Some 4-Point Hurwitz Numbers in Positive Characteristic

Irene I. Bouw and Brian Osserman

Abstract. In this paper, we compute the number of covers of curves with given ramification in positive characteristic for one class of examples. The proof combines the linear series approach with stable reduction of Galois covers.

1. Introduction

This paper considers the question of determining the number of covers between genus-0 curves with fixed ramification. More concretely, we consider covers $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ branched at $r$ ordered points $Q_1, \ldots, Q_r$ of fixed ramification type $(d; C_1, \ldots, C_r)$, where $d$ is the degree of $f$ and $C_i = e_1(i) \cdot \cdots \cdot e_s(i)$ is a conjugacy class in $S_d$. This means that there are $s_i$ ramification points in the fiber $f^{-1}(Q_i)$, with ramification indices $e_j(i)$. The Hurwitz number $h(d; C_1, \ldots, C_r)$ is the number of covers of fixed ramification type over $\mathbb{C}$, up to isomorphism. This number does not depend on the position of the branch points. If $p$ is a prime not dividing any of the ramification indices $e_j(i)$, the $p$-Hurwitz number $h_p(d; C_1, \ldots, C_r)$ is the number of covers of fixed ramification type whose branch points are generic over an algebraically closed field $k$ of characteristic $p$. In positive characteristic, we need to require that the branch points are generic, since the number of covers depends on the position of the branch points, in general. For a definition of generic branch points, see §4. Note that the condition that $f$ is a cover of genus-0 curves implies that $2d - 2 = \sum_{i,j} (e_j(i) - 1)$.

In this paper, we consider covers with ramification type $(p; e_1, e_2, e_3, e_4)$, where the degree $p$ is a prime number and $1 < e_1 \leq e_2 \leq e_3 < e_4 < p$ are odd integers, i.e. $C_i$ is the conjugacy class of a single cycle of order $e_i$. Our main result is the following.

Theorem 1.1. The $p$-Hurwitz number satisfies

$$h_p(p; e_1, e_2, e_3, e_4) = h(p; e_1, e_2, e_3, e_4) - p.$$ 

The computation of the characteristic-zero Hurwitz number translates to a combinatorial problem, by using Riemann’s Existence Theorem. Namely, $h(d; C_1, \ldots, C_r)$ is the cardinality of the set of Hurwitz factorizations

$$\{(\sigma_1, \ldots, \sigma_r) \in (C_1, \ldots, C_r) \mid \prod_i \sigma_i = 1, \quad \langle \sigma_i \rangle \subset S_d \text{ transitive } \}/ \sim,$$

where $\sim$ denotes uniform conjugacy by $S_d$. A variant is considering the Nielsen classes, where one specifies the transitive group $G$ generated by the $\sigma_i$, and considers the tuples $(\sigma_1, \ldots, \sigma_r)$ only up to uniform conjugacy by $G$. Nielsen classes correspond to isomorphism classes of $G$-Galois covers. In a few concrete cases there...
exist closed formulae for the Hurwitz number. For example, this is the case if all $C_i$ are the conjugacy class of a single cycle and the number of branch points is at most 4 ([?]).

In positive characteristic, the only general result known is that the $p$-Hurwitz number is less than or equal to the Hurwitz number, since every tame cover may be lifted from characteristic $p$ to characteristic zero ([?]). In this paper, we combine two methods to obtain information on $p$-Hurwitz numbers.

The first method is the deformation of admissible covers. These are certain covers between degenerate curves which deform to covers of smooth curves (§ 2). With this method one obtains a lower bound on the $p$-Hurwitz number from information on covers with less branch points. See for example [?] and [?]. The results of [?] use that the $p$-Hurwitz number in the single-cycle case and 3 branch points can be computed using linear series.

It is an interesting question under what assumptions one obtains all covers by deforming admissible covers. A cover which arises as the deformation of an admissible cover is said to have a good degeneration. In [?] one finds examples of covers with generic branch points which do not have a good degeneration. One might still hope that in special cases, for example in the single-cycle case, all covers with generic branch points have a good degeneration. This paper provides some (small) evidence for this.

The second method for constructing covers in positive characteristic is by degenerating covers from characteristic zero. In this approach it is more common to replace $f : P^1 \to P^1$ by its Galois closure $g : Y \to P^1$. Let $G$ be the Galois group of $g$. Let $R$ be a discrete valuation ring with fraction field $K$ of characteristic zero and residue field $k$ an algebraically closed field of characteristic $p > 0$. Suppose that $g$ may be defined over $K$. We say that $g$ has good reduction if after replacing $K$ by a finite extension, the cover $g$ extends to a cover of smooth curves $g_R : Y_R \to P^1_R$ over $R$. One asks how many covers of fixed ramification type have good reduction to characteristic $p$. In practice, it is mostly easier, though equivalent, to determine the number of covers with bad reduction.

The strongest results on the number of covers with bad reduction are due to Wewers ([?]) generalizing results of Raynaud ([?]) in the case of covers with three branch points such that $p$ strictly divides the order of the Galois group $G$. These results translate the calculation of the number of covers with bad reduction to a combinatorial problem (see § 4). Some partial results in the case of covers branched at four points can be found in [?] and special Galois groups are treated in [?], [?], and [?].

In this paper, we restrict to a case where we can combine these methods to compute both the Hurwitz number and the $p$-Hurwitz number. Let $p$ be a prime number. We fix odd, positive integers $1 < e_1 \leq e_2 \leq e_3 \leq e_4 < p$ such that

$$2p - 2 = \sum_{i=1}^{4} (e_i - 1).$$

The assumption that the $e_i$ are odd is made for simplicity; it ensures that the Galois group is always the alternating group $A_p$. We restrict to this case, as it illustrates the techniques used, without the combinatorics getting too complicated.

The strategy of the proof is as follows. In § 2, we recall a result from [?] which describes the possible degenerations of a cover with ramification type $(p; e_1, e_2, e_3, e_4)$
in characteristic zero. If the admissible cover is built up from single-cycle covers, one can determine the number of admissible covers in characteristic $p$ by the linear series method (ADD REF). This is described in §3. For admissible covers which are not built up from single-cycle covers, we use the theory of stable reduction of three-point covers to calculate the number of admissible covers in positive characteristic in this case as well (§ 5). Using the deformation of admissible covers, we therefore obtain a lower bound on the $p$-Hurwitz number $h_p(p; e_1, e_2, e_3, e_4)$.

We obtain an upper bound on the $p$-Hurwitz number as well, by using some techniques from [?] from which we conclude that the number $h(p; e_1, e_2, e_3, e_4) - h_p(p; e_1, e_2, e_3, e_4)$ of covers with bad reduction in the case of four generic branch points is positive and divisible by $p$, in our situation. Theorem 1.1 follows since the upper and lower bound coincides. It also follows that every cover with ramification type $(p; e_1, e_2, e_3, e_4)$ and generic branch points has a good degeneration.

2. The classical picture

We begin with a brief review of the situation in characteristic 0, as developed in §4 of [?]. In our situation, the Hurwitz number is given by(*)

$$h(d; e_1, \ldots, e_4) = \min_{i} \{e_i (d + 1 - e_i)\}.$$ 

This number is obtained by studying the possibilities for $\rho := \sigma_3 \sigma_4$, over all Hurwitz factorizations $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. There are two cases: the first is that $\rho$ is a (possibly trivial) single cycle, while the second is that $\rho$ is a product of two disjoint cycles.

In the first case, if a given $\ell$ occurs as a possible length of $\rho$, there are exactly $\ell$ Hurwitz factorizations such that $\rho$ has length $\ell$.

In the second case, we necessarily have that $\sigma_1$ and $\sigma_2$ are disjoint, and are therefore equal to the disjoint cycles appearing in $\rho$ (in particular, the lengths of the cycles in $\rho$ are predetermined as $e_1$ and $e_2$).

We find we have two situations to consider. If $d+1 \leq e_2 + e_3$, in the first case we have that $\ell$ varies from $e_2 - e_1 + 1$ to $2d + 1 - e_4 - e_4$, always with parity equal to the parity of $e_2 - e_1 + 1$. We obtain a total of $e_1 e_2$ Hurwitz factorizations in the first case. In the second case, we obtain $e_1 (d + 1 - e_1 - e_2)$ possibilities.

On the other hand, if $d+1 \geq e_2 + e_3$, in the first case $\ell$ varies from $e_4 - e_3 + 1$ to $2d+1 - e_3 - e_4$, again with fixed parity. There are then $(d+1 - e_3)(d+1 - e_4)$ possible Hurwitz factorizations. In the second case, there are $(e_3 + e_4 - d - 1)(d+1 - e_4)$ possibilities.

Our analysis of $\rho$ may be interpreted geometrically in terms of degenerations as follows: we let $X \to S$ be a family of rational curves with four sections $Q_1, \ldots, Q_4$, which has smooth generic fiber $X$, but has special fiber $X_0$ consisting of two smooth rational components $X_0^1, X_0^2$ joined at a single node. We moreover suppose that $Q_1, Q_2$ specialize to $X_0^1$, and $Q_3, Q_4$ specialize to $X_0^2$. If we let $\hat{X}$ be $X \setminus \{Q_1, \ldots, Q_4\}$, we also choose local monodromy generators for $\pi_1^{\text{tame}}(X)$ which are compatible with the degeneration to $X_0$. We then find that if we have a branched cover of $X$ corresponding to a Hurwitz factorization $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, we obtain an admissible cover on $X_0$ which is uniquely determined by a pair of covers of $X_0^1$ and $X_0^2$, with monodromy on $X_0^1$ given by $(\sigma_1, \sigma_2, \rho)$ above $Q_1, Q_2$ and the node, and monodromy on $X_0^2$ given by $(\rho^{-1}, \sigma_3, \sigma_4)$ above the node, $Q_3$, and $Q_4$. Moreover, any such pair of covers yields a unique admissible cover, which can always be deformed to a smooth cover of monodromy $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. 

---

(*) change to $d = p$
Although it is not important to our argument, we remark that in the case that \( \rho \) is a single cycle of length \( \ell \), we actually have a unique admissible cover up to isomorphism, which deforms to \( \ell \) distinct smooth covers. When \( \rho \) is a product of disjoint cycles, each admissible cover deforms to a unique smooth cover.

3. Reduction of admissible covers: the single-cycle case

We now fix a prime \( p \), and suppose that all our \( e_i \) are prime to \( p \). We will consider the situation in characteristic \( p \). Covers and admissible covers are still uniquely determined by monodromy data, but covers which exist in characteristic \( 0 \) may specialize to inseparable maps in positive characteristic. When studying degenerations, every admissible cover still deforms to a smooth cover, with the same number of smooth covers corresponding to each admissible cover as in characteristic \( 0 \), but a smooth cover need not specialize to an admissible cover in general.

We now consider which of the admissible covers we have discussed above still persist in positive characteristic. We will treat the case of admissible covers with \( \ell \) a single ramified point above the node, equivalent to the situation above that \( \rho \) is a single cycle of length \( \ell \geq 1 \). In this case, existence of an admissible cover is equivalent to the existence of a pair of covers branched over three points, having ramification \( (e_1, e_2, \ell) \) and \( (1, e_3, e_4) \) respectively. The existence and non-existence of such covers has been settled completely in Theorem 2.4 of [?]. (*)

The case that all \( e_i \) are less than \( p \) is particularly simple. In that case, a cover with three branch points exists if and only if its degree is less than \( p \) (see Theorem 4.2 of [?]). (*)

Proposition 3.1. When \( e_i < p \) for all \( i \), the number of admissible covers with a single ramified point over the node and bad reduction in characteristic \( p \) is

\[
(d - p + 1)(d + p + 1 - e_3 - e_4)
\]

if either \( d + 1 \geq e_2 + e_3 \) or \( d + 1 - e_1 \leq p \). Otherwise, \( (d - p + 1)(d + p + 1 - e_3 - e_4) > e_1 e_2 \) and all such admissible covers have bad reduction.

Proof. Since the degree of the cover with ramification \( (\ell, e_3, e_4) \) is always at least as large as the other, it is enough to calculate when its degree is less than \( p \), which is equivalent to the inequality

\[
e_3 + e_4 + \ell \leq 2p - 1.
\]

We note that we always have \( 2p + 1 - e_3 - e_4 \leq 2d + 1 - e_3 - e_4 \).

The range of “bad cases” is then precisely those corresponding to

\[
\ell = 2p + 1 - e_3 - e_4, \ldots, 2d + 1 - e_3 - e_4.
\]

If \( d + 1 < e_2 + e_3 \) but \( d + 1 - e_1 \leq p \), then we have \( 2p + 1 - e_3 - e_4 \geq e_2 - e_1 + 1 \), so the number of covers with bad reduction is given by

\[
\sum_{\ell = 2p + 1 - e_3 - e_4}^{2d + 1 - e_3 - e_4} \ell = (d - p + 1)(d + p + 1 - e_3 - e_4).
\]

If \( d + 1 - e_1 \leq p \) and \( d + 1 - e_1 > p \), we have \( 2p + 1 - e_3 - e_4 < e_2 - e_1 + 1 \), so all \( e_1 e_2 \) admissible covers have bad reduction, as desired. \( \square \)

Now and hereafter, we restrict to the case that \( d = p \).
Corollary 3.2. If $d = p$ the number of admissible covers with bad reduction is $2p + 1 - e_3 - e_4$.

Proof. We need only note that with $d = p$, the inequality $d + 1 - e_1 \leq p$ is automatically satisfied. □

4. Stable reduction

In this section, we recall some generalities on stable reduction of Galois covers of curves. The main references for this section are [?] and [?]. We start with a group-theory result which will be used in the sequel.

Proposition 4.1. Let $p$ be a prime number and $G$ a transitive group on $p$ letters. Suppose that $G$ contains a pure cycle of length $1 < e < p - 1$. Then $G$ is either $A_p$, $S_p$ or $G$ contains a unique minimal normal subgroup isomorphic to $PSL_2(2^r)$. The last case only occurs if $p = 1 + 2^r$ and $e = p - 2 = 2^r - 1$.

Proof. A transitive group on $p$ letters is either 2-transitive or solvable in which case it is a subgroup of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/(p - 1)$. Since we exclude $e = p - 1$, the last case does not occur, and $G$ is 2-transitive, hence primitive. The Theorem of Malgraff ([?]) states that $G$ is now at least $p - e + 1$-transitive.

The 2-transitive permutation group have been classified by Cameron ([?]). Suppose that $G \neq A_p, S_p$. Since $G$ is at least 3-transitive, we conclude that $G$ has a unique minimal normal subgroup $N$ which is isomorphic to a Matthieu group $M_{11}, M_{23}$ or to $N = PSL_2(2^r)$. It is easy to check that the Matthieu group $M_{11}$ and $M_{23}$ do not contain any single cycles, for example with the computer algebra package Gap. The group $PSL_2(2^r)$ can only occur if $p = 2^r + 1$. It only contains single cycles of order $e = p - 2 = 2^r - 1$. □

Let $R$ be a discrete valuation ring with fraction field $K$ of characteristic zero and residue field an algebraically closed field $k$ of characteristic $p > 0$. Let $f : V = \mathbb{P}^1 \rightarrow X = \mathbb{P}^1_K$ be a degree-$p$ cover branched at $r$ points $Q_1 = 0, Q_2 = 1, \ldots, Q_r = \infty$ over $K$ with odd ramification indices $e_1 \leq e_2 \leq e_3 \leq \ldots \leq e_r < p$ satisfying $\sum (e_i - 1) = 2p - 2$. We denote the Galois closure of $f$ by $g : Y \rightarrow \mathbb{P}^1$. The Galois group $G$ of $g$ is $A_p$ ([?], Theorem 4.3, or Proposition 4.1). We write $H = \text{Gal}(Y, V) \simeq A_{p - 1}$, and choose a Sylow $p$-subgroup $I \subset G$. We consider the case that the marked curve $(\mathbb{P}^1; Q_i)$ is generic. We assume that $g$ has bad reduction to characteristic $p$, and denote by $\bar{g} : \bar{Y} \rightarrow \bar{X}$ its stable reduction.

Since $(X; Q_i)$ has good reduction to characteristic $p$, there exists a model $\mathcal{X}_0 \rightarrow \text{Spec}(R)$ such that the $Q_i$ extend to disjoint sections. There is a unique irreducible component $X_0$ of $X$, called the original component, on which the natural contraction map $X \rightarrow \mathcal{X}_0$ is an isomorphism. For $b \in \{1, 2, \ldots, r\}$, let $\bar{X}_b$ be the irreducible component of $X$ to which $Q_b$ specializes. Then the restriction of $\bar{g}$ to $\bar{X}_b$ is separable, and $\bar{X}_b$ intersects the rest of $\bar{X}$ in a single point $\tau_b$. Let $\bar{Y}_b$ be an irreducible component of $\bar{Y}$ above $\bar{X}_b$, and write $\bar{g}_b : \bar{Y}_b \rightarrow \bar{X}_b$ for the restriction of $\bar{g}$ to $\bar{Y}_b$. We denote by $G_b$ the decomposition group of $\bar{Y}_b$. We call the components $\bar{X}_b$ for $b \in \{1, 2, \ldots, r\}$ := $\mathcal{B}$ the primitive tails. The following lemma states, in the terminology of [?], that $\bar{g}$ does not have any new tails.

Lemma 4.2. The curve $\bar{X}$ consists of exactly $r + 1$ irreducible components: the original component $\bar{X}_0$ and the primitive tails $\bar{X}_1, \ldots, \bar{X}_r$. 

Proof. In the case that \( r = 3 \) this is proved in [?], \S 4.4. The general case is a straightforward generalization. \hfill \Box

To the stable reduction \( \bar{g} \) of \( g \), we may associate a so-called patching datum \( P(\bar{g}) \). In our situation, it consists of a deformation datum \( \omega \) together with a set of primitive tail covers \( \bar{y}_b \) for \( b \in B \) satisfying certain compatibility relations ([?]). The deformation datum is a differential form on a cyclic cover of \( \bar{X}_b \); it describes the inseparability. The primitive tail covers are Galois covers which are branched at exactly two points. The specialization of \( Q_i \) is a tame branch point of order \( e_i \). The point \( \tau_b \) of \( \bar{X}_b \) with \( \bar{X}_b \) is a wild branch point. As part of the datum of a tail cover, we also choose a point \( \eta_b \in \bar{Y}_b \) above \( \tau_b \).

Note that \( p \) divides the order of the decomposition group \( G_b \) of \( \bar{Y}_b \) since \( \bar{y}_b \) has a wild branch point. This implies that \( H_b := G_b \cap H \) has index \( p \) in \( G_b \). We conclude that the induced cover \( f_b : \bar{Y}_b / H_b \to \bar{X}_b \) is a degree-\( p \) cover between projective lines. Note that \( f_b \) has a single (tame) ramification point, which has ramification index \( e_i \) and is totally branched over \( \tau_b \). For simplicity, we call the covers primitive tail covers of ramification type \( e_i \). Lemma 4.3 states that for every \( 1 < e < p \), there is a unique primitive tail cover. This is the key observation used in the characterization of the primitive tail covers.

In the computation of the number of covers with bad reduction, we need several groups of automorphisms. Let \( f_b : \bar{Y}_b \to \bar{X}_b \) be a primitive tail cover with Galois group \( G_b \), together with a wild ramification point \( \eta_b \), as above. We define \( \text{Aut}_{G_b}(\bar{y}_b) \) as the set of \( G_b \)-equivariant automorphisms of \( \bar{Y}_b \) and \( \text{Aut}_{G_b}^0(\bar{y}_b) \subset \text{Aut}_{G_b}(\bar{y}_b) \) as the set of automorphisms which fix \( \eta_b \). Similarly, we define \( \text{Out}_{G_b}(\bar{y}_b) \) as the set of automorphisms of \( \bar{Y}_b \) which normalize the \( G_b \)-action. Note that \( \text{Out}_{G_b}(\bar{y}_b) \) is the set of outer automorphisms of the tail cover \( \bar{y}_b \), i.e. those automorphisms which do not necessarily fix the isomorphism of \( G_b \) with the Galois group of \( \bar{y}_b \). The group \( \text{Aut}_{G_b}(\bar{y}_b) \) is a subgroup of \( \text{Out}_{G_b}(\bar{y}_b) \). Its index is at most the index of \( G_b \) in its automorphism group.

Lemma 4.3. Let \( 1 < e < p \) be an odd integer.

(a) There is a unique primitive tail cover \( \varphi_e : T_e \to \mathbb{P}^1 \) of ramification type \( e \). Its Galois group is \( A_p \).

(b) The wild branch point of \( \varphi_e \) has inertia group of order \( p(p - 1) / \gcd(p - 1, e - 1) = : p m_e \). The conductor is \( h_e := (p - e) / \gcd(p - 1, e - 1) \).

(c) The group \( \text{Aut}_{G_b}(\bar{y}_b) \) (resp. \( \text{Aut}_{G_b}^0(\bar{y}_b) \)) is cyclic of order \( p - e \) (resp. \( h_e \)).

(d) The group \( \text{Out}_{G_b}(\bar{y}_b) \) is cyclic of order \( p - e \).

Proof. Let \( e \) be as in the statement of the lemma. We define the primitive tail cover \( \varphi_e \) as the Galois closure of the degree-\( p \) cover \( \bar{\varphi}_e : \bar{T}_e := \mathbb{P}^1 \to \mathbb{P}^1 \) given by \( y^p + y^e = x, (x, y) \mapsto x \). One easily checks that this is the unique degree-\( p \) cover between projective lines with one wild branch point and the required tame ramification.

The decomposition group \( G_e \) of \( T_e \) is contained in \( S_p \). We note that the normalizer in \( S_p \) of a Sylow \( p \)-subgroup has trivial center. Therefore it follows from [?], Proposition 2.2.(i) that \( \gcd(h_e, m_e) = 1 \). Part (b) follows now directly from the Riemann–Hurwitz formula (as in [?], Lemma 4.10).

To prove (a), it remains to show that the Galois group of \( \varphi_e \) is \( A_p \). Suppose that this is not the case. Proposition 4.1 implies that \( e = p - 2 = 2^r - 1 \). Moreover,
\(G_b\) has a unique minimal normal subgroup which is isomorphic to \(PSL_2(2^e)\). In particular, \(G_b\) is a subgroup of \(Aut(PSL_2(2^e)) \simeq PGL_2(2^e) \simeq PSL_2(2^e) \times \mathbb{Z}/r\mathbb{Z}\). The normalizer in \(PGL_2(r^e)\) of a Sylow \(p\) subgroup \(I\) is \(\mathbb{Z}/p\mathbb{Z} \times /\mathbb{Z}/2r\mathbb{Z}\). The normalizer in \(PGL_2(r^e)\) of a Sylow \(p\) subgroup \(I\) is \(\mathbb{Z}/p\mathbb{Z} \times /\mathbb{Z}/2r\mathbb{Z}\). Part (b) shows that the inertia group \(I(\eta_b)\) of the point \(\eta_b\) is isomorphic to \(\mathbb{Z}/p\mathbb{Z} \times /\mathbb{Z}/p-1\mathbb{Z}\). Therefore \(PGL_2(2^e)\) contains a subgroup isomorphic to \(I(\eta_b)\) if and only if \(p = 17 = 2^4 + 1\), in which case \(I(\eta_b) = N_{PGL_2(2^e)}(I)\). This proves (a) except in the case that \(p = 17 = e_2\).

The previous discussion shows that the center of \(G_e\) is trivial (also in the exceptional case). Therefore \(G_e \cap Aut_{G_e}(\varphi_e)\) is trivial. We claim that \(Aut_{G_e}(\varphi_e) \simeq \mathbb{Z}/(p-e)\mathbb{Z}\). Choose a primitive \((p-e)\)th root of unity \(\zeta \in \mathbb{F}_p\). Then \(\mu(x, y) = (\zeta^p x, \zeta^y)\) is an automorphism of \(T_e\). Taking the quotient by the action of \(\mu\), we obtain a diagram

\[
\begin{array}{ccc}
T_e & \longrightarrow & T'_e = T_e/\langle \mu \rangle \\
\varphi_e \downarrow & & \downarrow \bar{\varphi}_e \\
\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1/\langle \mu \rangle
\end{array}
\]

One easily checks that the tame ramification of \(\bar{\varphi}_e\) is \(c-(p-e)\). Let \(\psi_e : T'_e \to \mathbb{P}^1\) be the Galois closure of \(\bar{\varphi}_e\). Since \(A_p\) does not contain an \(e-p-e\) cycle, it follows that the Galois group \(G'\) of \(\psi_e\) is strictly larger than \(G\). In the general case that \(G = A_p\), we conclude that \(G' \simeq S_p\).

As above, one checks by using the Riemann-Hurwitz formula that the wild ramification point of \(\bar{\psi}_e\) has ramification of order \(p(p-1)\) and conductor 1. Using Abhyankar’s Lemma one easily checks that \(\varphi_e\) is the pullback of \(\psi_e\) via the \((p-e)\)-cyclic cover \(\mathbb{P}^1 \to \mathbb{P}^1/\langle \mu \rangle\).

We first consider the case that \(G = A_p\). Since the Galois group \(G'\) of \(\psi_e\) is twice as large as the Galois group \(G\) of \(\varphi_e\), it follows that \(Out_{G_e}(\varphi_e)\) is isomorphic to \(\mathbb{Z}/(p-e)\mathbb{Z}\) and that \(Aut_{G_e}(\varphi_e)\) has index 2 in \(Out_{G_e}(\varphi_e)\). Since a wild ramification point of \(T'_e\) has inertia group of order \(p(p-1) = pm_e gcd(p-1, e-1)\), it follows that \(Aut_{G_e}(\bar{\varphi}_e)\) has order \(h_e = (p-e)/gcd(p-1, e-1)\). This implies (c) and (d) in this case.

Now suppose that \(G = PGL_2(2^4)\) and \(p = 17 = e + 2\). In this case we have that \(p - 2 = 2\). Therefore it follows also in this case that \([G' : G] = 2\). In particular, \(G\) is a normal subgroup of \(G'\). One checks, for example with the computer algebra package Gap, that there does not exist a transitive group on \(p\) letters whose order is \(2|\text{Gal}|\). We conclude that this case does not occur. This proves the lemma.

**Lemma 4.4.** Let \(1 < e_1 \leq e_2 < p\) be odd integers.

(a) There is a unique primitive tail cover \(\varphi_{e_1, e_2} : T_{e_1, e_2} \to \mathbb{P}^1\) of ramification type \(e_1, e_2\).

(b) The wild branch point of \(\varphi_{e_1, e_2}\) has inertia group of order \(p(p-1)/gcd(p-1, e_1, e_2 - 2) = pm_{e_1, e_2}\). The conductor is \(h_{e_1, e_2} := (p+1-e_1-e_2)/gcd(p-1, e_1 + e_2 - 2)\).

(c) The group \(Aut_{G_{e_1, e_2}}(\varphi_{e_1, e_2})\) is trivial. In particular, \(Aut_{G_{e_1, e_2}}^0(\varphi_{e_1, e_2})\) is trivial, as well.

(d) Suppose that \(e_1 = e_2\). Then \(Out_{G_{e_1, e_2}}(\varphi_{e_1, e_2})\) contains an element of order 2.

**Proof.** As in the proof of Lemma 4.3, we define \(\varphi_{e_1, e_2}\) as the Galois closure of a non-Galois cover \(\bar{\varphi}_{e_1, e_2} : T_{e_1, e_2} \to \mathbb{P}^1\) of degree \(p\). The cover \(\bar{\varphi}_{e_1, e_2}\), if it exists, is
given by an equation
\[ F(y) := y^{e_1}(y - 1)^{e_2} \tilde{F}(y) = x, \quad (x, y) \mapsto x, \]
where \( \tilde{F}(y) = \sum_{i=0}^{p-e_1-e_2} c_i y^i \) has degree \( p-e_1-e_2 \). We may assume that \( c_{p-e_1-e_2} = 1 \). The condition that \( \tilde{\phi}_{e_1,e_2} \) has exactly three ramification points \( y = 0, 1, \infty \) yields the condition \( F'(y) = \gamma t^{e_1-1}(t-1)^{e_2-1} \). Therefore the coefficient of \( \tilde{F} \) satisfy the recursion
\[
(4.1) \quad c_i = c_{i-1} \frac{e_1 + e_2 + i - 1}{e_1 + i}, \quad i = 1, \ldots, p-e_1-e_2.
\]
Therefore the \( c_i \) are uniquely determined by \( c_{p-e_1-e_2} = 1 \). Conversely, it follows that the polynomial \( F \) defined by these \( c_i \) has the required ramification. We conclude that there is a unique tail cover of type \( e_1\cdot e_2 \). This proves (a). Part (b) follows from the Riemann–Hurwitz formula, as in the proof of Lemma 4.3.

If \( e_1 \neq e_2 \), the curve \( T_{e_1,e_2} \) does not admit any nontrivial automorphisms which commute with \( \tilde{\phi}_{e_1,e_2} \). Moreover, Proposition 4.1 implies that the Galois group \( G_{e_1,e_2} \simeq A_p \) in this case. This proves (c) in this case.

Suppose that \( e_1 = e_2 \). It is clear that \( \text{Aut}_{G_{e_1,e_2}}(\tilde{\phi}_{e_1,e_2}) \) is contained in \( \mathbb{Z}/2\mathbb{Z} \), since any element of this group induces a nontrivial automorphism of \( \tilde{T}_{e_1,e_2} \) which interchanges \( y = 0 \) with \( y = 1 \), and fixes the point \( y = \infty \).

We claim that \( \mu(x,y) = (-x,1-y) \) is an element of \( \text{Aut}(\tilde{T}_{e_1,e_1}) \) which satisfies these conditions. Write
\[
G(y) := \mu^* F(y) = y^{e_1}(y-1)^{e_2} \sum_{i=0}^{p-2e_1} c_i y^i = y^{e_1}(y-1)^{e_2} \sum_{i=0}^{p-2e_1} d_i y^i.
\]
Since
\[
G'(y) = \mu^* F'(y) = \gamma y^{e_1-1}(y-1)^{e_2-1},
\]
it follows that the \( d_i \) also satisfy recursion (4.1). Since \( d_{p-2e_1} = -c_{p-2e_1} \), we conclude that \( G(y) = -F(y) \). This proves the claim on \( \mu \).

We define \( \psi_{e_1,e_2} : T_{e_1,e_2}/\langle \mu \rangle \to \mathbb{P}^1/\langle \mu \rangle \) as the quotient of \( \tilde{\psi}_{e_1,e_2} \) by \( \langle \mu \rangle \). Similar to proof in the single-cycle case, one checks that \( \psi_{e_1,e_2} \) has ramification type \( e_1-2(p-1)/-e_1 \). The wild ramification has order \( 2me_1,e_2 \) and conductor \( h_{e_1,e_2} \) if \( h_{e_1,e_2} \) is odd. If \( h_{e_1,e_2} \) is even, the ramification index is \( pm_{e_1,e_2} \) and the conductor is \( h_{e_1,e_2}/2 \). We write \( (h',pm') \) for the conductor and ramification index.

As in the proof of Lemma 4.3, it follows that \( \text{Aut}_{G_{e_1,e_2}}(\tilde{\phi}_{e_1,e_2}) \simeq \mathbb{Z}/2\mathbb{Z} \) if and only if the Galois group \( G' \) of the Galois closure \( \psi_{e_1,e_2} \) equals the Galois group of \( \phi_{e_1,e_2} \). We claim that \( G' \) is never a subgroup of \( A_p \), and hence that \( G' \neq G \).

Part (c) of the lemma follows from this.

We have seen that the conjugacy class describing the tame ramification of \( \tilde{\psi}_{e_1,e_2} \) is \( e_1 \cdot 2^r \) with \( r = (p-1)/2-e_1 \). This conjugacy class is contained in \( A_p \) if and only if \( r \) is even. This happens if and only if \( p \equiv 3 \pmod{4} \), since \( e_1 \) is odd.

The tame part of the wild ramification has conjugacy class \( m'(p-1)/m' \). This conjugacy class is contained in \( A_p \) if and only if \( m' \mid (p-1)/2 \). In the case that \( p \equiv 3 \pmod{4} \), one checks that \( h_{e_1,e_2} \) is odd, and hence that \( m' = 2m_{e_1,e_2} \). Since \( (p-1)/2 \) is odd, we conclude that \( m' \mid (p-1)/2 \). Therefore \( G' \not\subseteq G \). We conclude that \( [G' : G] = 2 \). Therefore \( \text{Out}_{G_{e_1,e_2}}(\tilde{\phi}_{e_1,e_2}) \) contains an element of order two. This proves the lemma.
5. Reduction of admissible covers: the two-cycle case

Let $f_0 : V_0 \to X_0$ be an admissible degeneration of $f$ in equal characteristic 0, as in §2. In particular, the branch points $Q_1, Q_2$ (resp. $Q_3, Q_4$) specialize to $X_0^1$ (resp. $X_0^2$). We denote by $g_0 : V_0 \to X_0$ the corresponding degeneration of the Galois closure $g$ of $f$. We choose an irreducible component $Y_0^1$ of $V_0$ above $X_0^1$, and write $g_0^1 : Y_0^1 \to X_0^1$ for the corresponding cover. Let $G^1$ be the decomposition group of $Y_0^1$.

We suppose that the monodromy of $g_0^1$ (resp. $g_0^2$) is given by $(\sigma_1, \sigma_2, \rho)$ (resp. $(\rho^{-1}, \sigma_3, \sigma_4)$), where $\rho$ is not a single cycle. It follows from [?] that $\sigma_1$ and $\sigma_2$ are disjoint cycles and $\rho = \sigma_1 \sigma_2$. Our assumptions on the $e_i$ now imply that $p$ does not divide the order of $G^1$. Hence $g_0^1$ has good reduction to characteristic $p$. Moreover, the cover $g_0^1$ is uniquely determined by the triple $(\rho, \sigma_3, \sigma_4)$ if $e_1 = e_2$. If $e_1 = e_2$ there are exactly 2 possibilities for the tuple $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ for given triple $(\rho, \sigma_3, \sigma_4)$. Therefore to count the number of admissible covers with bad reduction, it suffices to consider the reduction behavior of the cover $g_0^2 : Y_0^2 \to X_0^2$. Since the decomposition group $G^1$ is contained in $G^2$, it follows that $G^2 = G$. Moreover, $g_0^2$ has ramification type $(e_1, e_2, e_3, e_4)$.

In the rest of the section, we drop the subscript 0 and the superscript 2, and write $g : Y \to X$ for the three-point cover $g_0^2 : Y_0^2 \to X_0^2$. We are now exactly in the situation of [?], §4.4. The result of Wewers allows us therefore to compute the number of Galois covers of ramification type $(e_1, e_2, e_3, e_4)$ with bad reduction. From this we deduce the number of non-Galois covers of ramification type $(e_1, e_2, e_3, e_4)$ with bad reduction.

**Proposition 5.1.** The number of non-Galois covers with ramification type $(e_1, e_2, e_3, e_4)$ and bad reduction to characteristic $p$ is $p + 1 - e_1 - e_2$ if $e_1 \neq e_2$ and $(p + 1 - e_1 - e_2)/2$ otherwise.

Before proving the proposition, we first recall the result of Wewers. Let $g : Y \to \mathbb{P}^1$ be a $G$-Galois cover with ramification type $(e_1, e_2, e_3, e_4)$ and bad reduction to characteristic $p$, where the $e_i$ satisfy the conditions of the beginning of this section. Let $P(\bar{g}) = (\omega; \bar{g}_b)$ be the associated patching datum, where $\omega$ is a deformation datum and the $\bar{g}_b$ are tail covers, as in §4. We index the tails by $e_1 - e_2, e_3, e_4$ corresponding to the ramification type of the tame ramification.

Lemmas 4.3 and 4.4 imply that for given ramification type, there is a unique tail cover. Wewers ([?], page 1029) checks that there exists a unique deformation datum in this situation. Therefore the number of covers with ramification type $(e_1, e_2, e_3, e_4)$ and bad reduction is the number of lifts of the patching datum $P(\bar{g})$, in the terminology of [?]. Wewers ([?], Lemma 2.17) shows that the number of lifts of the patching datum equals the cardinality of the group $\text{Aut}_{\mathbb{P}^1}^0(\bar{g})$ of $G$-equivariant automorphisms which induce the identity on the original component $X_0$. 
Choose $\gamma \in \text{Aut}_G^0(\bar{g})$, and consider the restriction of $\gamma$ to the original component $\bar{X}_0$. Let $Y_0$ be an irreducible component of $Y$ above $\bar{X}_0$ whose inertia group is the fixed Sylow $p$-subgroup $I$ of $G$. Write $G_0 = \bar{I} \times H_0$ for the decomposition group of $Y_0$. Wewers ([?], proof of Lemma 2.18) shows that $\gamma_0 := \gamma|Y_0$ centralizes $H_0$ and normalizes $I$, i.e. $\gamma_0 \in C_{N_G(I)}(H_0)$. Since $Y|_{\bar{X}_0} = \text{Ind}_{G_0}^G \bar{Y}$ and $\gamma$ is $G$-equivariant, it follows that the restriction of $\gamma$ to $\bar{X}_0$ is uniquely determined by $\gamma_0$. We denote by $n'$ the number of $\gamma_0 \in C_{N_G(I)}(H_0)$ such that there exists a $\gamma \in \text{Aut}_G^0(\bar{g})$ with $\gamma|\bar{X}_0 = \gamma_0$.

Wewers ([?], Corollary 4.8) shows that the cardinality of $\text{Aut}_G^0(\bar{g})$ equals

$$\frac{p-1}{n'} \left| \frac{h_{e_1,e_2}}{|\text{Aut}_{G_{e_1,e_2}}^0(\bar{g}_{e_1,e_2})|} \frac{h_{e_3}}{|\text{Aut}_{G_{e_3}}^0(\bar{g}_{e_3})|} \frac{h_{e_4}}{|\text{Aut}_{G_{e_4}}^0(\bar{g}_{e_4})|} \right|$$

where $n'$ is defined above and the other notation is as in § 4.

**Lemma 5.2.** The integer $n'$ defined above equals

$$n' = m_{e_1,e_2} = \frac{p-1}{\gcd(p-1,e_1+e_2-2)}.$$

**Proof.** In our situation, we have that $G = A_p$, therefore $C_{N_G(I)}(H_0) \simeq \mathbb{Z}/\mathbb{Z}_{p-1}$. Let $\gamma_0$ be an element of this group. We ask whether we can find an element $\gamma \in \text{Aut}_G^0(\bar{g})$ such that the restriction of $\gamma$ to $Y_0$ is $\gamma_0$. It suffices to answer this question for each of the tails separately. We also write $\gamma_0$ for the unique $G$-equivariant automorphism of $\text{Ind}_{G_0}^G \bar{Y}_0$ induced by $\gamma_0$.

Let $b \in \{e_3,e_4\}$, i.e. we consider the case of a primitive tail whose tame ramification is described by a single cycle. We first remark that the restriction of $\gamma \in \text{Aut}_G^0(\bar{g})$ to $\bar{X}_b$ is an element of $\text{Aut}_G^0(\bar{y}_b)$, since $G = G_b = A_p$ in our situation. If the order of $\gamma_0$ divides $m_b = (p-1)/\gcd(p-1,e_b-1)$, the induced automorphism of $\text{Ind}_{G_0}^G \bar{Y}_0$ acts trivially on the fiber of $\tau_b$. Therefore we may extend $\gamma_0$ to $\bar{Y}_0$ by the identity, cf. [?], Remark 2.18. In general, the length of the orbits of $\gamma_0$ are the order of $\gamma_0$ divided by $m_b$, which divides $\gcd(p-1,e_b-1)/2$.

Now choose $\gamma_b \in \text{Aut}_{G_0}(\bar{y}_b)$ a generator. Lemma 4.3.(c) shows that $\gamma_b$ has order $(p-e_b)/2$. Since $\text{Aut}_G^0(\bar{y}_b)$ has order $h_b = (p-e_b)/\gcd(p-1,e_b-1)$, it follows that the length of the orbits of $\gamma_b$ on the fiber of $\bar{y}_b$ above $\tau_b$ is $\gcd(p-1,e_b-1)/2$. Therefore, there exists a power of $\gamma_b$ whose action on the fiber above $\tau_b$ is compatible with the action of $\gamma_0$.

It remains to study the tail $\bar{g}_{e_1,e_2}$. Since $\text{Aut}_{G_{e_1,e_2}}^0(\bar{g}_{e_1,e_2})$ is trivial (Lemma 4.4.(c)), it follows by the same argument as above, that we may extend $\gamma_0$ to an $G$-equivariant automorphism of the tail if and only if the order of $\gamma_0$ divides $m_{e_1,e_2}$. If this condition is satisfied, we may choose $\gamma|\bar{X}_{e_1,e_2} = \text{Id}$. □

**Remark 5.3.** The description of the group $\text{Aut}_G^0(\bar{g})$ in [?], Lemma 2.17 is not quite correct. The elements $\gamma_j$ defined in loc. cit. do not necessarily have to normalize to whole group $G$; it suffices if they normalize $G_j \cap G_0$. A more systematic study of the group $\text{Aut}_G^0(\bar{g})$ can be found in the manuscript [?].

We have seen in the introduction that the non-Galois covers in characteristic zero correspond to Hurwitz factorization which are triples of permutations up to uniform conjugacy by $S_p$. Since the Galois group of our covers is $G = A_p$, the $G$-Galois covers correspond to triples of permutations up to uniform conjugacy by
\textbf{Lemma 5.4.} We have that
\[
\left| \text{Out}_{G_i}^0(\bar{\gamma}) : \text{Aut}_{G_i}^0(\bar{\gamma}) \right| = \begin{cases} 
2 & \text{if } e_1 = e_2, \\
1 & \text{otherwise.} 
\end{cases}
\]

\textit{Proof.} Lemma 4.3.(d) states that \( \left| \text{Out}_{G_i}(\bar{\gamma}_b) : \text{Aut}_{G_i}(\varphi_b) \right| = 2 \) if \( b = e_3, e_4 \). Suppose that \( \left| \text{Out}_{G_i}(\bar{\gamma}_b) : \text{Aut}_{G_i}(\varphi_b) \right| = 2 \), and let \( \gamma \in \text{Out}_{G_i}(\bar{\gamma}_b) \setminus \text{Aut}_{G_i}(\varphi_b) \). We write \( \gamma_{e_1, e_2} : \text{Ind}_{G_i}^G Y_{e_1, e_2} \) for the restriction of \( \gamma \) to \( X_{e_1, e_2} \). As in the proof of Lemma 5.2, we conclude that \( \gamma_{e_1, e_2} \) is nontrivial. Since \( \gamma \) normalizes the action of \( G \), it follows that \( \gamma \) induces a nontrivial automorphism on \( \text{Ind}_{G_i}^G Y_{e_1, e_2} / H \), where \( H \cong A_{p-1} \) is a subgroup of \( G \cong A_p \) of index \( p \). Since \( \bar{\gamma}_{e_1, e_2} \) is wildly ramified, it follows that the order of \( g_{e_1, e_2} \) is divisible by \( p \). The uniqueness of the tails cover (Lemma 4.4.(a)) implies therefore that \( \text{Ind}_{G_i}^G Y_{e_1, e_2} / H \cong T_{e_1, e_2} \), where \( T_{e_1, e_2} \) is as in the proof of Lemma 4.4. As in the proof of Lemma 4.4 it follows that \( e_1 = e_2 \).

Conversely, in the case that \( e_1 = e_2 \) Lemma 4.4 states that \( \text{Out}_{G_i} g_{e_1, e_2} \) contains an element of order two. This proves the lemma.

We are now able to prove Proposition 5.1.

\textit{Proof.} It follows from (5.1), together with Lemmas 4.3.(c), 4.4.(c) and 5.2 that the number of Galois covers with ramification type \( (e_1, e_2, e_3, e_4) \) which have bad reduction is
\[
N := \frac{p - 1}{n'} \left| \frac{h_{1,2}}{\text{Aut}_{G_i}^0(\bar{\gamma}_{1,2})} \right| = \frac{p - 1}{m_{e_1, e_2}} h_{e_1, e_2} = p + 1 - e_1 - e_2.
\]

We have seen that the number of non-Galois covers with ramification type \( (e_1, e_2, e_3, e_4) \) which have bad reduction is \( N / \left| \text{Out}_{G_i}^0(\bar{\gamma}) \right| \). Therefore the proposition follows from Lemma 5.4.

The following corollary translates the result into a result for admissible degree-\( p \) covers in characteristic \( p \).

\textbf{Corollary 5.5.} The number of admissible covers in characteristic \( p \) with ramification type \( (e_1, e_2, e_3, e_4) \) such that the first two branch points specialize to the same component is \( \min_i \{ e_i(p + 1 - e_i) \} - p \).

\textit{Proof.} The number of admissible covers in the single-cycle case is computed in Corollary 3.2.

In the 2-cycle case, we use Proposition 5.1. We recall from the beginning of this section, that in the 2-cycle case the number of admissible covers with bad reduction as in the statement of the corollary is the number of three-point covers

\( G \). The difference between the two numbers is a factor \( 2 = |S_p : A_p| \). Since \( S_p \) is the automorphism group of \( A_p \), we can reformulate this as saying that the Hurwitz factorization correspond to isomorphism classes of Galois covers where we do not fix an isomorphism between the Galois group \( A_p \); this is sometimes called the case of mere covers. Adapting the proof of Wewers to the case of non-Galois covers, we see that the number of non-Galois covers with bad reduction is equal to the cardinality of the group \( \text{Out}_{G_i}^0(\bar{\gamma}) \) of automorphisms of \( Y \) which normalize the \( G \)-action and induce the identity on the original component \( X_0 \). Since the index of \( G = A_p \) in \( \text{Aut}(G) = S_p \) is two, we find that \( \left| \text{Out}_{G_i}^0(\bar{\gamma}) : \text{Aut}_{G_i}^0(\bar{\gamma}) \right| \leq 2 \).

with ramification type \((e_1, e_2, e_3, e_4)\) if \(e_1 \neq e_2\) and is twice this number if \(e_1 = e_2\). Therefore Proposition 5.1 implies that this number is \(p + 1 - e_1 - e_2\) in both cases.

We conclude that the total number of admissible covers with bad reduction is

\[
(2p + 1 - e_3 - e_4) + (p + 1 - e_1 - e_2) = p.
\]

\[
\square
\]

6. Reduction of Four-point Covers

In this section, we count the number of non-Galois covers with ramification type \((p; e_1, e_2, e_3, e_4)\) and bad reduction in the case that the branch points are generic. Equivalently, we compute the \(p\)-Hurwitz number \(h_p(e_1, e_2, e_3, e_4)\). As in \(\S\) 5, we first consider the case of Galois covers.

Suppose that \(r = 4\), and let \(g : Y \to X = \mathbb{P}^1_k\) be as in \(\S\) 4. In particular, we suppose that \((X; Q_i)\) is the generic \(r\)-marked curve of genus 0. It is no restriction to suppose that \(Q_1 = 0, Q_2 = 1, Q_3 = \lambda, Q_4 = \infty\), where \(\lambda\) is transcendental over \(\mathbb{Q}_p\). We suppose that \(g\) has bad reduction to characteristic \(p\), and denote by \(\tilde{g} : \tilde{Y} \to \tilde{X}\) the stable reduction. We have seen in \(\S\) 4 that we may associate to \(\tilde{g}\) a patching datum \((\omega; \tilde{g}_b)\). We have also seen that the primitive tail covers \(\tilde{g}_b\) for \(b \in B = \{1, 2, 3, 4\}\) are uniquely determined by \(e_b\) (Lemma 4.3). In the proof of the following proposition we show that there is a (in fact unique) deformation datum. By using some standard techniques in formal patching, we show that the patching datum may be “lifted” to a smooth four-point cover in characteristic zero. From this one concludes that there exists at least one four-point cover with bad reduction. For \(b \in B\), we define \((h_b, m_b) = ((p + 1 - e_b) / \gcd(p - 1, e_b - 1), (p - 1) / \gcd(p - 1, e_b - 1))\) (cf. Lemma 4.3).

**Proposition 6.1.** There exists a Galois cover with ramification type \((e_1, e_2, e_3, e_4)\) which has bad reduction to characteristic \(p\).

**Proof.** The first step in the proof is to construct a patching datum. The existence of the tail covers follows from Lemmas 4.3 and 4.4. It remains to construct the deformation datum. The compatibility condition between the tail covers and the deformation datum determines the key invariant of the deformation datum called the signature. We refer to [?] or [?] for a precise definition and more details. In our situation, the deformation datum is a logarithmic differential form \(\omega\) on the cover \(\tilde{Z}_0\) of \(\tilde{X}_0\) defined as a connected component of the projective curve with Kummer equation

\[
(6.1) \quad z^{p-1} = x^{p-e_1}(x - 1)^{p-e_2}(x - \lambda)^{p-e_3}.
\]

The degree of \(\tilde{Z}_0 \to \tilde{X}_0\) is

\[
m := \frac{p - 1}{\gcd(p - 1, e_1 - 1, e_2 - 1, e_3 - 1, e_4 - 1)}.
\]

The differential form \(\omega\) may be written as

\[
(6.2) \quad \omega = \epsilon \frac{z \, dx}{x(x - 1)(x - \lambda)} = \frac{x^{e_1-1}}{x^{p-1}}(x - 1)^{e_2} (x - \lambda)^{e_3} \frac{z^p \, dx}{x},
\]

where \(\epsilon \in k^\times\) is a unit. To show the existence of the deformation datum, it suffices to show that one may choose \(\epsilon\) such that \(\omega\) is logarithmic, or, equivalently, such that \(\omega\) is fixed by the Cartier operator \(\mathcal{C}\). It follows from standard properties of
the Cartier operator, (6.2) and the assumptions on the \(e_i\) that \(C_\omega = c^{1/p} e^{(1-p)/p} \omega\), where

\[
c = \min(e_3,e_1+e_2+e_3-p-1) \sum_{j=\max(0,e_1+e_3-p-1)}^{e_2} \binom{e_2}{e_1+e_2+e_3-p-1-j} \binom{e_3}{j} x^j.
\]

One easily checks that \(c\) is nonzero as polynomial in \(\lambda\). Therefore we may choose \(\epsilon \in k^\times\) such that \(\epsilon^{p-1} = c\). Then \(C_\omega = \omega\), and \(\omega\) defines a deformation datum.

We consider the patching datum \(P = (\omega; (\varphi_{e_i})_{e_i \in B})\) with \(\varphi_{e_i}\) as defined in Lemma 4.3. We claim that there exists a \(G\)-Galois cover \(g : Y \to X = \mathbb{P}^1\) with ramification type \((e_1,e_2,e_3,e_4)\) which has bad reduction and patching datum \(P(\bar{g}) = P\). This lifting result is proved in [\text{?}], Proposition 2.4.1. We sketch the proof in our situation, which is much easier than the general one due to the easy structure of the stable reduction (Lemma 4.2). We remark that away from the point \(\tau_b\), the primitive tail cover \(\bar{g}_b\) is tamely ramified. Therefore we can lift this cover of affine curves to characteristic zero.

Let \(X_0 = \mathbb{P}^1_R\) be equipped with 4 sections \(Q_1 = 0, Q_2 = 1, Q_3 = \lambda, Q_4 = \infty\), where \(\lambda \in R\) is transcendental over \(\mathbb{Z}_p\). Then (6.1) defines an \(m\)-cyclic cover \(\mathcal{Z}_0 \to X_0\). We write \(Z \to X\) for its generic fiber. Associated to the deformation datum is a character \(\chi : \mathcal{Z}/m\mathcal{Z} \to \mathbb{F}_p^\times\) defined by \(\chi(g) = g z/z (\mod z)\). The differential form \(\omega\) corresponds to a \(p\)-torsion point \(Q_0 \in J(\mathcal{Z}_0)[p]\chi\) on the Jacobian of \(\mathcal{Z}_0\). Since \(\sum_{i=1}^4 h_{e_i} = 2m\) and the branch points are generic, we have that \(J(\mathcal{Z}_0)[p]\chi \simeq \mathcal{Z}/p\mathcal{Z} \times \mu_p\) ([\text{?}]). After enlarging the discretely valued field \(K\), if necessary, we may choose a \(p\)-torsion point \(Q \in J(\mathcal{Z}_0 \otimes R K)[p]\chi\) lifting \(Q_0\). It corresponds to an étale \(p\)-cyclic cover \(W \to Z\). The cover \(\psi : W \to X\) is Galois, with Galois group \(N := \mathbb{Z}/p\mathcal{Z} \rtimes \mathcal{Z}/m\mathcal{Z}\). It is easy to see that \(\psi\) has bad reduction, and that its deformation datum is \(\omega\).

By using formal patching ([\text{?}] or [\text{?}]), one now checks that there exists a map \(g_R : \mathcal{Y} \to \mathcal{X}\) of stable curves over \(\text{Spec}(R)\) whose generic fiber is a \(G\)-Galois cover of smooth curves, and whose special fiber corresponds to the patching datum \(P\). Over a neighborhood of the original component \(g_R\) is the induced cover \(\text{Ind}_{\mathcal{Y}_0}^{\mathcal{Y}} \psi_R\). Over the tails, the cover \(g_R\) is induced by the lift of the tail covers. The fact that we can patch the tail covers with the cover over \(X_0\) follows from the observation that \(h_{e_i} < m_{e_i}\) (Lemma 4.3), since locally there a unique cover with this ramification ([\text{?}], Lemma 2.12).

The following proposition shows in the case that the branch point are generic the number of covers with bad reduction is divisible by \(p\).

**Proposition 6.2.** Suppose that \((X = \mathbb{P}^1_k; Q_i)\) is the generic \(r = 4\)-marked curve of genus zero. Then the number of covers of \(X\) of ramification type \((e_1,e_2,e_3,e_4)\) with bad reduction is divisible by \(p\).

**Proof.** We use the notation of the proof of Proposition 6.1. This proposition is proved in [\text{?}] Lemma 3.4.1 in a more general setting. In our situation, the statement of the proposition follows from the observation that the set of lifts \(Q\) of the \(p\)-torsion point \(Q_0 \in J(\mathcal{Z}_0)[p]\chi\) corresponding to the deformation datum is a \(\mu_p\)-torsor.

**Theorem 6.3.** Let \(p\) be an odd prime and \(k\) an algebraically closed field of characteristic \(p\). Suppose given odd integers \(1 \leq e_1 \leq e_2 \leq e_3 \leq e_4 < p\). There exists a dense open subset \(U \subset \mathbb{P}^1_k\) such that for \(\lambda \in U\) the number of degree-\(p\) covers with
ramification type \((e_1, e_2, e_3, e_4)\) is given by the formula

\[ h_p(e_1, \ldots, e_4) = \min_i (e_i (p + 1 - e_i)) - p. \]

Furthermore, every such cover has good degeneration under a degeneration of the base sending the first two branch points to the same component.

Proof. Propositions 6.1 and 6.2 imply that the number of degree-\(p\) covers with ramification type \((e_1, e_2, e_3, e_4)\) and bad reduction is at least \(p\). This implies that the generic Hurwitz number \(h_p(e_1, \ldots, e_4)\) is at most \(\min_i (e_i (p + 1 - e_i)) - p\). Corollary 5.5 implies that the number of admissible covers in characteristic \(p\) equals \(\min_i (e_i (p + 1 - e_i)) - p\). Since the number of separable covers can only decrease under specialization, we conclude that the number of admissible covers and the generic Hurwitz number both equal \(\min_i (e_i (p + 1 - e_i)) - p\). This proves the theorem. \(\square\)