FROBENIUS-UNSTABLE VECTOR BUNDLES AND THE GENERALIZED VERSCHIEBUNG

BRIAN OSSERMAN

Abstract. Let C be a smooth curve, and $M_r(C)$ the coarse moduli space of vector bundles of rank r and trivial determinant on C. We discuss the generalized Verschiebung map $V_r : M_r(C^{(p)}) \dashrightarrow M_r(C)$ induced by pulling back under Frobenius. We begin with a survey of general background results on the Verschiebung, as well as certain more specialized work in fixed genus, characteristic, and rank. We then discuss the case of vector bundles of rank 2 on a curve of genus 2 in some detail, explaining how to conclude the degree of V_2 from sufficient knowledge of the locus of semistable bundles which are destabilized by Frobenius. Finally, we sketch how one may obtain such results on the Frobenius-unstable locus by degeneration arguments.

1. Background on the Verschiebung

We fix the following notation:

Situation 1.1. We suppose C is a smooth, proper curve over an algebraically closed field k of characteristic p, and denote by $C^{(p)}$ the p-twist of C over k, and $F : C \to C^{(p)}$ the relative Frobenius morphism. Finally, for any $r > 1$ we denote by $M_r(C)$ and $M_r(C^{(p)})$ the coarse moduli spaces of semistable vector bundles of rank r and trivial determinant on C and $C^{(p)}$.

The basic existence result on the Verschiebung map may be summarized as follows:

Proposition 1.2. *In the above situation, and given $r > 1$, the operation of pulling back vector bundles under F induces a* **generalized Verschiebung** *rational map $V_r : M_r(C^{(p)}) \dashrightarrow M_r(C)$. If we denote by U_r the open subset of $M_r(C^{(p)})$ corresponding to bundles $\mathcal{E}$ such that $F^*(\mathcal{E})$ is semi-stable, then we obtain a morphism $V_r : U_r \to M_r(C)$.*

The proof of this proposition is routine, and would be trivial if not for the coarseness of the moduli spaces in question.

Definition 1.3. We say that a semistable vector bundle $\mathcal{F}$ is **Frobenius-unstable** if it is in the complement of U_r, which is to say, if $F^*\mathcal{F}$ is unstable.

There are several motivations to study this generalized Verschiebung. Of course, the importance of the Verschiebung is well-established in the case of rank 1, so one might naturally want to study its generalization. However, there is also a close relationship between the Verschiebung map and p-adic representations of the fundamental group of C, when the base field k for our curve C is finite (see the introduction to [4]). In particular, de Jong showed that curves in the moduli space of vector bundles which are fixed under some iterate of the Verschiebung will correspond to p-adic representations for which the geometric fundamental group has infinite image, which he conjectures in [1] cannot happen for ℓ-adic representations. He further showed that such curves would have to pass through the undefined locus of V_r. Another motivation comes from the fact that invariants such as the degree of the Verschiebung nearly always seem to be given by polynomials in p, with no obvious explanation for why this should be the case. One might hope that examination of enough different cases of this phenomenon would ultimately provide insight into the general situation.

We have the following general statements on the Verschiebung; we make use of only the first two in this talk.

Theorem 1.4. *We have the following additional statements on V_r:*

(i) *(Osserman-Pauly) The undefined locus of V_r is precisely U_r;*
(ii) *(de Jong) V_r is dominant;*
(iii) *(Pauly) If r is a prime number with $r \geq g(p-1)+1$, or (Mehta-Subramanian) C is ordinary, then V_r is generically étale.*

The argument that V_r cannot be extended away from U_r is based on a suggestion of Pauly, and requires invoking substantive theorems on the Picard group of the stack of vector bundles. The argument for dominance was provided by de Jong, and involves a careful examination of the map induced by V_r on hulls at a point of U_r corresponding to a direct sum of certain line bundles. Finally, the statements on generic étaleness are due to Pauly, see [11, Prop. 3.3], and Mehta-Subramanian, see [6]. Pauly also shows that V_r is not separable for a non-ordinary curve when p divides r.

Finally, we remark on some additional results in the literature of a more special nature. Laszlo and Pauly have recently given an explicit description of the Verschiebung map in the specific cases of genus 2, rank 2, and characteristics 2 and 3 in [4] and [5]. These arguments use techniques very special to low characteristic, but are nonetheless useful for producing guiding examples. In particular, in the case of characteristic 2, they were able to produce a line in the moduli space which is fixed under the square of the Verschiebung, giving an example of a characteristic 2 representation of the fundamental group of the curve for which the geometric fundamental group has infinite image. By work of de Jong, such an example is not possible for rank 2 in the case of a characteristic ℓ representation, and conjectured to be impossible for higher rank as well. We also note that Lange and Pauly obtain an upper bound for the degree of V_2 in the case of genus 2 via a different approach in [3], which agrees with the formula we will give.

2. The case of genus 2 and rank 2

We now specialize our situation:

Situation 2.1. We suppose that C has genus 2, that $r = 2$, and that $p > 2$.

Now, this situation, we know the following:

Theorem 2.2. (i) *(Narasimhan-Ramanan) $M_2 \cong \mathbb{P}^3$.*
(ii) *(de Jong, Laszlo-Pauly) The map V_2 is given by polynomials of degree p.*
(iii) *(Joshi-Xia) A Frobenius-unstable vector bundle $\mathcal{F}$ necessarily has a short exact sequence*

$$0 \to \mathscr{L} \to F^*\mathcal{F} \to \mathscr{L}^{-1} \to 0,$$

with $\mathscr{L}$ a theta characteristic; that is, $\mathscr{L}^{\otimes 2} \cong \Omega^1_C$.

The first statement is [8, Thm. 2, §7], noting that despite the Riemann surface language, the argument goes through unmodified in arbitrary odd characteristic. The second statement was proved by de Jong by considering the pullbacks of the Verschiebung to the Jacobians of C and $C^{(p)}$, which map to Kummer surfaces inside $M_2 \cong \mathbb{P}^3$. Laszlo and Pauly provided a different and more general proof in [5, Prop. A.2]. Finally, the last statement is [2, Prop. 3.3].

Because of the rigidity afforded by Joshi and Xia's result, the following *ad hoc* definition is useful:

Definition 2.3. Let C be a smooth, proper curve of genus 2. Given a Frobenius-unstable vector bundle $\mathcal{F}$ of rank 2 and trivial determinant, we say that $\mathcal{F}$ is a **reduced** point of the Frobenius-unstable locus if the first-order determinant-preserving infinitesmal deformations of $\mathcal{F}$ inject into the first-order infinitesmal deformations of $F^*\mathcal{F}$.

We will explain how to prove the following:

Theorem 2.4. *Let C be a smooth, proper genus 2 curve over an algebraically closed field k of characteristic $p > 2$, and suppose that the Frobenius-unstable locus for vector bundles of rank 2 and trivial determinant is composed of δ reduced points. Then:*

(i) *Each undefined point of V_2 may be resolved by a single blowup, and V_2 has degree $p^3 - \delta$;*
(ii) *The exceptional divisor associated to such an undefined point maps bijectively to $\mathbb{P}\operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1}) \subset M_2(C)$, where $\mathscr{L}$ is a theta characteristic on C, and specifically is the destabilizing line bundle for $F^*\mathcal{F}$, where $\mathcal{F}$ is the Frobenius-unstable vector bundle associated to the undefined point.*

Further, if C is general, the hypothesis holds with $\delta = \frac{2}{3}(p^3 - p)$, so that $\deg V_2 = \frac{p^3+2p}{3}$.

The last assertion on the Frobenius-unstable locus for a general curves follows from degeneration arguments of Mochizuki (see [7], [10]) or of the author (see [9]). We sketch these arguments at the end, but will mainly focus on how to translate information about the Frobenius-unstable locus to the consequent conclusions on the Verschiebung.

We have the following simple proposition; only the inequality aspect may be not so well-known.

Proposition 2.5. *Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a dominant rational map, defined on the complement of a finite set. Then if f is described in terms of homogeneous coordinates X_i on $\mathbb{P}^n$ by polynomials F_i, we have the inequality:*

$$\deg f \leq d^n - \delta,$$

where δ is the total length of the 'undefined locus' subscheme E_f of $\mathbb{P}^n$ cut out by the F_i. Moreover, the following are equivalent:

a) *The above inequality is an equality;*
b) *E_f is a locally complete intersection;*
c) *E_f is Gorenstein.*

In particular, we get equality when the length of the points of E_f are all 1 or 2.

We can conclude:

Corollary 2.6. *The degree of V_2 is bounded above by p^3; or more sharply, $p^3 - \delta$, where δ is the number of points at which V_2 it is undefined. If the undefined points are reduced, this upper bound is an equality.*

The remainder of the proof of the main part of Theorem 2.4 is therefore devoted to proving that a reduced Frobenius-unstable bundle corresponds to a reduced undefined point of V_2. We will see that the argument for this also naturally proves the statement on the image of the exceptional divisor. The idea of the proof is based on Langton's argument for properness of the moduli space of semi-stable vector bundles.

Suppose we have a family $\tilde{\mathcal{F}}$ of vector bundles corresponding to a smooth curve T in $M_2(C^{(p)})$, such that $F^*\tilde{\mathcal{F}}$ is generically stable, but on some fiber over $0 \in T$, if we write $\mathcal{F} := \tilde{\mathcal{F}}|_0$, we have $F^*\mathcal{F} \cong \mathcal{E}$ for one of our unstable $\mathcal{E}$. Applying adjointness to i_0, we obtain a map $\psi : F^*\tilde{\mathcal{F}} \to i_{0*}\mathscr{L}^{-1}$, and we find that $\ker\psi$ is a family of vector bundles with trivial determinant on C. We obtain a diagram as follows:

$$\begin{array}{ccccccc}
(\ker\psi)|_0 & \longrightarrow & F^*\mathcal{F} \cong \mathcal{E} & \xrightarrow{i_0^*\psi} & \mathscr{L}^{-1} & \longrightarrow & 0 \\
& \searrow & \nearrow & & & & \\
& & \mathscr{L} & & & & \\
& \nearrow & & & & & \\
0 & & & & & &
\end{array}$$

The dashed arrow exists and is surjective by exactness, so we find that $(\ker\psi)|_0$ is a vector bundle of trivial determinant with a surjective map to $\mathscr{L}$; that is, it is an element of $\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$. It is clear that if it is a non-trivial extension, it is stable, and hence describes the limit of $V_2(T)$ at $0 \in T$.

Next, we also show that if we carry out the same process with $T = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, we obtain a map

$$\mathrm{Def}(\mathcal{F}) \to \mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1}) \tag{2.1}$$

which is compatible with starting from the case that T is a smooth curve, and restricting to induced deformations. Moreover, further deformation theory computations shows that the kernel of this map is precisely the subspace of $\mathrm{Def}(\mathcal{F})$ which induces the trivial deformation of $F^*\mathcal{F} \cong \mathcal{E}$. In particular, the map is injective if $\mathcal{F}$ is a reduced Frobenius-unstable bundle, and this implies that the limit of $V_2(T)$ at $0 \in T$ is determined by the tangent vector of T, which is to say that it is only necessary to blow up $M_2(C^{(p)})$ once at $\mathcal{F}$ in order to resolve V_2.

3. Frobenius-unstable vector bundles

We conclude by sketching the degeneration argument for describing the locus of Frobenius-unstable vector bundles.

Theorem 3.1. *(Mochizuki) Let C be a general curve of genus 2, over an algebraically closed field k of characteristic $p > 2$. Then the locus of Frobenius-unstable bundles on C of rank 2 and trivial determinant on C consists of $\frac{2}{3}(p^3 - p)$ reduced points.*

There are two proofs of this result; the original argument of Mochizuki, as a special case of far more general results, and a subsequent proof due to the speaker. They are similar enough to sketch both at the same time.

The first step is to look for closely at Joshi and Xia's classification of unstable bundles $\mathcal{E}$ which could arise as $F^*\mathcal{F}$ for a semistable $\mathcal{F}$. One also checks that up to the choice of theta characteristic $\mathscr{L}$, we have that $\mathcal{E}$ is uniquely determined. We also have that their result has the converse that given an $\mathcal{E}$ as in their description, if $F^*\mathcal{F} \cong \mathcal{F}$, then $\mathcal{F}$ must be semistable (and in fact stable). Thus, the problem is reduced to finding all the bundles with trivial determinant pulling back to the given $\mathcal{E}$ under Frobenius.

We next consider Katz' notion of **p-curvature** of a connection ∇ on $\mathcal{E}$; this may be considered as a section of $\operatorname{End}(\mathcal{E}) \otimes F^*\Omega^1_{C^{(p)}}$, and Katz shows that bundles pulling back to $\mathcal{E}$ under Frobenius correspond to connections on $\mathcal{E}$ with vanishing p-curvature, up to modifying by automorphisms of $\mathcal{E}$. Restricting to connections with trivial determinant classifying bundles with trivial determinant which pull back to $\mathcal{E}$. These correspond to the **dormant torally indigenous bundles** of Mochizuki's theory.

The third step is to show that connections with vanishing p-curvature on $\mathcal{E}$ are well-behaved in families, and in particular that one can count them by degenerating to nodal curves. The key properness result is obtained by enlarging to the category of connections whose p-curvature is only required to have vanishing determinant, and showing that such connections are finite flat over the moduli space of curves. Smoothing results follow from deformation theory.

Finally, one degenerates to a rational nodal curve, pulls back to the normalization, and classifies the results by a careful examination of what one obtains in this case, relating the problem to rational functions with prescribed ramification.

It is only in the final step that the two proofs diverge, as the speaker degenerates to an irreducible curve, while Mochizuki degenerates to a totally degenerate curve – specifically, with two rational components glued to one another at three distinct points. There are two substantial advantages to considering the totally degenerate case: first, the translation to rational functions with prescribed ramification holds more functorially in this case; and second, counting rational functions with prescribed ramification is much easier with only three ramification points.

The only advantage to this particular argument to considering irreducible curves is that one may give the degeneration arguments somewhat more explicitly and naively. However, in forcing the speaker to examine the problem of counting rational functions with prescribed ramification more generally, it has led to a number of new results in other directions, including the solution to this question when all ramification indices are less than p, applications of Mochizuki's results to finiteness questions for the same problem, and most recently, to results on existence and non-existence of certain tamely branched covers of the projective line.

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