LIMIT LINEAR SERIES: PROGRESS AND GENERALIZATIONS

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Abstract. Eisenbud and Harris introduced the theory of limit linear series in the 1980's, and applied it spectacularly to prove new results on Weierstrass points and on the geometry of moduli spaces of curves. Over the last 30 years, limit linear series have remained the fundamental tool for studying the geometry of curves via degeneration arguments, with a wide range of further applications. After recalling the original ideas, we will discuss new foundational results on limit linear series, as well as generalizations to higher-rank vector bundles and to more general nodal curves than those considered by Eisenbud and Harris.

1. Linear series

Linear series arise naturally from the study of maps of varieties into projective space. Let X be a smooth proper curve over a field k. Then we define:

Definition 1.1. A **linear series** on X is a pair $(\mathscr{L}, V)$ where $\mathscr{L}$ is a line bundle on X, V is a subspace of $H^0(X, \mathscr{L})$. If $\deg \mathscr{L} = d$ and $\dim V = r + 1$, we say that $(\mathscr{L}, V)$ is a $\mathfrak{g}^r_d$.

The study of linear series is closely related to the study of maps from curves to projective spaces, along with the associated extrinsic geometry.

The following question is then quite natural:

Question 1.2. Given a curve X, and r, d, does X have a $\mathfrak{g}^r_d$? If so, what is the dimension of the space $G^r_d(X)$ of $\mathfrak{g}^r_d$s?

This question is answered for general curves by the famous Brill-Noether theorem, which was ultimately proved by Kempf, Kleiman, Laksov and Griffiths-Harris (based also work of Severi and Castelnuovo):

Theorem 1.3. *Given g, r, d, let*

$$\rho = (r + 1)(d - r) - rg.$$

If $\rho \geqslant 0$, then for all smooth, proper curves X of genus g, the space $G^r_d(X)$ is non-empty of dimension at least ρ. For a general curve X of genus g, we have $\dim G^r_d(X) = \rho$, and in particular $G^r_d(X) = \emptyset$ if $\rho < 0$.

2. Limit linear series

Theorem 1.3 was first proved via degeneration techniques, and a simplified proof of a generalized theorem was given by Eisenbud and Harris in the mid 1980's using their theory of "limit linear series." In a series of papers in *Inventiones Mathematicae*, Eisenbud and Harris also gave a range of other applications, including proving the existence of a range of types of Weierstrass points, and proving that the moduli space of curves of genus at least 24 is of general type.

Over the following 30 years, the theory of limit linear series has remained the fundamental tool for studying the geometry of curves via degeneration arguments. For instance, Farkas (*American Journal of Mathematics*, 2009) used limit linear series in his theory of Koszul divisors on moduli spaces of curves, producing the first infinite family of counterexamples to the slope conjecture, and he and Verra (*Annals of Mathematics*, 2014) used limit linear series in their analysis of the geometry of moduli spaces of spin curves. In a different direction, Cukierman and Fong (*Duke Mathematical Journal*, 1991) used limit linear series in their work on higher Weierstrass points. My own work has included the use of limit linear series:

- to prove (with Liu) connectedness of a certain family of Hurwitz spaces (*American Journal of Mathematics*, 2008);
- to prove a Riemann existence theorem in positive characteristic in some special cases (*Compositio Mathematica*, 2008);
- to enumerate rational functions with prescribed ramification in positive characteristic and over the real numbers (*Compositio Mathematica*, 2006, *Proceedings of the AMS*, 2006).

As the name suggests, the theory of limit linear series attempts to understand what happens to a linear series as a smooth curve degenerates to a singular one. Eisenbud and Harris considered the case of curves of compact type, meaning that the dual graph is a tree. They introduced a notion of limit linear series on such curves, and also defined a more tractable open subsets of "refined" limit linear series.

The main foundational results of Eisenbud and Harris were the following:

Theorem 2.1 (Eisenbud-Harris)**.** *(Specialization) Let $\pi : \mathcal{X} \to B$ be a one-parameter family of curves, smooth except at a single fiber X_0 of compact type. Then if we have a family of $\mathfrak{g}^r_d$s on the smooth fibers of π, we obtain an induced limit $\mathfrak{g}^r_d$ on X_0.*

(Smoothing) Conversely, if the space of refined limit $\mathfrak{g}^r_d$s on X_0 has the expected dimension ρ, then every refined limit $\mathfrak{g}^r_d$ arises as the limit of $\mathfrak{g}^r_d$s on the smooth members of π.

The smoothing result was proved via the following construction:

Theorem 2.2 (Eisenbud-Harris)**.** *Given $\pi : \mathcal{X} \to B$ as in Theorem 2.1, there is a moduli scheme $\mathcal{G}$ over B such that:*

(i) *the fibers of $\mathcal{G}$ recover $G^r_d(X_b)$ over points corresponding to smooth curves X_b;*
(ii) *the special fiber of $\mathcal{G}$ parametrizes refined limit linear series on X_0;*
(iii) *every component of $\mathcal{G}$ has dimension at least $\rho + 1$.*

Note: because the special fiber in the $\mathcal{G}$ of Theorem 2.2 parametrizes only refined limit linear series, in most cases $\mathcal{G}$ is not proper.

Now, limit linear series are defined in terms of $\mathfrak{g}^r_d$s on the components of X_0, so given Theorem 2.1, to study limits of linear series on X_η, it more or less suffices to study linear series on the separate components (which can be chosen to have smaller genus), at the price of considering also imposed ramification conditions. This naturally sets up induction arguments, leading to a simple proof of Theorem 1.3, and the other applications mentioned above. However, some headaches are caused by the fact that the $\mathcal{G}$ of Theorem 2.2 was not proper.

Until very recently, the following have been open:

Question 2.3. Can we describe a functor of points for a moduli space such as $\mathcal{G}$? Can we construct a proper version of $\mathcal{G}$, whose special fiber includes all limit linear series?

Question 2.4. Can we sharpen the smoothing theorem to include all limit linear series?

Question 2.5. Are spaces of limit linear series Cohen-Macaulay, and (in families) flat?

While these questions have intrinsic appeal, they also have direct applications to questions on smooth curves. For instance, it is natural to study the geometry of the space $G^r_d(X)$ on a general smooth curve X by degenerating the curve and studying the corresponding moduli space of limit linear series. However, without a flat proper moduli space construction, this is not possible. Additionally, a preprint of Khosla describes a family of effective divisors on $\mathcal{M}_g$ which he shows would give counterexamples to the slope conjecture, provided that one has a universal proper moduli space of linear series and limit linear series over a large enough open subset of $\overline{\mathcal{M}}_{g,1}$.

3. New foundations

Over the past few years, much of my work has focused on foundational development of limit linear series, including Questions 2.3, 2.4 and 2.5 above, as well as the questions of how to generalize the theory to nodal curves not of compact type, and to higher-rank vector bundles.

Much of this work has started from the discovery of a new way to think about limit linear series, equivalent to the Eisenbud-Harris theory, but better suited to abstract constructions. We now describe this new construction. Given a family $\pi : \mathcal{X} \to B$ as in Theorem 2.1, let $\mathcal{U}$ be the complement of X_0 in $\mathcal{X}$. For simplicity, suppose that X_0 consists of two smooth components Y_1 and Y_2 glued at a single node. If we have a family $(\mathscr{L}, V)$ of $\mathfrak{g}^r_d$s on $\mathcal{U}$, we consider how it extends over X_0. For any extension of $\mathscr{L}$ to all of $\mathcal{X}$, it is easy to see that there is a unique extension of V. Now, $\mathscr{L}$ can be extended over $\mathcal{X}$, but we see that this extension isn't unique: the Y_i are divisors supported on X_0, so we can twist any given extension by multiples of Y_1 or Y_2 to obtain new extensions. These extensions can be indexed by their **multidegree**, i.e. their degrees after restricting to the Y_i. These degrees will always sum to d, but twisting by Y_1 will increase the degree on Y_2 by 1 and decrease the degree by 1 on Y_1. The idea of Eisenbud and Harris was to consider the twists with multidegree $(d, 0)$ and $(0, d)$, for which no information is lost by restricting Y_1 and Y_2, respectively. In this way we obtain $\mathfrak{g}^r_d$s $(\mathscr{L}^i, V^i)$ on each Y_i.

Now the question becomes, which such pairs on X_0 can arise in this manner? For the compact-type case, Eisenbud and Harris discovered a compatibility condition in terms of ramification sequences at the nodes, and used this to define their theory of limit linear series. Our approach instead starts with the basic observation that if $\mathscr{L}_0$ is any extension of $\mathscr{L}$ to X_0, not necessarily concentrated on Y_1 or Y_2, then we have an extension V_0 of V to X_0, and furthermore the sections in V_0 can be viewed as obtained by gluing together sections in V^1 and sections in V^2. We thus make the following (slightly vague) definition:

Definition 3.1. A pair $(\mathscr{L}^1, V^1)$ and $(\mathscr{L}^2, V^2)$ is a **limit linear series** if for all line bundles $\mathscr{L}_0$ on X_0 obtained by gluing suitable twists of $\mathscr{L}^1$ and $\mathscr{L}^2$, the space of global sections of $\mathscr{L}_0$ obtained by gluing sections from V^1 and V^2 has dimension at least $r + 1$. That is, the kernel of the restriction map

$$\Gamma(X_0, \mathscr{L}_0) \to \big(\Gamma(Y_1, \mathscr{L}^1)/V^1\big) \oplus \big(\Gamma(Y_2, \mathscr{L}^2)/V^2\big)$$

has dimension at least $r+1$.

We emphasize that Definition 3.1 in no way supercedes the definition of Eisenbud and Harris, but rather complements it. Indeed, the power of the original definition is in its tractability: it beautifully sets up induction arguments, and leads to simple proofs of deep theorems such as the Brill-Noether theorem. However, the new definition is well suited to abstract arguments and to generalization. For instance, the condition of Definition 3.1 looks very much like a determinantal condition, and indeed, this leads to the following positive answer to Question 2.3:

Theorem 3.2. *Given a family $\pi : \mathcal{X} \to B$ as in Theorem 2.1, there is a scheme $\mathcal{G}$ proper over B whose fibers over $b \in B$ corresponding to smooth curves X_b are $G^r_d(X_b)$, and whose special fiber $\mathcal{G}_0$ is the space of limit linear series on X_0. Moreover, there is an explicit description of the functor of points of B.*

The construction of $\mathcal{G}$ involves the introduction of a new notion of "linked determinantal loci" generalizing classical determinantal loci of maps of vector bundles. By generalizing the classical bound on codimension of determinantal loci, I was further able to positively answer Question 2.4:

Theorem 3.3. *Every component of $\mathcal{X}$ has dimension at least $\rho+1$. In particular, the Eisenbud-Harris smoothing theorem holds also for non-refined limit linear series.*

However, one subtlety remained: while I had shown that the underlying sets of my construction and the Eisenbud-Harris construction agreed (inside the product of the $G^r_d(Y_i)$), it was not at all clear that the scheme structures agreed. In joint work with Murray, we answer Question 2.5:

Theorem 3.4. *If the space $\mathcal{G}_0$ of Theorem 3.2 has the expected dimension ρ, then it is Cohen-Macaulay, and given $\pi : \mathcal{X} \to B$ as in the theorem, $\mathcal{G}$ is likewise Cohen-Macaulay, and flat over B.*

We deduce:

Corollary 3.5. *If $\dim \mathcal{G}_0 = \rho$, the refined limit linear series are dense in $\mathcal{G}_0$, and the Eisenbud-Harris scheme structure is reduced, then the scheme structure on $\mathcal{G}_0$ is also reduced, so the two scheme structures coincide.*

This is the final technical ingredient necessary to analyze the geometry of the spaces $G^r_d(X)$ using limit linear series, and has already been used by Castorena, Lopez and Teixidor and by Chan, Lopez, Pflueger and Teixidor to study the genus and gonality of $G^r_d(X)$ in the case $\rho = 1$.

We also briefly mention work in progress with Lieblich, to construct limit linear series spaces in cases where the components of X_0 have nontrivial monodromy. This is necessary for instance in universal constructions such as those needed by Khosla, and also for certain desirable constructions over the real numbers. We believe we are able to do this, but haven't yet written out the details.

4. The higher-rank case

It is natural to expand the scope of our basic questions from line bundles with spaces of sections to vector bundles with spaces of sections. At its most basic, we ask:

Question 4.1. Given g, k, r, d, when does a general genus-g curve carry a (semi)stable vector bundle of rank r degree d with a k-dimensional space of global sections? When it does, what is the dimension of the corresponding moduli space?

There is a natural generalization of ρ to the higher-rank case, and in many examples, the moduli space is known to have components of dimension ρ, but the corresponding generalized Brill-Noether theorem fails in almost every way imaginable (components of higher dimension, nonemptiness when $\rho < 0$, and emptiness when $\rho \geqslant 0$). Despite many partial results, there is not even a general conjecture in the rank-2 case.

Question 4.1 is natural and fundamental. Despite the partial nature of progress to date, it has yielded a surprising array of applications:

- Mukai has applications to various areas, including study of Fano varieties, and curves on K3 surfaces, and classification of curves of genus 7, 8 and 9 (*Sugaku Expositions*, 2001; *Annals of Mathematics*, 2010);
- recently, Bhosle, Brambila-Paz and Newstead solved a conjecture of Butler on stability of the kernel of evaluation maps for general linear series on curves (*International Journal of Mathematics*, 2015);
- the first counterexample to the Harris-Morrison slope conjecture, given by Farkas and Popa (*Journal of Algebraic Geometry*, 2005), can be interpreted in terms of rank-2 Brill-Noether theory.

It was observed by Bertram, Feinberg and Mukai in the mid 1990's that if we consider vector bundles of rank 2 having canonical determinant, additional symmetries force the dimension of the moduli space to be at least $\rho - g + \binom{k}{2}$, while the naive expected dimension is $\rho - g$. (If $\binom{k}{2} > g$, we thus obtain examples also of components of the degree-$(2g-2)$ moduli space of dimension bigger than ρ.) Bertram, Feinberg and Mukai conjectured that a suitable version of the Brill-Noether theorem should hold in this case. The nonemptiness portion of the conjecture remains open, but the best results to date use a generalized theory of limit linear series developed by Teixidor.

In order to prove such results, one runs into the fundamental difficulty that the symmetries coming from canonical determinant are obscured in the case of limit linear series, so it is not clear how to prove the necessary smoothing theorems for this case. By generalizing a construction from my thesis, and (together with Teixidor) developing a theory of "linked symplectic Grassmannians," we were able to prove the necessary smoothing theorem, which was then used by Zhang to produce the aforementioned existence results. In fact, Teixidor and I also worked with special determinants in other degrees, and proved a range of new existence results for $d < 2g - 2$ as well, showing also that we have components of the (modified) expected dimension $\rho - g + \binom{k}{2}$.

Natural directions for further work include improving the existence results above by constructing new families of limit linear series, and also generalizing the smoothing theorem and existence results to cases of determinant $\mathscr{L}$ where $h^1(\mathscr{L}) > 1$.

5. The non-compact-type case

Another natural direction of generalization is to study (rank-1) limit linear series for nodal curves not necessarily of compact type. Reasons one might want to do this include:

- it's a fundamental problem to try to extend the moduli spaces of linear series over all of $\overline{\mathcal{M}}_g$;
- if one wants to compute the cohomology classes of subvarieties of $\overline{\mathcal{M}}_g$ having codimension greater than 1, generalizing limit linear series is a natural approach;
- there are problems (such as the maximal rank conjecture) on the behavior of linear series for general curves which have resisted existing techniques, so broadening the available tools could be helpful in attacking such problems;
- one could study linear series on curves over $\mathbb{Q}$ by looking at their (singular) reductions mod p. In many cases (such as a modular curve), these fibers may be well understood, but not of compact type.

There have been various papers thinking about different aspects of this question, but until the past few years, no systematic theory was proposed. A few years ago, Amini and Baker proposed a general definition with a somewhat tropical flavor, and showed that it satisfied specialization. However, they were unable to construct moduli spaces, prove smoothing theorems, or produce new examples of nodal curves with the expected dimension of limit linear series.

Using the alternate limit linear series construction mentioned I above, I independently developed a definition of limit linear series for arbitrary nodal curves. I have been able to prove specialization and to construct moduli spaces in full generality. For a more specialized class of curves (curves of "pseudocompact type", where the dual graph is a tree up to collapsing multiple edges), I was able to prove a smoothing theorem, and to give an equivalent definition generalizing the Eisenbud-Harris definition. Using the latter, I was also able to compute moduli space dimensions in several families, finding the expected dimension in each case. Interesting connections to tropical Brill-Noether theory emerged, shedding new light on the work of Cools, Draisma, Payne and Robeva.

This opens up a host of further problems, including generalizing the last few results to arbitrary nodal curves, and exploring the relationship to the Amini-Baker construction and to tropical Brill-Noether theory.