LIFTING TROPICAL INTERSECTIONS

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Abstract. Tropicalization commutes with intersection when the intersections have the expected dimension [...].

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1. Introduction

Tropicalization does not commute with intersection. The tropicalization of the intersection of two subvarieties of a torus $T$ over a nonarchimedean field is contained in the intersection of their tropicalizations, but this containment is sometimes strict. For instance, if $X$ is smooth in characteristic zero, and $X'$ is the translate of $X$ by a general torsion point $t$ in $T$, then $X$ and $X'$ meet transversely but their tropicalizations are equal. (*) In this case, the dimension of the tropical intersection rewrite
Trop(X) ∩ Trop(X') is strictly greater than the dimension of the algebraic intersection X ∩ X', so there are tropical intersection points that cannot be lifted to algebraic intersection points. Our main result, in its most basic form, says that the dimension of the tropical intersection is the only obstruction to lifting.

1.1. **Tropicalization commutes with proper intersections.** Following standard terminology from intersection theory in algebraic geometry, say that Trop(X) and Trop(X') meet properly at a point w if Trop(X) ∩ Trop(X') has pure codimension \( \text{codim}_X + \text{codim}_{X'} \) in a neighborhood of w.

**Theorem 1.1.1.** Suppose Trop(X) meets Trop(X') properly at w. Then w is contained in the tropicalization of X ∩ X'.

In other words, tropicalization commutes with intersections when the intersections have the expected dimension. This generalizes a well-known result of Bogart, Jensen, Speyer, Sturmfels and Thomas, who showed that tropicalization commutes with intersections when the tropicalizations meet transversely \([7, \text{Lemma 3.2}].\) However, Theorem 1.1.1 is still too restrictive for many applications. Frequently, X and X' are subvarieties of an ambient variety Z inside the torus. In this case, one cannot hope that Trop(X) and Trop(X') will meet properly in the above sense. Instead, we say that the tropicalizations Trop(X) and Trop(X') meet properly in Trop(Z) at a point w if the intersection Trop(X) ∩ Trop(X') ⊂ Trop(Z) has pure codimension \( \text{codim}_Z X + \text{codim}_Z X' \) in a neighborhood of w. We extend Theorem 1.1.1 to this situation, under an additional hypothesis on Trop(Z).

Recall that Trop(Z) is the underlying set of a polyhedral complex of pure dimension \( \text{dim} Z \), with a locally constant multiplicity function \( m \) on the interior of the facets. We say that a point w in Trop(Z) is simple if it lies in the interior of a facet, and the multiplicity \( m(w) \) is 1.

**Theorem 1.1.2.** Suppose Trop(X) meets Trop(X') properly in Trop(Z) at a simple point w. Then w is contained in the tropicalization of X ∩ X'.

Every point in the tropicalization of the torus is simple, so Theorem 1.1.1 is the special case of Theorem 1.1.2 where Z is the full torus T. We strengthen Theorem 1.1.2 further by showing that where Trop(X) and Trop(X') meet properly, the facets of Trop(X ∩ X') appear with the expected multiplicities. See Theorem 1.1.1 for a precise statement.

The proof of Theorem 1.1.2 is in two steps. Recall that the tropicalization of a subvariety X of T is the set of weight vectors w such that the initial degeneration \( X_w \) is nonempty in the torus torsor \( T_w \) over the residue field. We first show that we can pass from tropicalizations to initial degenerations:

**Theorem 1.1.3.** If Trop(X) meets Trop(X') properly in Trop(Z) at a simple point w, then \( X_w \) meets \( X'_w \) properly at a smooth point of \( Z_w \).

Indeed, if w is a simple point of Trop(Z) then \( Z_w \) is smooth, and standard tropical arguments imply the desired properness, so the main difficulty is showing that \( (X ∩ X')_w \) is nonempty, which requires working with extended tropicalizations and the intersection theory of toric varieties. See Section 3. The geometric tools

\[ (*) \]
over rank one valuation rings developed in Section 4 then show that we can lift points of $(X \cap X')_w$ to $X \cap X'$, so Theorem 1.1.2 is a consequence of the following theorem on proper intersections of initial degenerations.

**Theorem 1.1.4.** Let $w$ be a weight vector that maps the character lattice into the value group and suppose that $X_w$ meets $X'_w$ properly at a smooth point $\overline{\pi}$ of $Z_w$. Then $\overline{\pi}$ is contained in $(X \cap X')_w$.

(*) Theorem 1.1.4 is considerably stronger than Theorems 1.1.1 and 1.1.2; it often happens that initial degenerations at $w$ meet properly even when the tropicalizations do not. See Example [ REF ].

The proof of Theorem 1.1.4 is a straightforward application of a geometric argument involving the dimension of intersection, but due to the non-Noetherian setting, it requires the systematic extension of basic dimension theory from the Noetherian case to the situation of schemes over general rank one valuation rings. We use Noetherian approximation to prove the Krull principal ideal theorem in this context, and then give direct arguments along classical lines to prove that codimensions of intersections are well-behaved.

**Remark 1.1.5.** The main results of this paper can be interpreted as correspondence theorems for points in the spirit of Mikhalkin’s celebrated correspondence theorem for plane curves [7]. Mikhalkin considers curves of fixed degree and genus subject to constraints of passing through specified points. He shows, roughly speaking, that a tropical plane curve that moves in a family of the expected dimension passing through specified points lifts to a predictable number of algebraic curves passing through prescribed algebraic points. Here, we consider points subject to the constraints of lying inside subvarieties $X$ and $X'$ and prove the analogous correspondence—if a tropical point moves in a family of the expected dimension inside $\text{Trop}(X) \cap \text{Trop}(X')$ then it lifts to a predictable number of points in $X \cap X'$. One hopes that Mikhalkin’s Correspondence Theorem will eventually be reproved and generalized using Theorem 1.1.2 on suitable tropicalizations of moduli spaces of curves. Similarly, one hopes that Schubert problems can be answered tropically using Theorem 1.1.2 on suitable tropicalizations of Grassmannians and flag varieties. The potential for such applications underlines the importance of having the flexibility to work with intersections inside ambient subvarieties of the torus.

**Remark 1.1.6.** The basic tools for geometry over rank one valuation rings developed in Section 4 may be of independent interest and are useful elsewhere in tropical geometry. For instance, the understanding of the behavior of closed points in fibers provided by Corollary 4.2.5 directly fills the gap in the first proposed proof by Speyer and Sturmfels that every point in the initial degeneration $X_w$ is the specialization of a point of $X$ [7, Theorem 2.1]. See also [7] for more discussion and further references.

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2. Preliminaries

Let $K$ be an algebraically closed field, and let $\nu : K^* \to \mathbb{R}$ be a valuation with value group $G$. Let $R$ be the valuation ring, with maximal ideal $\mathfrak{m}$, and residue field $k = R/\mathfrak{m}$. Since $K$ is algebraically closed, the residue field $k$ is algebraically closed and the valuation group $G$ is divisible. In particular, $G$ is dense in $\mathbb{R}$ unless the valuation $\nu$ is trivial, in which case $G$ is zero. Typical examples of such nonarchimedean fields in equal characteristic are given by the generalized power series fields $K = k((t^G))$, whose elements are formal power series with coefficients in the algebraically closed field $k$ and exponents in $G$, where the exponents occurring in any given series are required to be well-ordered [Poonen93].

Let $T$ be an algebraic torus of dimension $n$ over $R$, with character lattice $M \cong \mathbb{Z}^n$, and let $N = \text{Hom}(M, \mathbb{Z})$ be the dual lattice. We write $N_G$ for $N \otimes G$, and $N_R$ for $N \otimes \mathbb{R}$, the real vector space of linear functions on the character lattice. We treat $M$ and $N$ additively and write $x^u$ for the character associated to a lattice point $u$ in $M$, which is a monomial in the Laurent polynomial ring $K[M]$. We write $T$ for the associated torus over $K$.

In this section, we give a brief review of fundamental facts on tropical geometry, intersection theory in both the tropical and algebraic geometry settings, and the relationship between tropical intersections and certain intersections inside toric varieties.

2.1. Initial degenerations. Each vector $w \in N_R$ determines a weight function on monomials, where the $w$-weight of $ax^u$ is $\nu(a) + \langle u, w \rangle$. The tilted group ring $R[M]^w \subset K[M]$ consists of the Laurent polynomials $\sum a_u x^u$ in which every monomial has nonnegative $w$-weight. We then define $T^w = \text{Spec } R[M]^w$, which is naturally a $T$-torsor over $\text{Spec } R$; its generic fiber is canonically identified with $T_K$, but its special fiber $T_w$ is a torsor over $T_k$. We have that $T_w$ is of finite type (*) over $R$ whenever $w \in N_G$, and indeed if and only if $w \in N_G$ in the situation that $\nu$ is non-trivial. (*)

Let $X$ be a closed subscheme of $T$ of pure dimension $d$.

Definition 2.1.1. Let $X_w$ denote the closure of $X$ in $T_w$. The initial degeneration $X_w$ is the special fiber of $X_w$.

In particular, the initial degeneration $X_w$ is a $k$-subscheme of $T_w$. The terminology reflects the fact that $X_w$ is cut out by residues of initial terms (lowest-weight monomials) of functions in the ideal of $X$.

One consequence of Gröbner theory is that the space of weight vectors $N_R$ can be decomposed into finitely many polyhedral cells so that the initial degenerations are essentially invariant on the relative interior of each cell; see [OSS03, Theorem 2.2.1]. Roughly speaking, these cells are cut out by inequalities whose linear terms have integer coefficients, and whose constant terms are in the value group $G$. More precisely, the cells are integral $G$-affine polyhedra, defined as follows.

Definition 2.1.2. An integral $G$-affine polyhedron in $N_R$ is the solution set of a finite number of inequalities
\[
\langle u, v \rangle \leq c, \quad \text{with } u \text{ in the lattice } M \text{ and } c \text{ in the value group } G.
\]

An integral $G$-affine polyhedral complex $\Sigma$ is a polyhedral complex consisting entirely of integral $G$-affine polyhedra; in other words, it is a finite collection of integral
G-affine polyhedra such that every face of a polyhedron in $\Sigma$ is itself in $\Sigma$, and the intersection of any two polyhedra in $\Sigma$ is a face of each. Note that if $G$ is zero, then an integral $G$-affine polyhedron is a rational polyhedral cone, and an integral $G$-affine polyhedral complex is a fan.

2.2. Tropicalization. Following Sturmfels, we define the tropicalization of $X$ to be

$$\text{Trop}(X) = \{ w \in N_\mathbb{R} \mid X_w \text{ is nonempty} \}.$$ 

The foundational theorems of tropical geometry, due to the work of many authors, are the following.\(^1\)

1. The tropicalization $\text{Trop}(X)$ is the underlying set of an integral $G$-affine polyhedral complex of pure dimension $d$.
2. The integral $G$-affine polyhedral structure on $\text{Trop}(X)$ can be chosen so that the initial degenerations $X_w$ and $X_{w'}$ are $T_0$-affinely equivalent whenever $w$ and $w'$ are in the relative interior of the same face.
3. The image of $X(K)$ under the natural tropicalization map $\text{trop} : T(K) \to N_\mathbb{R}$ is exactly $\text{Trop}(X) \cap N_G$.

Here two subschemes of the $T_0$-torsors $T_u$ and $T_w$ are said to be $T_0$-affinely equivalent if they are identified under some $T_0$-equivariant choice of isomorphism $T_w \cong T_w$. If the valuation is nontrivial, then it follows from (3) that $\text{Trop}(X)$ is the closure of the image of $X(K)$ under the tropicalization map.

We also note that, for any extension of valued fields $L/K$, the tropicalization of the base change $\text{Trop}(X_L)$ is exactly equal to $\text{Trop}(X)$ [7, Proposition 6.1]. (*) In particular, for any extension $L/K$ with nontrivial valuation, $\text{Trop}(X)$ is the closure of the image of $X(L)$. Furthermore, if we extend to some $L$ that is complete with respect to its valuation then the tropicalization map on $X(L)$ extends naturally to a continuous map on the nonarchimedean analytification of $X$, in the sense of Berkovich [7], whose image is exactly $\text{Trop}(X)$ [7]. It follows that $\text{Trop}(X)$ is connected if $X$ is connected, since the analytification of a connected scheme over a complete nonarchimedean field is connected. (\(^*)\)

Remark 2.2.1. The natural tropicalization map from $T(K)$ to $N_\mathbb{R}$ takes a point $t$ to the linear function $u \mapsto \nu \circ \text{ev}_t x^u$, and can be understood as a coordinatewise valuation map, as follows. The choice of a basis for $M$ induces isomorphisms $T \cong (K^*)^n$ and $N_\mathbb{R} \cong \mathbb{R}^n$. In such coordinates, the tropicalization map sends $(t_1, \ldots, t_n)$ to $(\nu(t_1), \ldots, \nu(t_n))$.

There is no canonical choice of polyhedral structure on $\text{Trop}(X)$ satisfying (2) in general. Nevertheless, in the remainder of the paper, we assume that such a polyhedral complex $\Sigma$ with underlying set $\text{Trop}(X)$ has been fixed, and we refer to the faces and facets of $\Sigma$ as faces and facets of $\text{Trop}(X)$.

The tropicalization of a subscheme of a torus torsor over $K$ is well-defined up to translation by $N_G$. In particular, if the value group is zero, then the tropicalization is well-defined, and is the underlying set of a rational polyhedral fan. An important special case is the tropicalization of an initial degeneration. The valuation $\nu$ induces
the trivial valuation on the residue field \(k\), and by Proposition 2.2.3 of \cite{SpeyerThesis}, there is a natural identification of \(\text{Trop}(X_w)\) with the star of \(w\) in \(\text{Trop}(X)\). This star is, roughly speaking, the fan one sees looking out from \(w\) in \(\text{Trop}(X)\); it is constructed by translating \(\text{Trop}(X)\) so that \(w\) is at the origin and taking the cones spanned by faces of \(\text{Trop}(X)\) that contain \(w\).

2.3. Tropical multiplicities. (*) If \(w\) is in the relative interior of a facet \(\sigma\) of \(\text{Trop}(X)\) then \(X_w\) is invariant under translation by the \(d\)-dimensional subtorus \(T_\sigma \subset T_\emptyset\) whose lattice of one parameter subgroups is parallel to the affine span of \(\sigma\) \cite[Proposition 2.2.4]{SpeyerThesis}. This action is free, so a geometric quotient \(X_w/T_\sigma\) exists \cite[Amplification 1.3, §1.3]{Fa1}, and \(X_w\) is a principal \(T_\sigma\)-bundle over \(X_w/T_\sigma\) \cite[Proposition 0.9, §0.4]{SpeyerThesis}. Since the dimensions of \(X_w\) and \(T_\sigma\) are equal, we have that \(X_w/T_\sigma\) is a zero dimensional \(k\)-scheme. The initial degeneration \(X_w\) at any other point \(w'\) in the relative interior of \(\sigma\) is \(T\)-affinely equivalent to \(X_w\), so the zero dimensional schemes \(X_w/T_\sigma\) and \(X_{w'}/T_\sigma\) are isomorphic. In particular, the length of this scheme depends only on the facet \(\sigma\).

**Definition 2.3.1.** The tropical multiplicity \(m_{\text{Trop}(X)}(\sigma)\) is the length of \(X_w/T_\sigma\) for \(w\) in the relative interior of \(\sigma\).

Henceforth, given \(X \subseteq T\), we consider \(\text{Trop}(X)\) to consist of the data of the above-mentioned polyhedral complex inside \(N_R\), together with the multiplicities on its facets.

The multiplicities on the facets of tropicalizations of initial degenerations agree with those on the tropicalization of the original variety. The facets of \(\text{Trop}(X_w)\) correspond to the facets of \(\text{Trop}(X)\) that contain \(w\), and the initial degeneration of \(X_w\) at a point in the relative interior of a given facet is \(T\)-affinely equivalent to the initial degeneration of \(X\) at a point in the relative interior of the corresponding facet of \(\text{Trop}(X)\). In particular, the multiplicities on the facets of \(\text{Trop}(X_w)\) are induced by those on \(\text{Trop}(X)\).

Points in the relative interiors of facets of multiplicity 1 will be particularly important for our purposes.

**Definition 2.3.2.** A simple point in \(\text{Trop}(X)\) is a point in the relative interior of a facet of multiplicity 1.

Initial degenerations at simple points of \(\text{Trop}(X)\) are isomorphic to \(d\)-dimensional tori. In particular, initial degenerations at simple points are smooth. The importance of these simple points, roughly speaking, is that intersections of tropicalizations of subvarieties of \(X\) at simple points behave just like intersections in \(N_R\) of tropicalizations of subvarieties of \(T\). See, for instance, the proof of Theorem 1.1.2 in Section 5.

2.4. Tropical intersection. Given \(X, X' \subseteq T\) of codimension \(j, j'\) respectively, there is a purely tropical definition, developed by (*), of the intersection of \(\text{Trop}(X)\) with \(\text{Trop}(X')\), which is a polyhedral complex of pure codimension \(j + j'\), supported inside \(\text{Trop}(X) \cap \text{Trop}(X')\), with appropriate intersection multiplicities assigned to its facets. This is called the stable tropical intersection. Using the fan displacement rule, we state the definition of tropical intersection multiplicity in the case that \(X, X'\) are instead contained in an ambient variety \(Y\), with codimensions \(j, j'\) respectively. For convenience, we consider \(\text{Trop}(X) \cap \text{Trop}(X')\) to be a polyhedral
complex, refined as necessary so that every face is a face of both \( \text{Trop}(X) \) and \( \text{Trop}(X') \). (*)

If \( \sigma \subseteq N_{\mathbb{R}} \) is any integral \( G \)-affine polyhedron, denote by \( N_{\sigma} \) the sublattice of \( N \) generated by the lattice points of the translation of \( \sigma \) to the origin. Let \( \tau \) be a face of \( \text{Trop}(X) \cap \text{Trop}(X') \) having codimension \( j + j' \) in \( Y \), and suppose that the simple points of \( \text{Trop}(Y) \) meet \( \tau \). Then there is a unique facet \( \eta \) (*) of \( \text{Trop}(Y) \) containing \( \tau \). Let \( v \) be a vector in \( N_{\eta} \) that is small relative to the size of the bounded faces of \( \text{Trop}(X) \) and \( \text{Trop}(X') \) containing \( \tau \), (*) and sufficiently general so that for any two facets \( \sigma \) and \( \sigma' \) in \( \text{Trop}(X) \) and \( \text{Trop}(X') \) respectively, the displaced facet \( \sigma' + v \) meets \( \sigma \) properly (if the intersection is non-empty). (*) The set of such vectors is open and dense in a neighborhood of the origin in \( N_{\mathbb{R}} \); it contains the complement of a finite union of linear spaces. Note that if \( \sigma \) intersects the displaced facet \( \sigma' + v \) then \( \sigma \) and \( \sigma' \) together span \( N_{\eta} \), so the index \([N_{\eta} : N_{\sigma} + N_{\sigma'}]\) is finite.

**Definition 2.4.1.** The **tropical intersection multiplicity** of \( \text{Trop}(X) \) and \( \text{Trop}(X') \) inside \( \text{Trop}(Y) \) along a face \( \tau \) of \( \text{Trop}(X) \cap \text{Trop}(X') \) having codimension \( j + j' \) in \( \text{Trop}(Y) \) is

\[
i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'); \text{Trop}(Y)) = \sum_{\sigma, \sigma'} \lfloor N : N_{\sigma} + N_{\sigma'} \| \cdot m_{\text{Trop}(X)}(\sigma) \cdot m_{\text{Trop}(X')}(\sigma'),
\]

where the sum is over facets \( \sigma \) and \( \sigma' \) of \( \text{Trop}(X) \) and \( \text{Trop}(X') \) respectively, both containing \( \tau \), such that \( \sigma \cap (\sigma' + v) \neq \emptyset \).

The summands appearing in the fan displacement rule formula for tropical intersection multiplicity depend on the choice of the displacement vector \( v \), but the balancing condition satisfied by tropicalizations (*) ensures that the end result is independent of all choices. (*) However, the fan displacement rule also computes intersection products in toric varieties, as we will discuss below, giving a separate proof of the independence of the choice of \( v \). The condition that \( \tau \) contain simple points of \( \text{Trop}(Y) \) is not necessary for the definition, which still makes sense as long as \( \tau \) is contained in a unique facet of \( \text{Trop}(Y) \). However, we only obtain good behavior in the simple case.

**Definition 2.4.2.** Suppose \( \text{Trop}(X) \) meets \( \text{Trop}(X') \) properly in \( \text{Trop}(Y) \), and the simple points of \( \text{Trop}(Y) \) are dense in \( \text{Trop}(X) \cap \text{Trop}(X') \). The **tropical intersection** \( \text{Trop}(X) \cdot \text{Trop}(X') \) assigns the multiplicity \( i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'); \text{Trop}(Y)) \) to each facet \( \tau \) of \( \text{Trop}(X) \cap \text{Trop}(X') \).

We will see in Lemma 3.2.3 below that the assigned multiplicities are all positive.

### 2.5. Intersection theory and toric varieties

We briefly review the basics of intersection theory in algebraic geometry, and the Fulton-Sturmfels description of the intersection theory of a complete toric variety. Recall that if \( Y \) is a smooth variety of dimension \( d \), an \( m \)-cycle on \( Y \) is a finite formal sum of \( m \)-dimensional closed subvarieties of \( Y \), with integer coefficients. The set of \( m \)-cycles naturally form an abelian group. In intersection theory, one often works with the **Chow homology group** \( A_m(Y) \) of cycles of dimension \( m \) modulo rational equivalence; see §1.3 of [1] for definitions. Given a pure-dimensional closed subscheme \( X \subseteq Y \), we obtain a cycle by, for each irreducible component \( Z \) of \( X \), setting the coefficient of \( Z \) to be the length of \( \mathcal{O}_{X,Z} \) over \( \mathcal{O}_{Y,Z} \). We denote the associated rational equivalence class by \([X]\). (*)

Y nonsingular unnecessarily strong for much of this?
Given $X, X' \subseteq Y$ closed subschemes of pure codimension $j, j'$, intersection theory yields a well-defined intersection class $[X] \cdot [X'] \in A_{d-j-j'}(Y)$, which gives $\oplus_m A_m(Y)$ the structure of a commutative graded ring, denoted $A_*(Y)$. If $j + j' = d$, we also have the intersection number $\deg([X] \cdot [X'])$ obtained by adding the coefficients of the 0-cycle $[X] \cdot [X']$.

In general, the dimension of every irreducible component of $X \cap X'$ is at least $d - j - j'$; in the case of equality, we say that $X$ meets $X'$ properly. In this case, intersection theory yields a refined intersection cycle $X \cdot X'$ which, as the name suggests, is a cycle, and not defined only modulo rational equivalence. The basic facts which we shall use about $X \cdot X'$ are that it is a sum of the irreducible components of $X \cap X'$, with the coefficient of a component $Z$ given by the intersection multiplicity $i(Z, X \cdot X'; Y)$. This multiplicity is computed by Serre’s formula as an alternating sum of Tor dimensions:

$$i(Z, X \cdot X'; Y) \sum (-1)^i \text{length}_{\mathcal{O}_{X', Y}} \text{Tor}^i_{\mathcal{O}_{X', Y}}(\mathcal{O}_X, \mathcal{O}_X').$$

This sum is always positive, and less than or equal to the length $\text{length}(\mathcal{O}_{Z, X \cap X'})$. See Chapter 8 of [1] for proofs of these and other fundamental facts about intersection theory in smooth algebraic varieties, and for further details and references. (*)

**Remark 2.5.1.** The above formula for intersection multiplicity makes sense for $Z$ an irreducible component of $X \cap X'$ even when $X$ does not meet $X'$ properly; however, in this case it turns out that the alternating sum always gives 0. (*)

We now review the Fulton-Sturmfels calculation of the intersection theory on a smooth, complete toric variety. Let $\Sigma$ be a complete, unimodular fan in $N_\mathbb{R}$, and $Y(\Sigma)$ the associated (smooth, complete) toric variety. Given a closed subscheme $X \subseteq Y(\Sigma)$ of pure codimension $j$, the Chow homology class $[X] \in A_{d-j}(Y(\Sigma))$ is uniquely determined by the weight function $m_{[X]}$ on codimension-$j$ cones of $\Sigma$ defined by

$$m_{[X]}(\sigma) = \deg([X] \cdot [V(\sigma)]).$$

Here $V(\sigma)$ denotes the $T$-invariant $j$-dimensional closed subvariety of $Y(\Sigma)$ corresponding to $\sigma$.

**Remark 2.5.2.** The weight functions which are actually realized as the functions associated to a Chow class on $Y(\Sigma)$ are the Minkowski weights, which satisfy a balancing condition corresponding precisely to the balancing condition on tropicalizations. However, we will not make direct use of either balancing condition, so we do not discuss them further.

Let $X, X' \subseteq Y(\Sigma)$ be closed subschemes of codimension $j, j'$ respectively. The theorem of Fulton and Sturmfels (*) is that the intersection class $[X] \cdot [X']$ is determined precisely by the fan displacement rule of Definition 2.4.1. That is, if we fix a sufficiently general $v \in N_\mathbb{R}$, and $\tau$ is a cone of $\Sigma$ of codimension $j + j'$, we have

$$m_{[X] \cdot [X']}(\tau) = \sum_{\sigma, \sigma'} [N : N_{\sigma} + N_{\sigma'}] \cdot m_{[X]}(\sigma) \cdot m_{[X']}(\sigma'),$$

where the sum is over cones $\sigma, \sigma'$ of $\Sigma$ having codimension $j, j'$ respectively, both containing $\tau$, and such that $\sigma \cap (\sigma' + v) \neq \emptyset$. (*)

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(*) need more cites to fulton?

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(*) cite be consistent with dimension notation, etc
2.6. **Tropical intersections and initial degenerations.** Suppose $X \subseteq T$ is a closed subscheme, and the valuation on $K$ is trivial, that $\text{Trop}(X)$ is a fan. Fix $\Sigma$ a complete unimodular fan containing a subfan which is a refinement of $\text{Trop}(X)$, and let $Y(\Sigma)$ be the induced smooth toric compactification of $T$.\(^2\) Also denote by $\overline{X}$ the closure of $X$ in $Y(\Sigma)$. The fundamental fact we shall use is that the weight function $m_{[\overline{X}]}$ determining the Chow homology class $[\overline{X}]$ corresponds precisely to the tropical multiplicity function on $\text{Trop}(X)$. That is, we have

$$m_{[\overline{X}]}(\sigma) = m_{\text{Trop}(X)}(\sigma')$$

if $\sigma$ is contained in a facet $\sigma'$ of $\text{Trop}(X)$, and is 0 otherwise [Katz09, Section 9]. (*)

We immediately conclude:

**Proposition 2.6.2.** If $X \subseteq T$ is any closed subscheme of pure dimension $d$, and $\tau$ a facet of $\text{Trop}(X)$, then

$$m_{\text{Trop}(X)}(\tau) = \sum_Z \text{length}_{\mathcal{O}_{T,Z}}(\mathcal{O}_{X,Z}) m_{\text{Trop}(Z)}(\tau),$$

where the sum is over components $Z$ of $X$, with their reduced structure.

**Proof.** By definition, we have the identity of cycle classes

$$[\overline{X}] = \sum_Z \text{length}_{\mathcal{O}_{Y,Z}}(\mathcal{O}_{X,Z}) [Z],$$

and also

$$\text{length}_{\mathcal{O}_{Y,Z}}(\mathcal{O}_{X,Z}) = \text{length}_{\mathcal{O}_{T,Z}}(\mathcal{O}_{X,Z}),$$

so we immediately conclude the desired statement from (2.6.1). □

Now suppose we have $X, X' \subseteq Y(\Sigma)$ of pure codimension $j, j'$ respectively, and suppose $\Sigma$ contains subfans refining $\text{Trop}(X')$ as well as $\text{Trop}(X)$. Putting together the Fulton-Sturmfels formula with the above gives us the following.

**Proposition 2.6.3.** If the valuation on $K$ is trivial, $X$ meets $X'$ properly, and every component of $\overline{X} \cap \overline{X}'$ meets the dense torus $T$, then for any $\tau \subseteq \text{Trop}(X) \cap \text{Trop}(X')$ of dimension equal to $j + j'$ we have

$$i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'), N_\mathbb{R}) = \sum_Z i(Z, X \cdot X', T) m_{\text{Trop}(Z)}(\tau),$$

where the sum is over components $Z$ of $X \cap X'$.

**Proof.** In light of the fact that $m_{\text{Trop}(X)} = m_{[\overline{X}]}$ and similarly for $X'$, the Fulton-Sturmfels theorem gives us that

$$i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'), N_\mathbb{R}) = \sum_Z i(Z, \overline{X} \cdot \overline{X}', Y(\Sigma)) m_{[Z]}(\tau),$$

\(^2\)Strictly speaking $Y(\Sigma)$ is not a toric variety in the usual sense, as defined for instance in [Fulton93]. It is a normal variety on which $T$-acts with a dense orbit $T_0$, but the dense orbit is not canonically identified with $T$. The choice of a point in the dense orbit induces a $T$-equivariant isomorphism with the honest toric variety usually associated to $\Sigma$, so the geometry of these toric compactifications of torus torsors is essentially indistinguishable from that of toric varieties, aside from the lack of a distinguished point in each orbit. We use standard results from toric geometry on these compactifications of torus torsors, such as the description of their intersection theory in [Fulton93], without further mention of this minor distinction.
where the sum is over components \( Z \) of \( \overline{X} \cap \overline{X}' \). By the description of intersection multiplicity in terms of alternating Tors we see that restriction to an open subset meeting a given component doesn’t change multiplicity, so if every component of \( \overline{X} \cdot \overline{X}' \) meets \( T \), then we get the desired statement. \( \square \)

3. The trivial valuation case

In order to prove Theorem 1.1.3 and its stronger form involving multiplicities (see Theorem 3.2.1 below), we need to study how the intersection of \( \text{Trop}(X) \) and \( \text{Trop}(X') \) near a point \( w \) relates to the intersection of the initial degenerations \( X_w \) and \( X'_w \). Since the stars of \( w \) in \( \text{Trop}(X) \) and \( \text{Trop}(X') \) are the tropicalizations of \( X_w \) and \( X'_w \), respectively, with respect to the trivial valuation, this is a special case of understanding how tropicalization relates to intersections when the (tropical) intersection is proper and the valuation is trivial. In order to prove non-emptiness of \( X_w \cap X'_w \), we need to make use of extended tropicalizations, so we begin by recalling the basic facts we will need.

3.1. Extended tropicalizations. The extended tropicalizations of \( Y(\Sigma) \) and its subvarieties were introduced by Kajiwara \cite{kajiwara08}, and their basic properties were developed further in Section 3 of \cite{analytification}, to which we refer the reader for further details. We recall the definition and some of the key properties, and develop them slightly further.

Let \( \Sigma \) be a fan, and \( Y(\Sigma) \) the associated toric variety. Recall that each cone \( \sigma \) in \( \Sigma \) corresponds to an affine open subvariety \( U_\sigma \) of \( Y(\Sigma) \) whose coordinate ring is the semigroup ring \( K[\sigma^\vee \cap M] \) associated to the semigroup of lattice points in the dual cone \( \sigma^\vee \) in \( M_\mathbb{R} \). Let \( \mathbb{R} \) be the real line extended in the positive direction \( \mathbb{R} = \mathbb{R} \cup \{+\infty\} \), which is a semigroup under addition, with identity zero. Then \( \text{Trop}(U_\sigma) = \text{Hom}(\sigma^\vee \cap M, \mathbb{R}) \) is the space of all semigroup homomorphisms taking zero to zero, with its natural topology as a subspace of \( \mathbb{R}^{\sigma^\vee \cap M} \). For each face \( \tau \) of \( \Sigma \), let \( N(\tau) \) be the real vector space \( N(\tau) = \mathbb{R}^{\tau^\vee \cap M} / \text{span}(\tau) \).

Then \( \text{Trop}(U_\sigma) \) is naturally a disjoint union of the real vector space \( N(\tau) \), for \( \tau \leq \sigma \), where \( N(\tau) \) is identified with the subset of semigroup homomorphisms that are finite exactly on the intersection of \( \tau^\perp \) with \( \sigma^\vee \cap M \).

The tropicalization \( \text{Trop}(Y(\Sigma)) \) is a space obtained by gluing the spaces \( \text{Trop}(U_\sigma) \) for \( \sigma \in \Sigma \) along the natural open inclusions \( \text{Trop}(U_\tau) \subset \text{Trop}(U_\sigma) \) for \( \tau \leq \sigma \). It is a disjoint union

\[
\text{Trop}(Y(\Sigma)) = \bigsqcup_{\sigma \in \Sigma} N(\sigma),
\]

just as \( Y(\Sigma) \) is the disjoint union of the torus orbits \( O_\sigma \). Points in \( N(\sigma) \) may be seen as weight vectors on monomials in the coordinate ring of \( O_\sigma \), and the tropicalization of a closed subvariety \( Z \subset Y(\Sigma) \) is defined to be

\[
\text{Trop}(Z) = \bigsqcup_{\sigma \in \Sigma} \{ w \in N(\sigma) \mid (Z \cap O_\sigma)_w \text{ is not empty} \}.
\]
In other words, $\text{Trop}(Z)$ is the disjoint union of the tropicalizations of its intersections with the $T$-orbits in $Y(\Sigma)$. This space is compact when $\Sigma$ is complete. Most importantly for our purposes, we have the following:

**Proposition 3.1.1.** If $Z$ meets the dense torus $T$, then $\text{Trop}(Z)$ is the closure in $\text{Trop}(Y(\Sigma))$ of the ordinary tropicalization $\text{Trop}(Z \cap T)$.

**Proof.** We have by the equivalence of (1) and (3) in Proposition 3.7 of [1] that $\text{Trop}(Z)$ is closed, so it suffices to prove that it is contained in the closure of $\text{Trop}(Z \cap T)$. Because the tropicalization map is not continuous, we use the fact, developed in [1], that the extended tropicalization map factors through the analytification, with the map from the analytification to $\text{Trop}(Z)$ continuous. Since the latter map is continuous, it suffices to prove that the analytification of $Z$ is contained in the closure of the analytification of $Z \cap T$, which is to say that a Zariski open subset of a variety is dense in its analytification. But this follows from Corollary 3.4.5 of Berkovich (\textsuperscript{*}).

If $\sigma$ is a face of $\Sigma$ that contains another face $\tau$, then we write $\sigma_\tau$ for the image of $\sigma$ in $N(\tau)$. It is a cone of dimension $\dim(\sigma) - \dim(\tau)$. We will use the following lemma in the proof of Proposition 3.2.2.

**Lemma 3.1.2.** Let $\sigma$ and $\tau$ be faces of $\Sigma$, and let $\overline{\sigma}$ be the closure of $\sigma$ in $\text{Trop}(Y(\Sigma))$. Then

$$\overline{\sigma} \cap N(\tau) = \sigma_\tau$$

if $\sigma$ contains $\tau$, and $\overline{\sigma}$ is disjoint from $N(\tau)$ otherwise.

**Proof.** Let $v$ be a point in $N(\tau)$, and let $\pi : \text{Trop}(U_\tau) \to N(\tau)$ be the continuous map that restricts to the canonical linear projections from $N(\gamma)$ onto $N(\tau)$ for $\gamma \leq \tau$. If $v$ is not in $\sigma_\tau$, then the preimage under $\pi$ of $N(\tau) \setminus \sigma_\tau$ is an open neighborhood of $v$ that is disjoint from $\sigma$, so $v$ is not in $\overline{\sigma}$.

For the converse, suppose $v$ is in $\sigma_\tau$ and let $w$ be a point in $\sigma$ that projects to $v$. Let $w'$ be a point in the relative interior of $\tau$. Then $w + \mathbb{R}_{\geq 0}w'$ is a path in $\sigma$ whose limit is $v$, so $v$ is in $\overline{\sigma}$.

### 3.2. Lower bounds on multiplicities

Throughout the remainder of this section, we assume the valuation $\nu$ is trivial. We have $X$ and $X'$ subvarieties of $T$ of codimension $j$ and $j'$, respectively, and $\Sigma$ is a complete unimodular fan in $N_\Sigma$ such that each face of $\text{Trop}(X)$ and each face of $\text{Trop}(X')$ is a union of faces of $\Sigma$. We write $X$ and $X'$ for the closures of $X$ and $X'$ in the smooth complete toric variety $Y(\Sigma)$.

Our goal is to prove the following refined version of Theorem 3.1.1 (\textsuperscript{*}) in the special case of the trivial valuation, giving in particular lower bounds on the multiplicities of the facets in $\text{Trop}(X \cap X')$. These lower bounds are extended to the general case in Section 5.

**Theorem 3.2.1.** Suppose $\nu$ is trivial and $\text{Trop}(X)$ meets $\text{Trop}(X')$ properly. Then

$$\text{Trop}(X \cap X') = \text{Trop}(X) \cap \text{Trop}(X'),$$

and for $\tau$ any facet of $\text{Trop}(X) \cap \text{Trop}(X')$, we have the following identity of multiplicities:

$$i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'), N_R) = \sum_Z i(Z, X \cdot X', T)m_{\text{Trop}, Z}(\tau),$$
where the sum is over components $Z$ of $X \cap X' \subseteq \text{Trop}(Z)$. In particular, we have

\[ m_{\text{Trop}(X \cap X')}(\tau) \geq i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'), N_\mathbb{R}) > 0. \]

We remark that in contrast to Theorem 3.2.1, the set-theoretic form of Theorem 3.2.2 relies on the refined knowledge of multiplicities. Given the background reviewed in Section 2, we already know that the class $[X] \cdot [X']$ is described by tropical intersection of $\text{Trop}(X)$ with $\text{Trop}(X')$, so the most substantive part of the argument is to prove that the closure $\overline{X} \cap \overline{X}'$ of the intersection of $X$ and $X'$ meets each closed $T$-invariant subvariety of $Y(\Sigma)$ properly, and (*) that each component of $\overline{X} \cap \overline{X}'$ is the closure of a component of $X \cap X'$. In general, this may be far from true: for instance, $\overline{X} \cap \overline{X}'$ could have components of larger than expected dimension, even if $X$ and $X'$ meet properly in $T$, and these components will contribute to the intersection cycle $\overline{X} \cdot \overline{X}'$. However, we will show that under the hypothesis that $\text{Trop}(X)$ meets $\text{Trop}(X')$ properly, this does not occur.

**Proposition 3.2.2.** Suppose $\text{Trop}(X)$ meets $\text{Trop}(X')$ properly. Then $\overline{X} \cap V(\tau)$ meets $\overline{X}' \cap V(\tau)$ properly in $V(\tau)$, for every $T$-invariant closed subvariety $V(\tau)$ in $Y(\Sigma)$.

In particular, $\overline{X}$ meets $\overline{X}'$ properly in $Y(\Sigma)$ and every component of $\overline{X} \cap \overline{X}'$ is the closure of some component of $X \cap X'$.

**Proof.** First, note that $\overline{X} \cap V(\tau)$ and $\overline{X}' \cap V(\tau)$ meet properly in $V(\tau)$ for all $\tau$ if and only if $\overline{X} \cap O_\tau$ meets $\overline{X}' \cap O_\tau$ properly in $O_\tau$ for all $\tau$. We prove the latter using compactified tropicalizations, as follows.

The tropicalization of $\overline{X} \cap \overline{X}' \cap O_\tau$ contained in the intersection of the tropicalizations

\[ \text{Trop}(\overline{X}) \cap \text{Trop}(\overline{X}') \cap N(\tau). \]

Therefore, it will suffice to show that if $\text{Trop}(\overline{X}) \cap \text{Trop}(\overline{X}') \cap N(\tau)$ is nonempty then it has codimension $j + j'$ in $N(\tau)$. Now $\text{Trop}(\overline{X})$ and $\text{Trop}(\overline{X}')$ are the closures of $\text{Trop}(X)$ and $\text{Trop}(X')$, respectively, and Lemma 3.1.2 implies that $\text{Trop}(\overline{X}) \cap N(\tau)$ is the union of the projected facets $\sigma_\tau$ such that $\sigma$ is in $\text{Trop}(X)$. Similarly, $\text{Trop}(\overline{X}') \cap N(\tau)$ is the union of those $\sigma'_\tau$ such that $\sigma'$ is in $\text{Trop}(X')$. In particular, $\text{Trop}(\overline{X}) \cap \text{Trop}(\overline{X}') \cap N(\tau)$ is nonempty if and only if there are facets $\sigma$ and $\sigma'$ of $\text{Trop}(X)$ and $\text{Trop}(X')$, respectively, that contain $\tau$. Now, if $\text{Trop}(X)$ and $\text{Trop}(X')$ meet properly along $\tau$, then $\sigma \cap \sigma'$ has codimension $j + j'$ in $N_\mathbb{R}$. Since the intersection of $\sigma_\tau$ and $\sigma'_\tau$ is $(\sigma \cap \sigma')_\tau$, if this intersection is nonempty then it has codimension $j + j'$ in $N(\tau)$, as required.

The following lemma says that – just as in the case of algebraic geometry – the tropical intersection multiplicity is always strictly positive wherever an intersection is proper.

**Lemma 3.2.3.** Suppose $\tau$ is a facet of $\text{Trop}(X) \cap \text{Trop}(X')$, having the expected codimension $j + j'$. Then $i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'); N_\mathbb{R}) > 0$.

**Proof.** (*) By definition, there exist cones $\sigma$ and $\sigma'$ containing $\tau$ such that $m_{\text{Trop}(X)}(\sigma)$ and $m_{\text{Trop}(X')}(\sigma')$ are positive. If, furthermore, $\text{Trop}(X)$ and $\text{Trop}(X')$ meet properly along $\tau$, then $\sigma$ and $\sigma'$ together span $N_\mathbb{R}$. In this case, we can compute
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$i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'); N_R)$ by the fan displacement rule with respect to a sufficiently general displacement vector

\[ v = w - w' \]

with \( w \) and \( w' \) in the relative interiors of \( \sigma \) and \( \sigma' \), respectively, since the set of all such differences is open in \( N_R \). Then \( \sigma' + v \) meets \( \sigma \) at \( w \), and the summand \( [N : N_\sigma + N_{\sigma'}] \cdot m_{\text{Trop}(X)}(\sigma) \cdot m_{\text{Trop}(X')}(\sigma') \) contributes positively to \( i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'); N_R) \). Since all the summands in the fan displacement rule are non-negative by definition, we obtain the desired positivity. \( \square \)

Remark 3.2.4. It is clear that the argument for Lemma 3.2.3 goes through in the full generality of Definition 2.4.1, without any of the additional hypotheses imposed in this section.

We can now easily prove the desired result.

Proof of Theorem 3.2.1. Since \( \text{Trop}(X) \) meets \( \text{Trop}(X') \) properly, \( X \) must meet \( X' \) properly in \( T \). Furthermore, by Proposition 3.2.2, \( \text{Trop}(X \cap X') \) has no boundary components. The identity of intersection multiplicities is then an immediate consequence of Proposition 2.6.3.

Now, by Proposition 2.6.2, for any facet \( \tau \) of \( \text{Trop}(X \cap X') \) we have that

\[ m_{\text{Trop}(X \cap X')} (\tau) = \sum_Z \text{length}_{O_{X \cap X'}, Z} \cdot m_{\text{Trop}(Z)} (\tau), \]

where the sum is over components \( Z \) of \( X \cap X' \). The length \( \text{length}_{O_{X \cap X'}, Z} \) is at least as large as the intersection multiplicity \( i(Z, X \cdot X'; T) \), by Proposition 8.2 of \([7]\), so the first part of the desired inequality follows. Finally, \( i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'), N_R) \) is strictly positive on every facet of \( \text{Trop}(X) \cap \text{Trop}(X') \) by Lemma 3.2.3, so we get the full inequality, and it also follows that \( \text{Trop}(X \cap X') \) contains \( \text{Trop}(X) \cap \text{Trop}(X') \), as claimed. \( \square \)

4. GEOMETRY OVER VALUATION RINGS OF RANK 1

In \([7]\), Nagata began extending certain results from dimension theory to non-Noetherian rings. Here we continue in Nagata’s spirit with a view toward tropical geometry, focusing on valuation rings of rank 1. The results of this section are applied in Section 6 to prove new tropical lifting theorems, but may be of more general interest.

4.1. Statements. A fundamental property of a regular scheme \( S \) is the theorem of Serre [7, Theorem V.3] that codimension is subadditive under intersection in \( S \). In other words, for any irreducible closed subschemes \( X \) and \( X' \) of \( S \), and any irreducible component \( Z \) of \( X \cap X' \), we have

\[ \text{codim}_S(X \cap X') \leq \text{codim}_S X + \text{codim}_S X'. \]

The same is then necessarily true for schemes smooth over \( S \). We will give a partial generalization of this statement to the non-Noetherian setting:

**Theorem 4.1.1.** Let \( R \) be a valuation ring of rank 1, and \( Y \) smooth over \( S := \text{Spec} R \). Then codimension is subadditive under intersection in \( Y \).

In this sense, valuation rings of rank 1 behave like regular local rings.
Remark 4.1.2. When the property of Theorem 4.1.1 holds, one may use dimension-counting arguments to deform points in a family over $S$. Such techniques are essential, for instance, in the theory of limit linear series developed by Eisenbud and Harris [14].

As in previous sections, we assume throughout that $R$ is a valuation ring of rank 1, but here we do not make additional hypotheses on the field of fractions $K$ or the residue field $k$. We now set

$$S = \text{Spec } R,$$

and we systematically study schemes of finite type over $S$.

Thm:lift to $K$

**Theorem 4.1.3.** Let $Y$ be smooth and finite type over $S$ with subschemes $X$ and $X'$ of pure codimension $j$ and $j'$, respectively. Suppose the special fibers $X_k$ and $X'_k$ intersect in codimension $j + j'$ at a point $x$ in $Y_k$. Then

1. There is a point in $X_K \cap X'_K$ specializing to $x$.
2. If $x$ is closed then there is a closed point in $X_K \cap X'_K$ specializing to $x$.

In §4.4 we will also prove a principle of continuity for intersection multiplicities in the same non-Noetherian setting.

### 4.2. A principal ideal theorem

Recall that Krull’s principal ideal theorem says (after translating to geometry) that every component of a Cartier divisor on a Noetherian scheme has codimension 1 [CITE]. Here we prove a relative version of the principal ideal theorem in the non-Noetherian setting, using Noetherian approximation.

Prop:krull

**Proposition 4.2.1.** Let $X$ be irreducible, flat, and finite type over $S$. Suppose $D$ is an effective Cartier divisor in $X$. If $D$ does not meet the generic fiber $X_K$, then every irreducible component of $D_k$ is an irreducible component of $X_k$.

**Proof.** The statement is local on $X$, so we may assume $X = \text{Spec } A$ is affine and $D$ is cut out by a single equation $f \in A$. Since $X$ is flat and of finite type over $S$, and $S$ is irreducible, it follows from Raynaud-Gruson [7, Corollary 3.4.7] that $X$ is of finite presentation over $S$. There is therefore a subring $R' \subseteq R$, finitely generated over the integers, such that $X$ and $D$ are defined over $R'$. In other words, $A' \subseteq A$, with $A'$ containing generators of $A$ over $R$, and $R'$ containing coefficients of relations, and such that $f \in A'$. We then have

$$A' \otimes_{R'} R = A,$$

and we write $X' = \text{Spec } A'$. Let $S' = \text{Spec } R'$ and let $D'$ be the subscheme of $X'$ cut out by $f$. By construction, $D' \times'_S S = D$. Then since $A' \subseteq A$ and $R' \subseteq R$, we have that $X'$ and $S'$ are still irreducible.

If $D$ does not meet the generic fiber $X_K$, then $D'$ does not meet the fiber of $X'$ over the generic point of $S'$. Let $s'$ be the image in $S'$ of the closed point in $S$. Since $X$ is irreducible and flat over $S$, $X_K$ and $X_k$ are both pure of the same dimension $d$,

---

3The assumption that schemes are of finite type over $S$ is critical for the main results of this section. The reader should keep in mind that the tilted torus $\text{Spec } R[M]$ is of finite type if and only if $w$ is in $N_G$. See Examples [REF].
Lemma 4.2.3. Let \( X \) be irreducible, and dominant and of finite type over \( S \), and let \( Z_1 \subseteq Z_2 \) be irreducible closed subsets of \( X \). Then
\[
\text{codim}_X Z_1 = \text{codim}_X Z_2 + \text{codim}_{Z_2} Z_1.
\]

In addition, if \( Z_1 \) is contained in the special fiber \( X_0 \), then
\[
\text{codim}_X Z_1 = 1 + \text{codim}_{X_0} Z_1.
\]

Note that \( X \) is flat over \( S \), so its special fiber \( X_0 \) is equidimensional; thus, codimension inside \( X_0 \) is equal to codimension inside any irreducible component of \( X_0 \).

This lemma is a special case of a more general result of Nagata for valuation rings of finite rank; see [EGA4, Theorem 13.1.3]. (*) We include the proof because it is quite transparent in this case.

Proof. We first claim that if \( Z_1 \) is contained in \( X_0 \), and \( Z_2 \) meets the generic fiber of \( X \), then there exists an irreducible closed subset \( Z \) strictly between \( Z_1 \) and \( Z_2 \) if and only if \( \text{codim}_{X_0} Z_1 > \text{codim}_X Z_2 \), and in this case \( Z \) may be chosen to be contained in \( X_0 \). Note that (the reduced subscheme on) \( Z_2 \) is flat over \( S \), so every component of \( Z_2 | X_0 \) has codimension in \( X_0 \) equal to \( \text{codim}_X Z_2 \). Since \( Z_1 \subseteq Z_2 | X_0 \), we conclude that \( \text{codim}_{X_0} Z_1 > \text{codim}_X Z_2 \) if and only if there is an irreducible closed subset of \( X_0 \) containing \( Z_1 \) and contained in \( Z_2 \). On the other hand, suppose that \( Z \) is an irreducible closed subset containing \( Z_1 \) and contained in \( Z_2 \), but meeting the...
generic fiber. Then $Z$ is flat over $S$, and we have
\[
\text{codim}_{X_0} Z_1 \geq \text{codim}_{X_0} Z|_{X_0} = \text{codim}_X Z > \text{codim}_X Z_1,
\]
proving the claim.

It follows immediately from the claim that if we have $Z_1 \subseteq Z_2$, with $Z_1$ contained in $X_0$ and $Z_2$ meeting the generic fiber, and $Z$ in between $Z_1$ and $Z_2$ meeting the generic fiber, then there exists $Z'$ between $Z_1$ and $Z_2$ contained in $X_0$. Inducting on this, we conclude that any chain between $Z_1$ and $Z_2$ can be replaced by one of the same length, with all but $Z_2$ contained in $X_0$. We thus obtain the assertion that $\text{codim}_X Z_1 = \text{codim}_{X_0} Z_1 + 1$. Applying the same statement with $Z_2$ in place of $X$, and using the classical identity
\[
\text{codim}_{X_0} Z_1 = \text{codim}_{X_0} (Z_2)_0 + \text{codim}_{(Z_2)_0} Z_1
\]
in the special fiber, we also obtain
\[
\text{codim}_X Z_1 = \text{codim}_X Z_2 + \text{codim}_{Z_2} Z_1,
\]
as desired. \qed

We now have the principal ideal theorem.

**Corollary 4.2.4.** Let $X$ be of finite type over $S$, and let $Z$ be a locally principal subscheme of $X$. Then every irreducible component of $Z$ has codimension at most 1 in each component of $X$ that contains it.

**Proof.** First note that every closed subscheme of $X$ can have only finitely many irreducible components, since each must have generic point in either the generic or closed fiber of $f$. The question is also local on $X$, and it suffices to work in a neighborhood of each generic point of $Z$, so we can reduce immediately to the case that $Z$ is irreducible, by restricting to the complement of all but one of its components. We may further restrict to any given irreducible component of $X$ containing $Z$, and we may further replace $X$ by $X_{\text{red}}$. We have thus reduced to the situation that $X$ is integral and $Z$ is irreducible. In this case, $Z$ must either be equal to $X$, or a Cartier divisor in $X$, so we may assume we are in the latter situation.

If $X$ is supported in the special fiber of $f$, we immediately conclude the desired statement from the classical principal ideal theorem. Suppose $X$ is not supported in the special fiber. Then $X$ is dominant, and hence flat, over $S$. If $Z$ meets the generic fiber, we may apply the classical principal ideal theorem to the generic fiber to conclude the desired statement. If $Z$ is contained in the special fiber, we conclude from Proposition 4.2.1 that it is in fact a component of the special fiber. It thus follows that $Z$ has height 1 in $X$, by Lemma 4.2.3. \qed

As another application of the principal ideal theorem, we have the following result on lifting closed points in the special fiber to closed points in the generic fiber. Our argument is in the spirit of Katz’s proof of Lemma 4.15 in [?].

**Corollary 4.2.5.** Let $X$ be a scheme of finite type over $S$, and let $x$ be a closed point of $X_k$. Then if $x$ is in the closure of $X_K$, there exists a closed point $x'$ in $X_K$ specializing to $x$.

**Remark 4.2.6.** For the initial degeneration of a subscheme of a torus $T$ over $K$ associated to a weight vector $w \in N_G$, the corollary says that every closed point in
the special fiber $X_w(k)$ lifts to a point in the general fiber $X(K)$. The proof of the corollary therefore fills the gap in the proof of [SpeyerSturmfels04, Theorem 2.1].

**Proof.** We may replace $X$ by a component of $X$ which meets $X_K$ and contains $x$, so we may assume $X$ is integral, It then follows that $X$ is flat over $S$. Moreover, because the morphism is of finite type, we may pass to Zariski open subsets of $X$ and thus assume $X = \text{Spec} \, A$ is also affine. The proof is by induction on the dimension of $X_K$. If the dimension is 0, then the generic point of $X$ is already necessarily closed, so we can take that to be $x'$. Suppose now that the dimension of the generic fiber is $d > 0$, and we know the desired statement for fiber dimension $d - 1$. By semicontinuity of fiber dimension, the fiber dimension at $x$ is necessarily at least $d$, and in particular strictly positive. We may thus choose $f \in A$ vanishing at $x$, but whose vanishing set $D$ does not contain any component of the special fiber. $D$ can have at most finitely many irreducible components which meet the generic fiber; we may thus replace $X$ by a Zariski open subset neighborhood of $x$, and assume that there is no irreducible component of $D$ which meets the generic fiber but does not contain $x$. By Proposition [prop:krull], we conclude that $D$ must meet the generic fiber, so if we let $Z$ be any irreducible component of $D$ meeting the generic fiber, we have by semicontinuity of fiber dimension that the generic fiber of $Z$ has dimension $d - 1$, and applying the induction hypothesis to $Z$ we obtain a closed point $x'$ in the generic fiber of $Z$ over $S$ which specializes to $x'$, giving us the desired statement. □

4.3. **Subadditivity of codimension.** We now use the principal ideal theorem over valuation rings of rank 1 to the desired statement on subadditivity of codimension under intersection. In contrast to Serre’s theorem in the regular case, our proof follows the much easier argument for varieties smooth over a field.

Smooth schemes over valuation rings of higher rank do not typically have the same subadditivity property, as demonstrated by the following example.

**Example 4.3.1.** Let $B$ be a valuation ring of rank greater than 1, and $b \in B$ an element generating an ideal of height greater than 1. Let $A = B[x]$, and consider the ideals of $A$ generated by $x$ and by $x - b$. These each have height 1, but the ideal $(x, x - b) = (x, b)$ has height strictly greater than 2. (*)

The following proposition is the main additional technical step in our proof of Theorem [thm:int reg].

**Proposition 4.3.2.** Let $Y$ be irreducible and finite type over $S$, and let $X$ and $X'$ be irreducible closed subschemes. Then for every irreducible component $Z$ of $X \times_S X'$, and every irreducible component $Y'$ of $Y \times_S Y'$ containing $Z$, we have

$$\operatorname{codim}_{Y'} Z \leq \operatorname{codim}_Y X + \operatorname{codim}_Y X'.$$

Furthermore, equality holds unless $X$ and $X'$ are both contained in the special fiber of $Y$.

**Proof.** Since $X \times_S X'$ is homeomorphic to $X_{\text{red}} \times_S X'_{\text{red}}$ (Proposition 5.1.7 of [EGa1]), we may assume that $X$ and $X'$ are reduced. It follows that if both $X$ and $X'$ meet the generic fiber of $Y$, then they are both flat over $S$, and thus so is $X \times_S X'$. Then $Z$ must also dominate $S$. In this case, we can compute all the necessary codimensions inside the generic fiber, and we have the desired equality from classical dimension theory over a field.
Theorem 4.3.3. \( \text{thm:smooth-lci} \)

Let \( f : X \to Y \) be an immersion, locally of finite presentation, of one smooth \( S \)-scheme into another. Then \( X \) is a local complete intersection sub scheme of \( Y \).

In particular, if \( f : X \to S \) is smooth, then the diagonal \( \Delta : X \to X \times_S X \) is a locally complete intersection subscheme.

Proof. This is essentially the implication (a) implies (d) in Proposition 7 of §2.2 of [17]. We note that the assertion is local, so we may assume that \( f \) is in fact a closed immersion. If (at given points \( x \) and \( f(x) \)) we denote by \( r \) and \( n \) the relative dimensions of \( X \) and \( Y \) over \( S \), then given \( \text{loc. cit.} \) it is enough to verify that the codimension of \( X \) inside \( Y \) is \( n - r \). Let \( Z \) be any irreducible component of \( X \) containing \( x \), and \( Z' \) an irreducible component of \( Y \) containing \( Z \); we claim that the codimension of \( Z \) in \( Z' \) is \( n - r \). Since \( X \) and \( Y \) are smooth, we have that \( Z \) and \( Z' \) must dominate an irreducible component \( Z \) of \( S \), and we can compute the codimension after passing to the generic fiber. But in the generic fiber, we are simply of finite type over a field, so we can naively compute the codimension as \( n - r \), as desired.

For the second assertion, note that the diagonal is locally of finite presentation by Corollary 1.4.3.1 of [17], so we can apply the first assertion. \( \square \)

We now can easily conclude the desired results.

Proof of Theorem 4.1.1. Suppose \( Y \) is smooth over \( S \), and \( X \) and \( X' \) are two irreducible closed subschemes of \( Y \). By Proposition 4.3.2, if \( Y' \) is the connected (hence irreducible, by smoothness) component of \( Y \times_S Y \) containing the diagonal \( \Delta \), we have that every irreducible component \( Z \) of \( X \times_S X' \) satisfies

\[ \text{codim}_{Y'} Z \leq \text{codim}_Y X + \text{codim}_Y X'. \]
The transitivity of codimension in Lemma 4.2.3 allows us to inductively apply Corollary 4.2.4 to see that codimension can only decrease when intersecting with any locally complete intersection subscheme. Since $X \cap X'$ can be realized as $(X \times_S X') \cap \Delta$, and the diagonal $\Delta$ is a local complete intersection, we conclude the desired statement from Theorem 4.3.3. □

**Proof of Theorem 4.4.2.** Let $Z$ be a component of $X \cap X'$ containing $x$. We have from Theorem 4.1.1 that $\operatorname{codim}_X Z \leq \operatorname{codim}_X X + \operatorname{codim}_X X' = j + j'$. But $\operatorname{codim}_{X_k} X_k \cap X_k' = j + j'$, so we conclude that $Z$ cannot be contained in $X_k \cap X_k'$, and in particular the generic point of $Z$ is a point of $X_k \cap X_k'$ specializing to $x$. (*) On the other hand, if $x$ is closed in its fiber, we can find a point of $Z$ closed in $Z_k$ and specializing to $x$ by Corollary 4.1.3. □

### 4.4. Intersection multiplicities over valuation rings of rank 1

Due to a lack of suitable references in the non-Noetherian setting, we prove a principle of continuity (see Fulton, Corollary 10.2.2 [9]) for non-Noetherian base schemes. We continue to focus on the case that $S = \text{Spec} \ R$, for $R$ a valuation ring of rank 1, but the proof generalizes immediately; see Remark 4.4.5. We recall Serre’s definition of intersection multiplicity (for simplicity, we restrict to the case of 0-dimensional intersections).

**Definition 4.4.1.** Suppose $Y_0, Z_0$ are closed subschemes of a regular scheme $X_0$, with $Y_0 \cap Z_0$ containing an isolated point $P$. Then the **intersection multiplicity** $i_{X_0, P}(Y_0, Z_0)$ of $Y_0$ and $Z_0$ at $P$ is defined to be

$$
\sum_{i=0}^{\infty} (-1)^i \text{length}_{\mathcal{O}_{X_0, P}} \operatorname{Tor}^i_{\mathcal{O}_{X_0}}(\mathcal{O}_{Y_0}, \mathcal{O}_{Z_0}).
$$

If further $Y_0 \cap Z_0$ has dimension 0 and $X_0$ is smooth and quasi-compact over a field $k$, the **total intersection multiplicity** $i_{X_0}(Y_0, Z_0)$ of $Y_0$ and $Z_0$ is

$$
\sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Tor}^i_{\mathcal{O}_{X_0}}(\mathcal{O}_{Y_0}, \mathcal{O}_{Z_0}).
$$

It follows from the regularity hypothesis that these numbers are finite. It is clear (*) that when the total intersection multiplicity is defined, we have

$$
i_{X_0}(Y_0, Z_0) = \sum_{P \in Y_0 \cap Z_0} [\kappa(P) : k] i_{X_0, P}(Y_0, Z_0).
$$

The definition is typically made under the hypothesis that $Y_0 \cap Z_0$ has the “expected dimension”. While this is indeed the situation of interest for us, this hypothesis is not technically necessary, (*) so we omit it.

The principle of continuity states then when intersecting flat subschemes, intersection multiplicities are constant under specialization or generalization.

**Theorem 4.4.2.** Let $\pi : X \to S$ be a smooth quasiprojective morphism, and $Y, Z \subseteq X$ closed subschemes, also flat over $S$. Suppose that $Y \cap Z$ is finite over $S$. Then we have

$$
i_{X_k}(Y_K, Z_K) = i_{X_k}(Y_k, Z_k).
$$

We recall the following fact, which is an easy consequence of the finiteness theorem of Gruson and Raynaud:

**Proposition 4.4.3.** Let $X$ be of finite presentation over $S$. Then $\mathcal{O}_X$ is coherent.
Proof. We first show that $\mathcal{O}_X$ is coherent for $X = \text{Spec} A[t_1, \ldots, t_n]$, and any $n \geq 0$. Since the question is local, it is enough to show that for all affine open subsets $U$ of $X$, and all sheaf homomorphisms of the form $f : \mathcal{O}^\bullet_U \to \mathcal{O}_X|_U$, then the kernel of $f$ is finitely generated. The image of $f$ is visibly a finitely generated quasicoherent ideal sheaf $\tilde{I}$ of $\mathcal{O}_X|_U$ for some $\mathcal{O}_X(U)$-ideal $I$. Then $\tilde{I}$ is torsion-free over $A$, hence flat over $A$, so by Raynaud-Gruson (Theorem 3.4.6 of [8]) we conclude that $\tilde{I}$ is finitely generated. We thus want to see that the surjection $\mathcal{O}_X(U)^m \to \tilde{I}$ has finitely generated kernel, which follows from finite presentation of $I$ (see Theorem 2.6 of [8]).

For the general case, again using that coherence is local we may assume that $X$ is affine, and thus that $X = \text{Spec} A[t_1, \ldots, t_n]/I$ for some finitely generated ideal $I$. But as above we know that $I$ is finitely presented, hence $\tilde{I}$ is coherent on $\text{Spec} A[t_1, \ldots, t_n]$, and by 5.3.10 of [7] we conclude that $\mathcal{O}_X$ is coherent, as desired. □

Proof of Theorem 11.9.3. Again using Raynaud-Gruson and the hypotheses that $X$ is of finite presentation and $Y, Z$ are flat, we conclude that $Y$ and $Z$ are also of finite presentation. By Proposition II.4.3 we have that $\mathcal{O}_X$ is coherent, so we conclude from the finite generation of $I_Y$ and $I_Z$ that $\mathcal{O}_Y$ and $\mathcal{O}_Z$ are also coherent, when considered as $\mathcal{O}_X$-modules. Now, $\text{Tor}_n^\mathcal{O}_X(Y, Z)$ is computed locally as the homology of $P_\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_Z$, where $P_\bullet$ is a resolution of $\mathcal{O}_Y$ by free $\mathcal{O}_X$-modules. Note that by the coherence of $\mathcal{O}_Y$ and $\mathcal{O}_X$, we can in fact require that the $P_\bullet$ have finite rank. It follows that for each $i$ we have that $\text{Tor}_i^\mathcal{O}_X(Y, Z)$ is coherent.

Since we have assumed $X$ to be quasiprojective, and $Y \cap Z$ is finite over $S$, there exists an affine open subset of $X$ containing $Y \cap Z$, and we may restrict to this subset without affecting our hypotheses or the total intersection multiplicities. If we then fix a global resolution $P_\bullet$ of $\mathcal{O}_Y$, by the flatness of $Y$ over $S$ we have that the base change to $X$, of the same complex computes $\text{Tor}_n^\mathcal{O}_X(Y, Z)$ for $s$ equal to either the closed or generic point of $S$. Each $\text{Tor}_i^\mathcal{O}_X(Y_s, Z_s)$ vanishes in degree above the relative dimension of $X$ over $S$, by Corollary 17.2.10 and Theorem 17.3.1 in Chapter 0, [7].

Now, $\text{Tor}_n^\mathcal{O}_X(Y, Z)$ is supported on $Y \cap Z$ (see 6.5.1 of [7]), and hence is a coherent $\mathcal{O}_{Y \cap Z}$-module. Since $Y \cap Z$ is assumed finite over $S$, we have by Corollary 6.1.11 of [7] and Theorem 2.2.2 of Exposé III of [7] (note that in our case, pseudo-coherence is equivalent to coherence; see also the beginning of Exposé II of [7]) that $\pi_* \text{Tor}_n^\mathcal{O}_X(Y, Z)$ is similarly coherent on $S$, and (by affineness of $\pi$) can be computed as the homology of $L_\bullet := \pi_*(P_\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_Z)$. Once again, the same assertions hold after base change to any $s \in S$. But now it is standard (using that $\mathcal{O}_S$ is coherent; see Remark 11.9.3 (ii) of Chapter 0 of [7]) that we can replace $L_\bullet$ by a complex $M_\bullet$ of free, finite rank $\mathcal{O}_S$-modules having the same cohomology.

The $M_i$ are then each free of finite rank, and we claim that for $i$ sufficiently large we have that $M_i$ is exact. Indeed, this follows from Nakayama’s lemma and the vanishing of $\text{Tor}_i^\mathcal{O}_X(Y_k, Z_k)$ in the closed fiber, using that the complex consists of flat modules. If we choose $i$ sufficiently large, we can thus truncate $M_\bullet$ above $i$, and if we replace $M_i$ by its image in $M_{i-1}$, we do not change the cohomology. This image is still coherent and torsion-free, hence free of finite rank. By flatness of the complexes we have that their cohomology also agrees after base change (see Remark 6.10.3 (ii) of [7]). It is then clear that for $s$ equal to either the closed or
generic point of $S$, we have
\[ i_{X_k}(Y_s, Z_s) = \sum_i (-1)^i \text{rk} \mathcal{M}_i, \]
which proves the desired identity. \hfill \Box

In the case we are interested in, that the fraction field $K$ of $R$ is algebraically closed, we can convert Theorem 4.4.2 into a statement on individual intersection multiplicities, which holds regardless of the dimension of the intersection.

**Corollary 4.4.4.** Let $\pi : X \to S$ be a smooth separated quasicompact morphism, and $Y, Z \subseteq X$ closed subschemes, also flat over $S$. Suppose also that $K$ is algebraically closed. (*) Let $W$ be an irreducible component of $Y_k \cap Z_k$. Then we have
\[ i_{X_k}(W, Y_k, Z_k) = \sum W m(W, \tilde{W}) \cdot i_{X_k}(\tilde{W}, Y_K, Z_K), \]
where the sum is over irreducible components $\tilde{W}$ of $Y_K \cap Z_K$ whose closures contain $W$, and $m(W, \tilde{W})$ denotes the multiplicity of $W$ in the closure of $\tilde{W}$.

(*)

**Proof.** We first give the proof in the case that $W$ is a point. We claim that there is a Zariski open subset $U \subseteq X$ such that $W \subseteq U$, and $Y \cap Z \cap U$ is finite over $S$. (*) By upper semicontinuity of fiber dimension, the locus $W_1$ of $Y \cap Z$ on which fibers are positive-dimensional is closed in $X$, and does not contain $W$ because $W$ is by hypothesis a point and a component of $Z_k \cap Z_k$. On the complement of $W_1$, we have $Y_K \cap Z_K$ finite, so every point is $K$-valued, and we let $W_2$ be the union of the closures of points of $Y_K \cap Z_K$ which do not specialize to $W$. We then have that $W_2$ is closed as well, and disjoint from $W$ by definition. Let $U = X \setminus (W_1 \cup W_2)$. Now, $Y \cap Z \cap U$ is quasifinite over $S$ by construction, so it is enough by Theorem 8.11.1 of [1] to show that it is proper. (*) However, $Y \cap Z \cap U$ is visibly separated and of finite type, and by construction every point of $Y \cap Z \cap U$ in $X$ is $K$-valued, and extends to an $R$-valued point of $Y \cap Z \cap U$; one checks that this implies the valuative criterion for universal closedness, Theorem 7.3.8 of [1]. This concludes the proof of the claim. It is then clear that $Y_K \cap Z_K \cap U$ consists precisely of the points of $Y_K \cap Z_K$ specializing to $W$, and because both $k$ and $K$ are algebraically closed, the statement of the corollary follows immediately from Theorem 4.4.2. Note here that $m(W, \tilde{W}) = 1$ because every $\tilde{W}$ is a $K$-valued point, and its closure is a section.

For the general case, we reduce to the 0-dimensional case. Noting that the statement is local on $X$, we may assume $X$ is affine, that $W$ is the only component of $Y_k \cap Z_k$, and that every component of $Y_K \cap Z_K$ contains $W$ in its closure. Let $\tilde{W}_1, \ldots, \tilde{W}_m$ be the irreducible components of $Y_K \cap Z_K$. Localizing further, we can also assume that $W$ and the $\tilde{W}_i$ are all smooth (although we cannot assume that the closure of the $\tilde{W}_i$ are smooth, as $W$ may appear with multiplicity higher than 1). (*) Now, let $L_k$ be a linear subspace of $X_k$ meeting $W$ transversely at some point $w$, and nowhere else. Localizing $X$ further, we may assume that $W \cap L_k = \{w\}$, and that for each $i$, every point of $\tilde{W}_i \cap L_K$ specializes to $w$. Let $L$ be a linear subspace of $X$ with special fiber $L_k$. Applying [17, Example 8.1.10] (*) inductively to the hyperplanes defining $L$ on each fiber, we have that
\[ i_{X_k}(W, Y_k, Z_k) = i_{X_k \cap L_k}(w, Y_k \cap L_k, Z_k \cap L_k) \]
and for each $\tilde{w}Y_K \cap Z_K \cap L_K$,

$$\sum_{i: \tilde{w} \in \tilde{W}_i} i_{X_K}(\tilde{W}_i, Y_K, Z_K) i_{X_K}(\tilde{w}, \tilde{W}_i, L_K) = i_{X_K \cap L_K}(\tilde{w}, Y_K \cap L_K, Z_K \cap L_K).$$

We apply the 0-dimensional case to the intersection of $Y \cap L$ with $Z \cap L$ and to the intersections of the closure of $\tilde{W}_i$ with $L_i$ and find that

$$i_{X_K \cap L_k}(w, Y_K \cap L_K, Z_K \cap L_K) = \sum_{\tilde{w} \in Y_K \cap Z_K \cap L_K} i_{X_K}(\tilde{w}, Y_K \cap L_K, Z_K \cap L_K)$$

and for each $i$,

$$m(W, \tilde{W}_i) = i_{X_k}(w, W, L_k)m(W, \tilde{W}_i) = \sum_{\tilde{w} \in W_i \cap L_K} i_{X_K}(\tilde{w}, \tilde{W}_i, L_K).$$

details?

(*) Combining these four identities gives the desired formula.

\[\square\]

\textbf{Remark 4.4.5.} In fact, the proofs of Proposition 4.4.3 and Theorem 4.4.2 go through unchanged in the case that $S = \text{Spec } R$ for $R$ an arbitrary valuation ring. It is also easy to see that total intersection multiplicity is invariant under extension of the base field, so it follows immediately from Proposition 7.1.4 of \[\text{[?]}\] that Theorem 4.4.2 in fact holds for an arbitrary base scheme.

\section{Main Results and Consequences}

We prove the main theorems, both as stated in the introduction and in refined forms addressing intersection multiplicities. We then give their generalizations to intersections of more than two subschemes, and also discuss what we can say when the tropicalizations do not necessarily intersect properly, but the initial degenerations still do.

\subsection{The main theorems.}

Throughout this section, $Y$ is a subvariety of the torus $T$ that contains $X$ and $X'$, and $w$ is a simple point of $\text{Trop}(Y)$ at which $\text{Trop}(X)$ and $\text{Trop}(X')$ meet properly. Let $j$ and $j'$ be the codimensions of $X$ and $X'$, respectively, in $Y$.

Let $\sigma$ be the facet of multiplicity 1 in $\text{Trop}(Y)$ that contains $w$, with $(N_{\sigma})_R$ the subspace of $N_R$ parallel to $\sigma$. Let $\Sigma$ be a complete unimodular fan in $(N_{\sigma})_R$ such that each face of $\text{Star}_w(\text{Trop}(X))$ and $\text{Star}_w(\text{Trop}(X'))$ is a union of faces of $\Sigma$. Let $Y(\Sigma)$ be the corresponding smooth complete toric compactification of $Y_w \cong T_\sigma$.

\textbf{Proof of Theorem 1.1.3.} First, since tropicalization is invariant under extensions of valued fields, we may assume the valuation on $K$ is nontrivial. Then points in $N_G$ are dense in the set of simple points of $\text{Trop}(Y)$ at which $\text{Trop}(X) \cap \text{Trop}(X')$ properly, so because $\text{Trop}(X \cap X')$ is closed, we may assume $w$ is in $N_G$. (*)

By Theorem 5.2.1, the tropicalization of the intersection of $X_w$ and $X'_w$ is exactly

$$\text{Trop}(X_w \cap X'_w) = \text{Star}_w(\text{Trop}(X)) \cap \text{Star}_w(\text{Trop}(X')).$$

In particular, $X_w \cap X'_w$ is nonempty of codimension $j + j'$ in $Y_w$, and since $Y_w$ is smooth, we have the desired result.

\[\square\]

\textbf{Proof of Theorem 1.1.3.} For $w \in N_G$, we have $T_w$ of finite type over $\text{Spec } R$. (*) Hence, by Theorem 1.1.3, there is a closed – hence $K$-rational – point in $X \cap X'$ specializing to $x$, (*) which gives the desired result.

\[\square\]
Theorem 5.1.1. Let \( \tau \) be a facet of \( \text{Trop}(X) \cap \text{Trop}(X') \) of codimension \( j + j' \) in \( \text{Trop}(Y) \), and containing simple points of \( \text{Trop}(Y) \). Then \( \tau \) is contained in \( \text{Trop}(X \cap X') \), and we have

\[
i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'), \text{Trop}(Y)) = \sum_Z i(Z, X \cdot X', Y) m_Z(\tau),
\]

where the sum is over components \( Z \) of \( X \cap X' \) with \( \tau \subseteq \text{Trop}(Z) \), and \( m_Z(\tau) \) is the multiplicity of \( \tau \) in \( \text{Trop}(Z) \). In particular, we have

\[
m_{X \cap X'}(\tau) \geq i(\tau, \text{Trop}(X) \cdot \text{Trop}(X'), \text{Trop}(Y)).
\]

Proof. (*)

The case that \( \text{Trop}(X) \) meets \( \text{Trop}(X') \) everywhere properly has a particularly simple statement equating the tropicalization of the refined intersection cycle with the stable tropical intersection. Because one typically only defines the refined intersection cycle for intersections in a smooth ambient variety, we add that hypothesis as well.

Corollary 5.1.2. Suppose that \( Y \) is smooth, and \( \text{Trop}(X) \) meets \( \text{Trop}(X') \) properly in \( \text{Trop}(Y) \), and the simple points of \( \text{Trop}(Y) \) are dense in \( \text{Trop}(X) \cap \text{Trop}(X') \). Then

\[
\text{Trop}(X \cdot X') = \text{Trop}(X) \cdot \text{Trop}(X').
\]

5.2. Intersection of multiple subschemes. In applications, and particularly in the context of enumerative geometry, one frequently wants to intersect more than two subschemes. Once intersection theory is developed suitably, treatment of multiple intersections follows immediately from results on intersection of pairs, and our situation is no exception. In particular, intersection multiplicities along components of intersection may be defined by inductive intersection, (*) and (*) cite tropical case?

In our case, we have:

Theorem 5.2.1. Given \( X_1, \ldots, X_r \subseteq Y \) of codimensions \( j_1, \ldots, j_r \), suppose \( \tau \) is a facet of \( \bigcap_i \text{Trop}(X_i) \) of codimension \( \sum_i j_i \) in \( \text{Trop}(Y) \), and containing simple points of \( \text{Trop}(Y) \). Then \( \tau \) is contained in \( \text{Trop}(\bigcap_i X_i) \), and we have

\[
i(\tau, \text{Trop}(X_1) \cdots \text{Trop}(X_r), \text{Trop}(Y)) = \sum_Z i(Z, X_1 \cdots X_r, Y) m_Z(\tau),
\]

where the sum is over components \( Z \) of \( \bigcap_i X_i \) with \( \tau \subseteq \text{Trop}(Z) \), and \( m_Z(\tau) \) is the multiplicity of \( \tau \) in \( \text{Trop}(Z) \). In particular, we have

\[
m_{\bigcap_i X_i}(\tau) \geq i(\tau, \text{Trop}(X_1) \cdots \text{Trop}(X_r), \text{Trop}(Y)).
\]

Proof. (*)

5.3. Non-proper intersections. Our argument relies in several places on the hypothesis that \( \text{Trop}(X) \) meets \( \text{Trop}(X') \) properly. However, we still obtain some nontrivial consequences under weaker hypotheses. For instance, we have
Corollary 5.3.1. Suppose that $Y$ is smooth, and $Y_w$ is smooth for every $w \in \text{Trop}(X) \cap \text{Trop}(X')$. Suppose further that for every $w \in \text{Trop}(X) \cap \text{Trop}(X')$, we have that $X_w$ meets $X'_w$ properly in $Y_w$ if the intersection is non-empty. Then the set of $w$ such that $X_w \cap X'_w \neq \emptyset$ has codimension $j + j'$ in $\text{Trop}(Y)$.

Proof. (*)

6. Examples

We conclude with some examples where hypotheses of Theorem 6.1.2 are not satisfied, and as a result the conclusion likewise fails. The most trivial example demonstrates that the hypothesis on the dimension of $\text{Trop}(Y) \cap \text{Trop}(Z)$ is necessary. (*)

Example 6.0.2. Suppose $n = 2$, and $X = T$, and $Y$ and $Z$ are lines inside $T$. One can easily choose $Y$ and $Z$ distinct so that their tropicalizations are the same, in which case $Y \cap Z$ certainly does not surject onto $\text{Trop}(Y) \cap \text{Trop}(Z)$. Similarly, one can translate so that $\text{Trop}(Y)$ and $\text{Trop}(Z)$ meet in a ray, which is not the tropicalization of any variety, let alone of $Y \cap Z$.

The next three examples demonstrate the necessity of considering intersection points at simple points of the ambient variety $X$. All three examples are based on considering skew lines inside a smooth quadric surface.

Example 6.0.3. We first consider what happens in the constant coefficient case: for instance, let $Q \subseteq (K^*)^3$ be the smooth quadric surface given by

$$4y^2 - z^2 - 3xy + 3xz - 3y = 0.$$  

This contains the two lines $L_1, L_2$ given by $y = z = 1$ and $x - 1 = 2y - z = 0$, respectively. $L_1$ and $L_2$ do not intersect, but $\text{Trop}(L_1)$ meets $\text{Trop}(L_2)$ at $(0, 0, 0)$, which is the expected dimension of intersection inside $\text{Trop}(X)$. In this case, because we are in the constant coefficient case we have that $\text{Trop}(X)$ is a fan centered at $(0, 0, 0)$, so the intersection does not occur at a facet. The corresponding tropical degeneration is not a torus torsor, and there is no geometric reason for the degenerations of $L_1$ and $L_2$ to meet, so they do not (and indeed they cannot, since in the constant coefficient case the geometry of the tropical degeneration is the same as that of the original varieties over $K$).

Example 6.0.4. Next, let $Q$ be the smooth quadric given by

$$2y^2 + z^2 + 3txy - 3yz - 3tx - 3ty + 3t = 0,$$

where $t \in K^*$ has valuation 1. Let $L_1$ and $L_2$ be as in Example 6.0.4. Then as before, $\text{Trop}(L_1)$ and $\text{Trop}(L_2)$ meet with the expected dimension inside $\text{Trop}(X)$, even though $L_1$ and $L_2$ are disjoint, and remain disjoint under tropical degeneration.

Here, one checks that $(0, 0, 0)$ does in fact lie in a facet of $\text{Trop}(X)$, but one of multiplicity 2. Although the tropical degeneration $X_{(0,0,0)}$ of $X$ is a torus torsor, it is a torus torsor over a disjoint union of two points, and the tropical degenerations of $L_1$ and $L_2$ lie in disjoint component of $X_{(0,0,0)}$.

Example 6.0.5. Finally, consider the smooth quadric given by

$$(y - 1)^2 - (z - 1)^2 + (x - 1)(y - 1) + t(y - 1) + t(z - 1) = 0,$$

where again $t \in K^*$ has valuation 1. In this case, we let $L_1$ be as in Example 6.0.4, but let $L_2$ be the line $x - 1 = y - z + t = 0$. Once again, $\text{Trop}(L_1)$ and $\text{Trop}(L_2)$
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meet at $(0,0,0)$, with expected dimension inside $\text{Trop}(X)$. Moreover, they meet at $(1,1,1)$ in the corresponding tropical degeneration. However, this intersection point does not lift, which is possible because the tropical degeneration of $X$ is a cone with its singularity at $(1,1,1)$.

APPENDIX A. TOPOLOGY OF FINITE TYPE MORPHISMS

(*). Because the result may be of independent interest, we note that in fact Corollary 4.2.5 can be proved for an arbitrary base scheme:

**Theorem.** Let $X$ be a scheme of finite type over another scheme $S$, let $s,s'$ be points of $S$ with $s'$ specializing to $s$, and let $x$ be a closed point of the fiber $X_s$. Suppose there exists $x' \in X_{s'}$ specializing to $x$. Then there exists a point $x''$, closed in $X_{s'}$, such that $x'$ specializes to $x''$ and $x''$ specializes to $x$.

**Proof.** We first note that the proof of Proposition 4.2.1 goes through unmodified for an arbitrary irreducible base scheme. (*) Similarly, the proof of Corollary 4.2.5 works for an arbitrary valuation ring. It thus suffices to show that the statement of the theorem reduces to the case that $S$ is the spectrum of a valuation ring. As before, we may replace $X$ by the closure of $x'$, and then $S$ by the closure of $s'$, so we may assume $X$ and $S$ are integral, and $s',s$ are the generic and closed points of $S$ respectively. Now, let $A$ be a valuation ring with a dominant morphism $\text{Spec } A \to X$ mapping the closed point to $x$ (Proposition 7.1.4 of [EGA2]). Then set $B = A \cap K(S)$; the inclusion $B \hookrightarrow A$ is then a local homomorphism of valuation rings, and we have a commutative diagram

$$
\begin{array}{ccc}
\text{Spec } A & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec } B & \longrightarrow & S
\end{array}
$$

Then set $S' = \text{Spec } B$, $X' = X \times_S \text{Spec } B$. The maps from $\text{Spec } A$ induce a map $\text{Spec } A \to X'$ which gives us a pair of points $\tilde{x}, \tilde{x}' \in X'$ mapping to $x,x'$ in $X$, and with $\tilde{x}'$ specializing to $\tilde{x}$. We see moreover that $\tilde{x}$ must be closed in the special fiber over $S'$, since it maps to a closed point of $f^{-1}(s)$. Thus, if we have the desired statement over a valuation ring, we conclude that we have $\tilde{x}''$ closed in the generic fiber and with $\tilde{x}'$ specializing to $\tilde{x}''$ specializing to $\tilde{x}$. Let $x''$ be the image of $\tilde{x}''$ in $X$; we necessarily have that $x'$ specializes to $x''$, which specializes to $x$. However, since there was no extension of residue fields at the generic point under the map $\text{Spec } B \to S$, we have that the generic fibers are isomorphic, so we conclude that $x''$ is closed in the generic fiber, giving the desired statement. \qed

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