RECENT PROGRESS ON VECTOR BUNDLES WITH SECTIONS

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Abstract. Higher-rank Brill-Noether, at its most basic, seeks to answer the following question: how many global sections can a (semistable) vector bundle of given rank and degree have on general curve of genus g? When there exist vector bundles with a k-dimensional space of sections, one then asks how many such bundles there are. The classical rank-1 version was settled completely in the 1970's and 1980's, but even the rank-2 case is still wide open, with many partial results but no comprehensive conjecture. In the 1990's, Bertram, Feinberg and Mukai observed that the canonical determinant case has special behavior, with a higher expected dimension, and I will discuss recent work which studies the special determinant case more systematically. This work is in two parts: producing new expected dimensions on smooth curves by exploiting symmetries to produce lower bounds on dimensions, and developing the general theory of higher-rank limit linear series in order to use degeneration arguments to prove existence results. Some of this is joint work with Montserrat Teixidor i Bigas.

1. Background

We consider the following basic question:

Question 1.1. If C is a general (smooth, projective) curve of genus g, and we are given $r, d > 0$, what is the maximum for

$$\dim \Gamma(C, \mathscr{E}),$$

where $\mathscr{E}$ is a (semi)stable rank-r vector bundle of degree d on C? For k less than or equal to this maximum, what is the dimension of the space of vector bundles with $\dim \Gamma(C, \mathscr{E}) \geqslant k$?

Recall that $\mathscr{E}$ is semistable if for all subbundles $\mathscr{F} \subseteq \mathscr{E}$ of rank r' and degree d', we have $d'/r' \leqslant d/r$, and $\mathscr{E}$ is stable if we always have strict inequality.

The case of rank 1 is the situation considered in classical Brill-Noether theory, introduced in the 1870's and fully solved roughly a century later. There has been a great deal of work on higher rank since then, but despite many partial results, there are no comprehensive conjectures even for rank 2. Aside from its fundamental nature, Question 1.1 has various directions of motivation:

- it can be used to study maps from curves to Grassmannians;
- despite how little is known, Mukai has successfully applied it to various areas, including study of Fano varieties, and curves on K3 surfaces, and classification of curves of genus 7, 8 and 9 [Muk01],[Muk01];
- recently, Bhosle, Brambila-Paz and Newstead applied it to solve a conjecture of Butler on stability of the kernel of evaluation maps for general linear series on curves [BBPN];
- just as in the classical rank-1 case, higher-rank Brill-Noether theory can be used to produce effective divisors on the moduli space of curves, and thus to study its geometry [FP05].

In both the classical and higher-rank cases, the starting point is the *expected dimension*

$$\rho = r^2(g-1) + 1 + k(d + r(1-g) - k).$$

Then we can answer Question 1.1 in the rank-1 case as follows:

Theorem 1.2 (Brill-Noether)**.** *With notation as in Question 1.1, the maximum value of* $\dim \Gamma(C, \mathscr{E})$ *is simply the maximal* k *with* $\rho \geqslant 0$*, and for any such* k*, the dimension of choices of* $\mathscr{E}$ *with* $\dim \Gamma(C, \mathscr{E}) \geqslant k$ *is equal to* $\min\{\rho, g\}$*.*

The proof combined intersection theory calculations for the nonemptiness portion with degeneration techniques for the emptiness and dimension statements.

However, as soon as we consider rank 2 or more, the situation becomes far more complicated. Although in many cases we still get some components of dimension ρ, the spaces in question may be empty when ρ is nonnegative, and nonempty when ρ is negative, and there are often components of dimension larger than ρ. A major new component of the picture, first observed in the 1990's independently by Bertram and Feinberg and Mukai, is that the expected behavior depends also on the determinant of the vector bundle in question. We introduce notation as follows:

Notation 1.3. Given a line bundle $\mathscr{L}$ of degree d on a curve C, let $G^k_{r,\mathscr{L}}(C)$ be the moduli space of pairs $(\mathscr{E}, V)$ where $\mathscr{E}$ is a stable rank-r vector bundle on C of determinant $\mathscr{L}$, and $V \subseteq \Gamma(C, \mathscr{E})$ is a k-dimensional space of global sections.

Then one sees immediately that when $G^k_{r,\mathscr{L}}(C)$ is nonempty, every component has dimension at least $\rho - g$. However, Bertram-Feinberg and Mukai observed the following:

Proposition 1.4. *If* ω *is the canonical line bundle on* C*, then every component of* $G^k_{2,\omega}(C)$ *has dimension at least*

$$\rho - g + \binom{k}{2}.$$

In particular, in cases where $\binom{k}{2} > g$, they observed that one obtains moduli spaces in the varying-determinant situation of strictly larger than the naive expected dimension ρ.

Bertram, Feinberg and Mukai also conjectured that with this modified expected dimension, the spaces $G^k_{2,\omega}(C)$ should behave as in the rank-1 case, and in particular should be nonempty if and only if $\rho - g + \binom{k}{2} \geqslant 0$.

2. New expected dimensions

My initial work on the subject consisted of attempts to generalize and systematize the work of Bertram, Feinberg, and Mukai, by considering more general types of special determinant. My hope was (and is) that at least in rank 2, the fixed determinant spaces $G^k_{2,\mathscr{L}}(C)$ may have the good behavior which is lacking in the varying determinant case. I have mainly focused on the rank-2 case, on the theory that the final answer is likely to have a structure which is inductive with respect to the rank.

My first result is as follows:

Theorem 2.1 ([Oss13a])**.** *Let* $\mathscr{L}$ *be a line bundle of degree* d*, and let* δ *be the minimal degree of an effective divisor* Δ *such that* $h^1(C, \mathscr{L}(-\Delta)) \geqslant 1$*. Then every component of* $G^k_{2,\mathscr{L}}(C)$ *has dimension at least* $\rho - g + \binom{k-\delta}{2}$*.*

As a special case we obtain the following, which recovers Proposition 1.4 in the special case that $\mathscr{L} = \omega$.

Corollary 2.2. *If* $h^1(C, \mathscr{L}) \geqslant 1$*, then every component of* $G^k_{2,\mathscr{L}}(C)$ *has dimension at least* $\rho - g + \binom{k}{2}$*.*

A more difficult but parallel argument leads to the following result.

Theorem 2.3. *Let* $\mathscr{L}$ *be a line bundle of degree* d*, and suppose that* $h^1(C, \mathscr{L}) \geqslant 2$*. Then every component of* $G^k_{2,\mathscr{L}}(C)$ *has dimension at least* $\rho - g + 2\binom{k}{2}$*.*

There is a natural generalization of Theorem 2.3, but it requires a good understanding of certain subtle nondegeneracy conditions, which I have only been able to work out in a few special cases. Specifically, I show the following:

Theorem 2.4 ([Oss13c]). *Suppose that $\mathscr{L} \in \mathrm{Pic}^d(C)$, with $h^1(C, \mathscr{L}) \geqslant m$. Let $\mathscr{E}$ be a vector bundle of rank r on C with determinant $\mathscr{L}$, and $V \subseteq H^0(C, \mathscr{E})$ a k-dimensional space of global sections. Suppose that in addition, one of the following conditions is satisfied.*

(I) *$k = r$, and V is not contained in any subbundle of $\mathscr{E}$ of rank $r-2$.*
(II) *$k = r+1$, $m = 1$, and no r-dimensional subspace of V is contained in any subbundle of $\mathscr{E}$ of rank $r-2$.*
(III) *$r = 3$, $k = 5$ or 6, $m = 1$, and no 2-dimensional subspace of V is contained in any subbundle of $\mathscr{E}$ of rank 1.*

Then every component of $\mathscr{G}^k_{r,\mathscr{L}}(C)$ passing through the point corresponding to $(\mathscr{E}, V)$ has dimension at least

$$\rho - g + m\binom{k}{r}. \tag{2.4.1}$$

For some cases in which k is small with respect to r, the dimension bound of (2.4.1) can be verified to be sharp via direct constructions. The simplest case is $k = r = 2$:

Proposition 2.5. *If C is general and $\Gamma(C, \mathscr{L}) \neq 0$, then $G^2_{2,\mathscr{L}}(C)$ is irreducible of dimension $\rho - g + h^1(C, \mathscr{L})$.*

Grzegorczyk and Newstead generalized the above as follows:

Theorem 2.6 (Grzegorczyk-Newstead). *For $g \geqslant 2$, if $\deg \mathscr{L} \geqslant r+1$ and $\mathscr{L}$ has a section with distinct zeroes, then $G^{r,L}_{r,\mathscr{L}}(C)$ is irreducible of dimension $\rho - g + h^1(C, \mathscr{L})$.*

If further $h^0(C, \mathscr{L}) \geqslant r+1$, then $G^{r+1,L}_{r,\mathscr{L}}(C)$ has a component of dimension $\rho - g + (r+1)h^1(C, \mathscr{L})$.

Remark 2.7. $G^{k,L}_{r,\mathscr{L}}(C)$ is a space closely related to $G^k_{r,\mathscr{L}}(C)$, but with a stability condition coming from the theory of coherent systems.

If $\deg \mathscr{L} < r+1$, then $G^{r,L}_{r,\mathscr{L}}(C)$ is empty.

For the last statement, the component of dimension $\rho - g + (r+1)h^1(C, \mathscr{L})$ is the only component whose general member is a pair $(\mathscr{E}, V)$ with V generating $\mathscr{E}$.

3. Degeneration techniques

In the 1980's, Eisenbud and Harris developed a theory of limit linear series which they used to give a very simple proof of the Brill-Noether theorem. In the early 1990's, Teixidor i Bigas generalized the Eisenbud-Harris theory to the case of vector bundles, and has since applied it to give many of the most important results to date in higher-rank Brill-Noether theory, including existence of components of moduli spaces having the expected dimension ρ, and the injectivity of the generalized Petri map for vector bundles of rank 2 with canonical determinant. It is thus natural to try to use these techniques in order to prove that the expected dimensions produced above are in fact correct at least in some cases, by producing components of $G^k_{2,\mathscr{L}}(C)$ of the prescribed dimension.

However, one quickly runs into a foundational difficulty: the higher expected dimensions for the special determinant case are the result of certain symmetries, and smoothing results for limit linear series are proved using dimension counts. The usual arguments only allow one to smooth limit linear series when one has a family of dimension ρ (or $\rho - g$ in the fixed determinant case), but in the situation we are interested in, the dimension will always be bigger. In order to prove useful smoothing theorems in the special determinant case, it is necessary to give modified dimension counts for limit linear series which take the extra symmetries into account, but it is completely unclear how to do this in the context of the Eisenbud-Harris-Teixidor limit linear series. I am currently completing two preprints (the latter joint with Teixidor i Bigas) that address this, proving new foundational smoothing theorems, and applying them to prove new existence results.

In [Oss14a], I introduce two new notions of limit linear series in higher rank, which I call "linked linear series" and which generalize a construction in my thesis [Oss06]. I also introduce a moduli space (stack) structure on Teixidor's definition (which had previously only been described in terms of a stratification), and prove some fundamental comparison results between the three spaces. This incorporates foundational work on dimension theory for morphisms and stacks [Oss13b], on a notion of "ℓ-stability" of vector bundles on reducible curves which is very convenient in degeneration contexts [Oss14b], and on a generalization of determinantal loci to pushforwards.

The linked linear series spaces are in some sense more natural, and in particular the symmetries coming from special determinant conditions are more visible for them. Thus, in [OT14] (with Teixidor i Bigas), we use them along with a theory of "linked symplectic Grassmannians" to prove the following fundamental smoothing theorem:

Theorem 3.1 (O.-Teixidor i Bigas)**.** *Given g, d, k, with $k \geqslant 2$, let X_0 be a projective nodal curve with dual graph a chain, and $\mathscr{L}_0$ a line bundle of degree d on X_0 such that there exists a morphism $\mathscr{L}_0 \to \omega_{X_0}$ not vanishing uniformly on any component of X_0. Suppose there exists an ℓ-stable chain-adaptable limit linear series of rank 2 and fixed determinant $\mathscr{L}_0$, such that the space of such limit linear series on X_0 has the expected dimension $\rho - g + \binom{k}{2}$ at the corresponding point.*

Then for a general smooth curve X of genus g and a general special line bundle $\mathscr{L}$ of degree d, the space $G^k_{2,\mathscr{L}}(X)$ is nonempty, with a component of expected dimension $\rho - g + \binom{k}{2}$.

In the theorem, chain-adaptable is a technical condition on higher-rank limit linear series which allows comparison between limit linear series and linked linear series, and which is satisfied for all families of limit linear series considered to date by Teixidor i Bigas in her work. We then apply Theorem 3.1 to prove:

Theorem 3.2 (O.-Teixidor i Bigas)**.** *Given g, d, k nonnegative, with $k \geqslant 2$ and $g-2 \leqslant d \leqslant 2g-2$, suppose that*

$$4g \geqslant \begin{cases} k^2 + 2k(2g-2-d): & d \text{ even} \\ k^2 + 4 + 2k(2g-2-d): & d \text{ odd.} \end{cases}$$

Suppose further that

$$(g, d, k) \neq (1, 0, 2), (2, 2, 2), (3, 2, 2) \text{ or } (4, 6, 4).$$

Then for a general smooth curve X of genus g and a general special line bundle $\mathscr{L}$ of degree d, the space $G^k_{2,\mathscr{L}}(X)$ is nonempty, with a component of expected dimension $\rho_{\mathscr{L}} := \rho - g + \binom{k}{2}$.

This thus gives a large family of examples in a range of degrees for which the modified expected dimension of Corollary 2.2 is seen to be sharp.

On the other hand, in the special case $\mathscr{L} = \omega$, the result still falls substantially short of the existence statements predicted by Bertram-Feinberg and Mukai. However, the families of limit linear series constructed in the proof of Theorem 3.2 are in some sense the "simplest" cases, where the underlying vector bundles are semistable on every component of the reducible curve. Our results may be optimal for this situation, but can be improved (still using the same general theory) by writing down families of limit linear series which have unstable bundles on some components but still satisfy a suitable global semistability condition. Naizhen Zhang has been pursuing this approach in his thesis, focusing on the case $\mathscr{L} = \omega$. In this case, he has improved the existence results in Theorem 3.2 by an amount which gets arbitrarily large as g increases.

Recent work of Lange, Newstead and Park uses an intersection theory approach parallel to the classical rank-1 case to address the same existence questions, but they run into very complicated combinatorial issues, and their results are ultimately not as strong as Zhang's. Thus, degeneration techniques still appear to be the most promising approach to the existence question, and in particular there still seems to be substantial room for further improvements using the same basic techniques.

References

[BBPN] Usha Bhosle, Leticia Brambila-Paz, and Peter Newstead, *On linear systems and a conjecture of D. C. Butler*, preprint.

[FP05] Gavril Farkas and Mihnea Popa, *Effective divisors on $\overline{\mathscr{M}}_g$, curves on $K3$ surfaces, and the slope conjecture*, Journal of Algebraic Geometry **14** (2005), no. 2, 241–267.

[Muk01] Shigeru Mukai, *Non-abelian Brill-Noether thoery and Fano 3-folds*, Sugaku Expositions **14** (2001), no. 2, 125–153.

[Oss06] Brian Osserman, *A limit linear series moduli scheme*, Annales de l'Institut Fourier **56** (2006), no. 4, 1165–1205.

[Oss13a] ———, *Brill-Noether loci in rank 2 with fixed determinant*, International Journal of Mathematics **24** (2013), no. 1350099, 24 pages.

[Oss13b] ———, *Relative dimension of morphisms and dimension for algebraic stacks*, preprint, 2013.

[Oss13c] ———, *Special determinants in higher-rank Brill-Noether theory*, International Journal of Mathematics **24** (2013), no. 1350084, 20 pages.

[Oss14a] ———, *Limit linear series moduli stacks in higher rank*, preprint, 2014.

[Oss14b] ———, *Stability of vector bundles on curves and degenerations*, preprint, 2014.

[OT14] Brian Osserman and Montserrat Teixidor i Bigas, *Linked symplectic forms and limit linear series in rank 2 with special determinant*, preprint, 2014.