K^*)^n$ (where K is an algebraically closed field with non-Archimedean valuation and algebraically closed residue field, for instance Puiseux series over $\mathbb{C}$), then

$$\text{Trop}(X \cap X') \subseteq \text{Trop}(X) \cap \text{Trop}(X'), \tag{1.1}$$

but in general it is easy to see that one need not have equality.

Example 1.2. Suppose X, X' are lines in the plane. Then it is possible for X, X' to be disjoint but $\text{Trop}(X) = \text{Trop}(X')$. Or we can have $\text{Trop}(X) \cap \text{Trop}(X')$ a single ray, and we must always have strict containment in (1.1). However, if $\text{Trop}(X) \cap \text{Trop}(X')$ is a single point, then $X \cap X'$ must also intersect at a single point, so we have equality in (1.1).

If X is a line and X' is the parabola $y = x^2$, then we can have $\text{Trop}(X)$ intersecting $\text{Trop}(X')$ at one or two points, and $X \cap X'$ is also always one or two points. In fact, in this case we always have equality in (1.1) (and the number of intersection points is the same, if counted with appropriate multiplicity).

These examples may be suggestive, and indeed a result of Bogart, Jensen, Speyer, Sturmfels and Thomas asserts that if $\text{Trop}(X)$ and $\text{Trop}(X')$ intersect transversely in a suitable sense then (1.1) is always an equality. This handles the well-behaved examples discussed above, except in the case of a parabola and a line, when the intersection is a single point, but not tropically tranverse.

Our results build on their work by weakening the transversality hypothesis, taking intersection multiplicities into account, and – crucially, from the point of view of applications to correspondence theorems – allowing the intersections to occur inside an ambient variety. That is, we suppose $X, X' \subseteq Y \subseteq (K^*)^n$, and under suitable hypotheses on the behavior of $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ inside $\mathrm{Trop}(Y)$, we prove equality in (1.1). Initially, examples of this type are discouraging: for instance, if one starts with X, X' being disjoint lines in $(K^*)^3$ with $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ consisting of a single point, it is easy to write down smooth quadric surfaces $Y \subset (K^*)^3$ containing X, X'. Then the intersection of $\mathrm{Trop}(X)$ with $\mathrm{Trop}(X')$ occurs as transversely as possible inside $\mathrm{Trop}(Y)$, but equality fails in (1.1). However, it turns out that there is a natural hypothesis which prevents this from happening.

2. Results

Denote by k the residue field of K, and R the valuation ring. Recall that given a subvariety $X \subseteq (K^*)^n$, and $w \in \mathbb{R}^n$, we obtain a family $\mathcal{X}_w$ over R with generic fiber X, and special fiber $X_w \subseteq (k^*)^n$, the initial degeneration of X at w. As a set, $\mathrm{Trop}(X)$ consists of those w for which X_w is nonempty. Our convention will be that the data of $\mathrm{Trop}(X)$ is this set, together with a polyhedral decomposition such that at any two points w, w' in the interior of the same face, we have that $X_w, X_{w'}$ are isomorphic, and finally, integer multiplicities associated to each facet of the polyhedral decomposition as follows: if w lies in the interior of a facet σ, then X_w is acted on by a subtorus of the same dimension, so the quotient is a scheme of finite length over k, and the multiplicity of σ is defined to be the length of the quotient.

We say that a point $w \in \mathrm{Trop}(X)$ is a **simple point** if it lies in the interior of a facet of multiplicity 1; this implies in particular that X_w is isomorphic to a subtorus of $(k^*)^n$, and in particular is smooth. Following traditional terminology from intersection theory in algebraic geometry, we say that $\mathrm{Trop}(X)$ intersects $\mathrm{Trop}(X')$ **properly** in $\mathrm{Trop}(Y)$ if

$$\mathrm{codim}_{\mathrm{Trop}\,Y} \mathrm{Trop}(X) \cap \mathrm{Trop}(X') = \mathrm{codim}_{\mathrm{Trop}\,Y} \mathrm{Trop}(X) + \mathrm{codim}_{\mathrm{Trop}\,Y} \mathrm{Trop}(X').$$

Our main result is the following:

Theorem 2.1 (O.-Payne)**.** *Given $X, X' \subseteq Y \subseteq (K^*)^n$, suppose that $\mathrm{Trop}(X)$ intersects $\mathrm{Trop}(X')$ properly in $\mathrm{Trop}(Y)$, and the simple points of $\mathrm{Trop}(Y)$ are dense in $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$. Then*

$$\mathrm{Trop}(X \cap X') = \mathrm{Trop}(X) \cap \mathrm{Trop}(X').$$

Furthermore, under these hypotheses there is a purely tropical definition of intersection multiplicities for $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ inside $\mathrm{Trop}(Y)$, and these agree (in a natural sense) with the algebrogeometric intersection multiplicities of $X \cap X'$.

We mention that examples show that the statement of the theorem fails if the simple point hypothesis is not satisfied, even if $\mathrm{Trop}(X) \cap \mathrm{Trop}(X')$ is contained inside facets of $\mathrm{Trop}(Y)$.

Our results are sufficiently sharp that we obtain the natural generalization to multiple intersections by inductive application of Theorem 2.1. Note that if $Y = (K^*)^n$, every point is simple, so we recover the result of Bogart-Jensen-Speyer-Sturmfels-Thomas under the weaker hypothesis of proper intersection, and taking multiplicities into account. The definition of tropical intersection multiplicity in our situation is essentially the stable intersection multiplicity introduced by Mikhalkin, which (not coincidentally) also agrees with the fan displacement rule used by Fulton and Sturmfels to describe the intersection theory of toric varieties.

The proof of Theorem 2.1 is in two steps: we first prove that under the stated hypotheses, if $w \in \mathrm{Trop}(X) \cap \mathrm{Trop}(X')$, then X_w and X'_w have non-empty proper intersection in Y_w, which is smooth by the simple point hypothesis. We then prove that the intersection $X_w \cap X'_w$ lifts to $X \cap X'$.

Remark 2.2. Because it is quite possible for X_w and X'_w to have non-empty proper intersection even if the hypotheses of Theorem 2.1 are not satisfied, our results in fact deal with a substantially wider range of cases than stated. However, the statement is the sharpest we are able to obtain under hypotheses stated purely in terms of the tropicalizations.

3. From tropicalizations to initial degenerations

The first step in our argument is to lift from the intersection of the tropicalizations to the intersection of the initial degenerations. We prove:

Theorem 3.1 (O.-Payne)**.** *Given $X, X' \subseteq Y \subseteq (K^*)^n$, suppose that* $\operatorname{Trop}(X)$ *intersects* $\operatorname{Trop}(X')$ *properly in* $\operatorname{Trop}(Y)$ *at a point $w \in \mathbb{R}^n$, and suppose also that w is a simple point of* $\operatorname{Trop}(Y)$*. Then X_w, X'_w have nonempty proper intersection inside Y_w.*

Of course, we also prove a version of this taking multiplicities into account, and indeed the multiplicities are crucial to proving the nonemptiness of $X_w \cap X'_w$. We have in particular that the multiplicity of $\operatorname{Trop}(X \cap X')$ along a facet σ is at least equal to the tropical intersection multiplicity of $\operatorname{Trop}(X)$ and $\operatorname{Trop}(X')$ along σ, computed by the fan displacement rule.

The first step is to reduce to the case that $Y = (K^*)^n$; indeed, this reduction is immediate from the hypothesis that w is a simple point. We see that X_w and X'_w intersect properly, because $\operatorname{Trop}(X_w)$ is the star of w in $\operatorname{Trop}(X)$, and similarly for $\operatorname{Trop}(X'_w)$, so this follows from the hypothesis that $\operatorname{Trop}(X)$ intersects $\operatorname{Trop}(X')$ properly at w. The next observation is that the tropical intersection multiplicity of any component of $\operatorname{Trop}(X)$ and $\operatorname{Trop}(X')$ containing w is strictly positive, since we can always displace $\operatorname{Trop}(X')$ slightly so that the interior of a facet of $\operatorname{Trop}(X')$ meets the interior of a facet of $\operatorname{Trop}(X)$.

Next, we let Σ be a complete unimodular fan which has subfans refining the stars of w in $\operatorname{Trop}(X)$ and in $\operatorname{Trop}(X')$. Let $Y(\Sigma)$ be the corresponding toric variety, and let $\overline{X}_w, \overline{X}'_w$ be the closures of X_w, X'_w in $Y(\Sigma)$. Then the Chow homology classes of $\overline{X}_w, \overline{X}'_w$ in $Y(\Sigma)$ are determined by $\operatorname{Trop}(X_w)$ and $\operatorname{Trop}(X'_w)$, and Fulton-Sturmfels says that the intersection class of $\overline{X}_w \cdot \overline{X}'_w$ in the Chow homology group agrees precisely with $\operatorname{Trop}(X) \cap \operatorname{Trop}(X')$, using the tropical intersection multiplicity.

The main point remaining is then to show that $\overline{X}_w \cdot \overline{X}'_w$ does not have any components supported in the boundary, which follows from the tools of extended tropicalizations, using also results on the analytification to understand the behavior of closure. We conclude Theorem 3.1.

4. From initial degenerations to original varieties

It remains to show that we have good behavior in lifting from the initial degenerations to the original subvarieties of the torus. We prove:

Theorem 4.1 (O.-Payne)**.** *Given $X, X' \subseteq Y \subseteq (K^*)^n$, and $w \in \mathbb{R}^n$, suppose that has X_w intersects X'_w properly at a smooth closed point $y \in Y_w$. Then there is a closed point $\tilde{y} \in X \cap X'$ specializing to y.*

Here, the idea is quite intuitive: let $\mathcal{Y}_w$ be the family over $\operatorname{Spec} R$ used to define Y_w; recall that $\mathcal{Y}_w$ has generic fiber Y, and special fiber Y_w. Let $\mathcal{X}_w, \mathcal{X}'_w$ be the families over $\operatorname{Spec} R$ corresponding to X, X'. Finally, let $m = \dim Y - \operatorname{codim}_Y X - \operatorname{codim}_Y X'$. This is the "expected dimension" of $X \cap X'$. Then because y is a smooth point of Y_w, it is a smooth point of $\mathcal{Y}_w$, and it should be the case that every component of $\mathcal{X}_w \cap \mathcal{X}'_w$ has codimension in $\mathcal{Y}_w$ at most

$$\operatorname{codim}_{\mathcal{Y}_w} \mathcal{X}_w + \operatorname{codim}_{\mathcal{Y}_w} \mathcal{X}'_w = \operatorname{codim}_Y X + \operatorname{codim}_Y X',$$

and thus dimension at least $m+1$ (since $\dim \mathcal{Y}_w = \dim Y + 1$). The hypothesis of the theorem is that the components of $X_w \cap X'_w$ have dimension m at y, so we conclude that none of them can constitute entire components of $\mathcal{X}_w \cap \mathcal{X}'_w$, and therefore that there are points of the generic fiber of $\mathcal{X}_w \cap \mathcal{X}'_w$ specializing to y.

To make this idea precise involves generalizing classical dimension theory results from varieties over a field to schemes over a non-Noetherian valuation ring R of rank 1. Using Noetherian approximation, we show that the Krull principal ideal theorem holds for schemes of finite type over R, and we then mimic classical arguments to conclude that intersections inside a scheme smooth over R satisfy the usual subaddivity of codimension. Both of these results fail over more general non-Noetherian schemes, and even in the case of valuation rings of rank 2. We also generalize standard "principal of continuity" results on intersection multiplicities in families to the non-Noetherian setting.

Finally, for tropical geometry it is important to know that the point $\tilde{y}$ which specializes to y can be chosen to be a closed point, so that it is K-rational. Using the same tools, we prove the following general statement on the topology of morphisms of finite type:

Theorem 4.2 (O.-Payne)**.** *Let $f : X \to S$ be a morphism of finite type, and suppose we have $s, s' \in S$ with s' specializing to s. Given x a closed point of $f^{-1}(s)$, suppose there exists $x' \in f^{-1}(s')$ specializing to x. Then there exists a point x'' closed in $f^{-1}(s')$, with x' specializing to x'' and x'' specializing to x.*

Using this, we conclude Theorem 4.1 as well as Theorem 2.1.