Contemporary Mathematics

Two Denegeration Techniques for Maps of Curves

Brian Osserman

Abstract. In this paper, we discuss the theories of admissible covers (Harris-Mumford) and limit linear series (Eisenbud-Harris), two techniques for using degenerations to study maps between curves. While the theories are superficially very similar, they have certain fundamental differences which we will also explore.

1. Motivational problems

We begin by laying out a number of questions on curves which should serve as motivation for the degeneration theories we discuss. Not all the problems mentioned require the use of admissible covers or limit linear series, or even degeneration arguments in general. However, they may all be approached by these methods, typically producing very natural solutions.

One fundamental question is the following:

Question 1.1. What kinds of maps exist from curves to other curves? To projective spaces?

The first part of the question leads naturally to the theory of branched covers of curves, while the second leads to the theory of linear series. In the case of maps to the projective line, both theories apply and the interplay between them is often fruitful, particularly in positive characteristic, where even basic questions remain open [**12**], [**11**], [**9**]. The fundamental distinction in this context is that the theory of branched covers treats prescribed branching on the target, while the theory of linear series deals with ramification on the source. However, we will see that the techniques and results of both have a number of important distinctions.

First, we discuss some of the more specific directions in which one can take Question 1.1.

One could make the question more precise by, for instance, fixing the target curve and the branch behavior of a map, and asking if there is a source curve giving such a map; this is closely tied to the theory of fundamental groups [**1**]. Or, one could fix the dimension of the projective space and the degree of a prospective map

2000 *Mathematics Subject Classification.* 14H30.

Key words and phrases. Covers, linear series.

This paper was supported by fellowships from the Clay Mathematics Institute and National Science Foundation.

(or even more sharply, its ramification behavior), and ask what curves admit such a map to projective space, and what the dimension of the space of such maps is; this is Brill-Noether theory [**6**].

One can also ask such questions in a more enumerative vein: in cases above where there are a finite expected number of maps, one could ask how many there are. One is thus led, for instance, to Hurwitz numbers (see, e.g., [**13**]) and to numbers of maps to projective spaces with prescribed ramification [**10**].

Finally, questions such as what sort of Weierstrass points can exist on curves can be rephrased in terms of the ramification behavior of certain specific maps to projective space [**4**], and thus fall under the same general rubric.

A question with a rather different flavor is the following:

Question 1.2. Is it possible to deform any curve to any other curve of the same genus?

This is the question of the connectedness of the moduli space of curves, which naturally lends itself to degenerations arguments. Although it was first solved by Deligne and Mumford [**2**], the first algebraic argument was provided by Fulton, using the theory of admissible covers [**7**, Appendix].

Another basic problem in the moduli of curves is the following:

Question 1.3. What kinds of curves and families of curves can we actually write down?

For instance, one could ask when we can write down a family of curves containing the 'general' curve of a given genus: that is, containing every curve in a dense subset of the moduli space. Alternatively, the Brill-Noether theory mentioned above describes what kind of maps to projective space a general curve of a given genus admits, and one could ask simply to write down a single curve of each genus having the behavior described by that theory. For small genus, it is easy to write down such a general curve in either sense; for instance, any curve of genus 2 may be written in the form $y^2 = f(x)$ for $f(x)$ a sextic having distinct roots. However, one finds that as the genus increases, it is no longer possible to write down general curves so easily, and that indeed, for sufficiently large genus, no one has been able to give explicit equations for a curve provably having the behavior prescribed by the Brill-Noether theorem.

The difficulty in writing down general families may be expressed more precisely in modern language by the fact that for sufficiently large genus, the moduli spaces of curves are no longer unirational. This then leads naturally to the question of what the Kodaira dimension of moduli spaces is. This question turns out to be closely related to the earlier questions, because one studies the Kodaira dimension of the moduli space by close study of particular divisors. Some of the most natural and productive divisors to consider are described by the curves which are not general in the sense of Brill-Noether theory, so to approach Question 1.3 in the sense just described, both Harris and Mumford [**7**] and Eisenbud and Harris [**5**] returned to Question 1.1. Indeed, it was largely with this motivation that they developed the theories of admissible covers and limit linear series, respectively.

2. Admissible covers

Spaces of branched covers of smooth curves are straightforward to define, and natural objects to study. They parametrize the data of pairs of smooth curves

(C, D), with marked points on each, and a map between them mapping marked points to marked points, and such that the ramification occurs at the marked points. These are taken up to isomorphism, which in practice means that all maps obtained by composition with automorpism of the cover are considered equivalent. In the case that the base is $\mathbb{P}^1$, and the branching is simple, the corresponding parameter space is known as the Hurwitz scheme, but neither restriction is necessary to get a well-behaved moduli space. As is often the case, in order to better study branched covers, one quickly finds it desireable to compactify the space in some well-behaved manner.

Of course, in order to compactify spaces of maps between curves with marked points, one must first know how to compactify the spaces of curves with marked points themselves, so we begin by reviewing this. Certainly, families of curves may become singular, and when they do so, it is often not possible to replace the singular curve in the family with a smooth one. However, if we allow curves with nodes, this turns out to be enough: it is a theorem that for any family of curves degenerating to a singular curve, the limit curve may be replaced by a nodal curve without altering the other curves in the family. This also turns out to handle the case with marked points: remarkably, if one allows nodal curves, it is not even necessary to allow marked points to come together in order to obtain a compact moduli space. The conceptual reason for this is that if one had a family where two marked points came together in the special fiber, one could blow up the family at that point. This would introduce a new component in the special fiber, and the marked points would then approach different points in this new component. However, in order for the moduli spaces to be well-behaved, one has to restrict to **stable curves**: these are nodal curves with finitely many automorphisms. Equivalently, stable curves have the property that every component of geometric genus 0 has a total of at least three marked points or points lying above nodes in its normalization, and every component of geometric genus 1 has at least one marked point or node. Restricting to such curves then leads to a proper moduli space.

The concept behind admissible covers is very similar. In order to compactify the moduli space, one allows both the source and target curves to become singular. There are varying precise definitions of admissible covers, depending on the context, but the main idea is roughly as follows.

Definition 2.1. A map $f : C \to D$ of curved with marked points is defined to be an **admissible cover** of degree d if:

- C and D are both stable curves, and f has degree d.
- The set of nodes of C are precisely the preimage under f of the set of nodes of D.
- The set of smooth ramification points of C are the marked points of C.
- Lift f to $\tilde{f} : \tilde{C} \to \tilde{D}$ on the normalizations of C and D. Then for each node of C, the ramification indices of $\tilde{f}$ at the two points of $\tilde{C}$ lying above the node must coincide.

There is a somewhat more technical definition for a family of admissible covers, but we will not concern ourselves with that. We now consider an example of admissible covers.

Example 2.2. We consider the question of which admissible covers there are with (the underlying curve) C being stable of genus 3, D having genus 0, and f of degree 2.

Of course, if C is irreducible and smooth, we will have precisely the hyperelliptic curves of genus 3 as possibilities for C, and f the hyperelliptic covering map.

If C is a union of two smooth components C_1 and C_2 at a single node, the components must have genus 1 and 2 (we may suppose respectively) in order for the underlying curve to be stable; and we see that D must be a union of two $\mathbb{P}^1$'s at a node. Because C has only a single node, it must map to the node of D, and it must be the only point of C to map to that node; it follows that C_1 and C_2 must each map two-to-one to a (distinct) component of D, with one of the ramification points of C_1 gluing to one of the ramification points of C_2. We see that for given C_1 and C_2, there are only finitely many admissible covers (up to automorphism of the C_i) that are of this form.

Next, if C is a union of two smooth components C_1, C_2 at two nodes, we necessarily have two components of genus 1. D must have at least one node, hence at least two components, and it follows that it has exactly two, with C_1 and C_2 each mapping two-to-one to a distinct component, as before. However, in this case there are two nodes above the node of D, so each of the points on the C_i mapping to the node must be unramified, and we find that they must be hyperelliptic conjugates on each component. In this case, given C_1 and C_2 the admissible cover is actually unique up to automorphism.

Similarly, it is straightforward to first check that it is not possible for C to have any nodal irreducible components, and then to write down a list of the remaining possibilities.

Remark 2.3. One may define admissible covers very naturally in the context of log structures. This approach appears to be helpful in proving existence results for moduli spaces of admissible covers. It also leads to a very simple understanding of smoothability of of admissible covers, and the local structure of the space of admissible covers at the boundary. See [**8**, §3] for details, but note the correction to the statement of Theorem §3.22 that the morphisms of the stack should consist of isomorphisms α, β between possibly distinct triples $(C; D; \pi)$ and $(C'; D'; \pi')$ such that $\pi' \circ \alpha = \beta \circ \pi$.

3. Limit linear series

We now discuss linear series and limit linear series, starting with general definitions before exploring some concrete examples. Let C be a smooth, proper curve. A **linear series** of degree d and dimension r on C is a pair $(\mathcal{L}, V)$ of a line bundle $\mathcal{L}$ of degree d on C and a subspace $V \subset H^0(C, \mathcal{L})$ of global sections of $\mathcal{L}$ of dimension $r+1$. Given a linear series on C, at each point $P \in C$, one can look at the order of vanishing of the sections in V at P. It is easy to check (for instance, by finding a basis with strictly increasing order of vanishing at P) that there will be $r+1$ orders of vanishing, forming a sequence $0 \leq a_0^{(\mathcal{L},V)}(P) < \cdots < a_r^{(\mathcal{L},V)}(P) \leq d$, called the **vanishing sequence** of $(\mathcal{L}, V)$ at P. For notational convenience, we define the **ramification sequence** $\alpha_i^{(\mathcal{L},V)}(P)$ at P in terms of the vanishing sequence by the formula

$$\alpha_i^{(\mathcal{L},V)}(P) = a_i^{(\mathcal{L},V)}(P) - i.$$

Finally, we say that P is a **base point** of a linear series if the first term of the vanishing sequence (equivalently, ramificatication sequence) is positive, or equivalently, if every section in V vanishes at P.

We pause momentarily for motivation. It is a standard fact that if $(\mathcal{L}, V)$ has no base points, and we choose a basis for V, we obtain a map from C to $\mathbb{P}^r$. Conversely, given a map from C to $\mathbb{P}^r$, if we choose coordinates on $\mathbb{P}^r$ we can recover $\mathcal{L}$ by pulling back $\mathcal{O}(1)$ on $\mathbb{P}^r$, and the $r+1$ sections giving the basis of V by pulling back the coordinate sections. Maps to $\mathbb{P}^r$ are thus equivalent to basepoint-free linear series together with a choice of basis of V, and change of basis corresponds simply to applying an automorphism of $\mathbb{P}^r$, so we find that basepoint-free linear series on C of dimension r are equivalent to maps from C to $\mathbb{P}^r$, up to automorphism of the image space.

Example 3.1. Suppose that $C = \mathbb{P}^1$. The only line bundle on C of degree d is then $\mathcal{O}(d)$, and its global sections are equivalent to polynomials of degree d. A linear series is thus an $(r+1)$-dimensional space of polynomials of degree d, and it will be basepoint-free if it has no common factors. A choice of basis gives $r+1$ polynomials of degree d, which if they have no common factors define a map of degree d to $\mathbb{P}^r$.

Example 3.2. Suppose that $r = 1$. Then the vanishing sequence at a point P consists only of two terms. The first is 0 if the linear series is basepoint-free, and in this case, the second is simply the ramification index at P of the induced map from C to $\mathbb{P}^1$, as a map of curves. Note that here we use convention for the ramification index between maps of curves for which x^e has ramification index e, not $e-1$.

We now define limit linear series on reducible curves. For technical reasons, we must restrict to curves of **compact type**, which are nodal curves whose dual-graph is a tree, or equivalently, whose Jacobian is proper. On such a curve C, with irreducible components $C_1, \dots, C_n$, we define a **limit linear series** (or limit series) of degree d and dimension r to be a collection of linear series $(\mathcal{L}_i, V_i)$ of the same degree and dimension, with one on each of the C_i, and such that at each node P, if P is the intersection of C_i and C_j, the following compatibility condition is satisfied:

$$\alpha_m^{(\mathcal{L}_i, V_i)} + \alpha_{r-m}^{(\mathcal{L}_j, V_j)} = d,$$

for any m between 0 and r.

Example 3.3. Returning to the case of $r = 1$, one checks that this compatibility condition is equivalent to the condition that, after removing base points at the node P, the ramification indices (as maps as curves) at P on each component are equal, as in the definition of admissible covers.

4. Properness and deformations

In making degeneration arguments, there are typically two essential properties that must be checked. We will refer to these as properness and deformability. Properness is simply the property that given any degenerating family of curves, the associated family of objects is proper. Heuristically, if the objects of interest exist for a general curve in the family, their limit should exist for the degenerate curve as well. Deformability is less standardized from a technical standpoint, but conceptually perfectly clear: if the objects of interest exist for the degenerate curve, one should be able to "smooth them out" to a general curve, either in a prespecified family containing the degenerate curve, or abstractly, to some family containing the degenerate curve (and with smooth general fiber). Often, this boils down to flatness of the family of objects of interest, but not always, as we will see shortly.

We first briefly address properness. In characteristic 0, properness is reasonably well-behaved for both admissible covers and limit series. In either case, one obtains properness results for practical applications by explicit construction of limits of families, possibly after appropriate base change [**7**, p. 60], [**3**, Thm. 2.6].

Deformability provides a more interesting comparison. Admissible covers are always deformable without hypotheses, which can be seen easily via the now-standard deformation theoretic approach of considering thickenings over Artinian bases [**7**, p. 61]. However, in general they are not uniquely deformable, with the number of possible deformations depending on the ramification indices at the nodes. In contrast, limit series are not always deformable [**3**, Ex. 3.2], and the condition one uses to conclude deformability from the general theory is that the space of limit series on the degenerate curve have the expected dimension. This condition arises from the fairly explicit nature of the construction of schemes parametrizing linear series. Via this construction, one obtains a lower bound on the dimension of the resulting scheme of limit series, and it follows that if the dimension of a fiber is small enough, each point in the fiber must be on a component dominating the base of the family, which is to say precisely that each limit series on the degenerate curve must deform to the general curve in the family. In characteristic 0, there are generalized Brill-Noether results [**3**, Thm. 4.5] which for many applications allow one to automatically assert the expected dimension hypothesis and conclude deformability, but one typically finds that in the context of limit series arguments with only a finite (expected) number of limit series on each curve, deformations are unique. Thus, the flavor is quite different.

Example 4.1. We can conclude from the properness and deformability properties for admissible covers that the reducible possibilities for C described in Example 2.2 actually give the closure of the hyperelliptic locus in the moduli space of stable curves of genus 3.

5. Relationship between the theories

The theories of limit series and admissible covers are certainly closely related. For instance, limit series with $r = 1$ may always be interpreted as giving admissible covers with base having genus 0, and vice versa [**3**, §5 a)], although the correspondence is not unique in either direction. Even for $r > 1$, one can explicitly construct a geometric realization of a family of limit series as a family of maps; this is described in [**3**, §5 b)]. And as remarked earlier, the overlap in results is substantial, most notably when computing the Kodaira dimensions of moduli spaces of curves.

That said, there are some fairly clear distinctions as well. Admissible covers specify branching on the target curve, while limit series specify ramification on the source. They only overlap directly in the case of the target being of genus 0, with admissible covers generalizing to higher-genus curves, and limit series generalizing to higher-dimensional projective spaces. Furthermore, the technical restriction of limit series to source curves of compact type can also pose obstacles to attacking certain problems with this theory.

However, the distinctions become far sharper in positive characteristic. Here, it turns out that it is frequently useful to apply the perspective of linear series to obtain results on branched covers, and vice versa (see, for instance [**12**]). Certain fundamental differences emerge quickly. For instance, there are always only finitely many branched covers of a given curve with prescribed tame branching. However, even in characteristic 0, the expected dimension behavior of Brill-Noether theory only holds for general curves and ramification configurations, and in positive characteristic there are examples even for genus 0: the family $x^{p+2} + tx^p + x$ is easily verified to give an infinite family of rational functions on $\mathbb{P}^1$, all with the same ramification. On the other hand, degenerations behave somewhat better for limit series than for admissible covers. It is not yet practical to prevent admissible covers from degenerating from separable to inseparable, which means that in positive characteristic, spaces of admissible covers are typically not proper. The same problem can certainly occur for limit series, but there are certain positive results controlling this phenomenon (see [**11**, Thm. 6.1]).

Acknowledgements

I would like to thank Akio Tamagawa, Shinichi Mochizuki and Joe Harris for their patient explanations of the theories discussed here, as well Herb Clemens, Rob Lazarsfeld, and Ravi Vakil, for their invitation to give the talk upon which this paper is based. I would also like to thank the referee for his or her suggestions.

References

1. Jean-Benoit Bost, Francois Loeser, and Michel Raynaud (eds.), *Courbes semi-stables et groupe fondamental en geometrie algebrique*, Birkhauser, 1998.
2. Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Institut Des Hautes Etudes Scientifiques Publications Mathematiques (1969), no. 36, 75–109.
3. David Eisenbud and Joe Harris, *Limit linear series: Basic theory*, Inventiones Mathematicae **85** (1986), 337–371.
4. ———, *Existence, decomposition, and limits of certain Weierstrass points*, Inventiones Mathematicae **87** (1987), 495–515.
5. ———, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Inventiones Mathematicae **90** (1987), 359–387.
6. Phillip Griffiths and Joseph Harris, *On the variety of special linear systems on a general algebraic curve*, Duke Mathematical Journal **47** (1980), 233–272.
7. Joe Harris and David Mumford, *On the Kodaira dimension of the moduli space of curves*, Inventiones Mathematicae **67** (1982), 23–88.
8. Shinichi Mochizuki, *The geometry of the compactification of the Hurwitz scheme*, Publ. RIMS **31** (1995), no. 3, 355–441.
9. Brian Osserman, *Linear series and existence of branched covers*, in preparation.
10. ———, *The number of linear series on curves with given ramification*, International Mathematics Research Notices **2003**, no. 47, 2513–2527.
11. ———, *Rational functions with given ramification in characteristic p*, preprint.
12. Akio Tamagawa, *Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups*, Journal of Algebraic Geometry **13** (2004), no. 4, 675–724.
13. Ravi Vakil, *Genus 0 and 1 Hurwitz numbers: Recursions, formulas, and graph-theoretic interpretations*, Trans. Amer. Math. Soc. **353** (2001), 4025–4038.

Department of Mathematics, University of California, Berkeley, CA 94720
E-mail address: osserman@math.berkeley.edu