VECTOR BUNDLES WITH SECTIONS

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Abstract. Classical Brill-Noether theory studies, for given g, r, d, the space of line bundles of degree d with $r + 1$ global sections on a curve of genus g. After reviewing the main results in this theory, and the role of degeneration techniques in proving them, we will discuss the situation for higher-rank vector bundles, where even the most basic questions remain wide open. Focusing on the case of rank 2, we will discuss the role of spaces with fixed determinant, and how the dimension theory of Artin stacks may be useful in applying degeneration techniques.

1. Classical Brill-Noether theory

Classical Brill-Noether theory studies linear series on a general smooth projective curve C of genus g. We recall:

Definition 1.1. A **linear series** of dimension r and degree d on C is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle of degree d on C, and $V \subseteq H^0(C, \mathcal{L})$ is an $(r + 1)$-dimensional space of global sections.

$G^r_d(C)$ is the space of all linear series of dimension r and degree d on C; it is a projective scheme over $\mathrm{Pic}^d(C)$.

$G^r_d(C)$ is a compactification (in the weak sense) of the space of non-degenerate morphisms $C \to \mathbb{P}^r$ of degree d. Beyond the immediate applications to such morphisms, Eisenbud and Harris successfully applied linear series techniques to topics such as the existence of curves with particular types of Weierstrass points (which can be rephrased in terms of the geometry of the canonical imbedding), and the geometry of the moduli space of curves (in particular, that it is of general type for $g \geq 24$).

The space $G^r_d(C)$ can be constructed inside a relative Grassmannian scheme over $\mathrm{Pic}^d(C)$, which shows that the dimension of any component of $G^r_d(C)$ is at least

$$\rho = (r + 1)(d - r) - rg.$$

The foundational results of Brill-Noether theory can be summarized as follows.

Theorem 1.2. *Let C be a smooth projective curve of genus g, and fix $d, r > 0$. Then:*

1. *$G^r_d(C)$ is non-empty if $\rho \geq 0$;*
2. *$G^r_d(C)$ is connected if $\rho \geq 1$;*
3. *$G^r_d(C)$ is pure of dimension ρ if C is general;*
4. *$G^r_d(C)$ is smooth if C is general.*

Note that (3) implies in particular that $G^r_d(C)$ is empty if $\rho < 0$ and C is general. These statements were proved over the course of a decade from the early 1970's through the early 1980's. Many of them go back to Brill and Noether in 1874, and the original arguments follow ideas due to Castelnuovo and Severi. Kempf and Kleiman-Laksov settled (1) in 1971; Griffiths and Harris, building on further work of Kleiman, proved (3) in 1980, Fulton and Lazarsfeld proved (2) in 1981, and Gieseker proved (4) (reproving (3) in the process) in 1982 by proving an assertion of Petri dating back to 1925. Assertions (1) and (3) together are frequently called the "Brill-Noether theorem."

2. Higher rank

A natural generalization of Brill-Noether theory is to replace line bundles by vector bundles of arbitrary rank. Instead of compactifying spaces of morphisms to projective space, we then compactify spaces of morphism to Grassmannians. If we consider pairs $(\mathcal{E}, V)$ on C where $\mathcal{E}$ has rank r and degree d, and V is a k-dimensional space of global sections of $\mathcal{E}$, the same construction as before gives us an Artin stack $G^k_{r,d}(C)$ parametrizing such pairs, which then has the property that each component has dimension at least

$$\rho - 1 = r^2(g-1) - k(k - d + r(g-1)).$$

(the discrepancy of 1 arises because every pair has at least a 1-dimensional automorphism group). When restricting to appropriate stable loci, one can work with moduli schemes are before, but automorphism behavior in the higher-rank case makes it more appropriate to work with stacks when dealing with semistable and unstable pairs. As in the case of classical Brill-Noether theory, one obtains many effective divisors on $\mathcal{M}_g$, and there is ample reason to expect them to be interesting: the original counterexample of Farkas and Popa to the slope conjecture can be described as a rank-2 Brill-Noether locus.

It is then reasonable to ask whether the obvious generalizations of the statements of Theorem 1.2 hold; however, it turns out that none of them (except possibly connectedness) hold, even when restricting to stable or semistable loci on a general curve. Spaces can be empty with ρ positive, nonempty with ρ negative, reducible with components of different dimensions, and so forth. Teixidor, Newstead, and many others have produced systematic results for various extremal cases, and partial results in a range of other cases, but there are not yet any comprehensive conjectures on dimension or existence even in the case of bundles of rank 2. We do mention however that despite the known pathologies, the naive expected dimension $\rho - 1$ is not completely wrong: indeed, most known results show either that in certain cases the dimension is in fact as expected, or that there exist components of the expected dimension.

Some apparent difficulties are explained when one works with stacks rather than coarse moduli spaces. These include some cases where the dimension is higher than expected, but explained by the presence of extra automorphisms, and more extreme, some cases in which ρ is negative but spaces are nonetheless nonempty. Working with stacks also avoids many of the singularities that arise intrinsically on the strictly semistable locus of the moduli space of curves.

However, some apparent pathologies are real, particularly when it comes to dimension. For instance, the existence of bundles of the form $\mathcal{L} \oplus \mathcal{L}'$ for $\mathcal{L}$ of very high degree and $\mathcal{L}'$ of very low degree violates expected dimension hypotheses regardless of whether one takes automorphisms into account. Pathologies of this type are typically addressed by imposing some type of stability conditions, and we will not focus on this sort of issue.

3. Canonical determinant

Instead, we will focus our attention on vector bundles with special determinants. If one fixes a line bundle of degree d on C, one has the stack $G^k_{r,\mathcal{L}}(C)$ parametrizing pairs $(\mathcal{E}, V, \varphi)$ where $\varphi : \det \mathcal{E} \overset{\sim}{\to} \mathcal{L}$. The construction is just as before, and the naive expected dimension is $\rho - g$ (the generic automorphism group becomes finite due to the fixing of the determinant). In the mid 1990's, Bertram, Feinberg, and Mukai studied the case of rank 2 and canonical determinant ω, and showed that in this case one has a modified expected dimension

$$\rho_\omega = 3g - 3 - \binom{k+1}{2} = \rho - g + \binom{k}{2}$$

which is always strictly larger than $\rho - g$ when $k \geq 2$, and which is sometimes strictly larger than ρ itself. Every component of $G^k_{2,\omega}(C)$ has dimension at least ρ_ω. Thus, one produces components of $G^k_{2,2g-2}(C)$ of dimension larger than ρ.

It is convenient to introduce also $G^{k,s}_{r,\mathcal{L}}(C)$, the (coarse) moduli scheme of pairs with $\mathcal{E}$ stable. Based on a family of examples in low genus, Bertram and Feinberg and Mukai conjectured the following:

Conjecture 3.1. *Fix $k \geq 1$, and C a smooth projective curve of genus g. Then:*

1. *$G^{k,s}_{2,\omega}(C)$ is non-empty when $\rho_\omega \geq 0$;*
2. *$G^{k,s}_{2,\omega}(C)$ has dimension ρ_ω when C is general;*
3. *$G^{k,s}_{2,\omega}(C)$ is smooth when C is general.*

Recently, Teixidor has proved (2) and (3) using degeneration techniques. However, (1) remains open, despite partial results of Park using intersection theory on the moduli space.

The argument for the modified expected dimension revolves around symmetries in the vanishing conditions used to define $G^k_{2,2g-2}(C)$. One uses the fact that the determinant is ω to impose a vanishing condition not on an arbitrary subbundle, but on a subbundle isotropic for a symplectic form on an ambient bundle. Using modified dimensions for symplectic Grassmannians yields a different, smaller codimension.

4. Results and speculation

I have embarked on an attempt to systematically produce modified expected dimension results for fixed determinant loci in rank 2. For instance, by generalizing the argument for the canonical determinant case, I prove the following:

Theorem 4.1. *Let C be a smooth projective curve of genus g, and $\mathcal{L}$ a line bundle of degree d on C. Let $\delta \geq 0$ be the smallest degree of an effective divisor Δ such that $h^1(C, \mathcal{L}(-\Delta)) > 0$. Suppose $k \geq \delta$. Then every irreducible component of $G^{k,s}_{2,\mathcal{L}}(C)$ has dimension at least*

$$\rho^1_{\mathcal{L}} = \rho - g + \binom{k-\delta}{2}.$$

The result for canonical determinant is simply the case that $\delta = 0$ and $d = 2g-2$. More generally, an interesting special case is when $\delta = 0$:

Corollary 4.2. *Let C be a smooth projective curve of genus g, and $\mathcal{L}$ a line bundle of degree d on C, and suppose that $h^1(C, \mathcal{L}) \geq 1$. Then every irreducible component of $\mathcal{G}_C(2, \mathcal{L}, k)$ has dimension at least*

$$\rho^1_{\mathcal{L}} = \rho - g + \binom{k}{2}.$$

The argument is similar to the case of the canonical determinant, except that one has to study Grassmannians of subspaces isotropic for an alternating form which need not be non-degenerate.

Despite the general phrasing, the above results are not comprehensive, even in rank 2. To the contrary, as $h^1(C, \mathcal{L})$ get larger, one should obtain larger and larger expected dimensions. A first result in this direction is the following:

Theorem 4.3. *Let C be a smooth projective curve of genus g, and $\mathcal{L}$ a line bundle of degree d on C, and suppose that $h^1(C, \mathcal{L}) \geq 2$. Then every irreducible component of $\mathcal{G}_C(2, \mathcal{L}, k)$ has dimension at least*

$$\rho^2_{\mathcal{L}} = \rho - g + 2\binom{k}{2}.$$

This result relies on a study of "doubly symplectic Grassmannians" – moduli spaces of subspaces of a fixed vector space which are simultaneously isotropic for two different symplectic forms. One has to study the singularities of such spaces and show that the points of interest lie in the smooth locus.

There are various obvious possibilities for generalizing these theorems, but preliminary investigation suggests the situation may be more complicated as one generalizes. We will however engage in some vague speculation. A naive, maximally optimistic guess for the geography of Brill-Noether theory in rank 2 might be something like the following:

Indefensible Speculation 4.4. Let C be a smooth curve of genus g, and $\mathcal{L}$ a line bundle of degree d on $\mathcal{L}$. For any $k \geq 1$:

(1) there is an expected dimension $\rho_{\mathcal{L}}$ depending only on $g, \mathcal{L}, k$, and such that every component of $G^k_{2,\mathcal{L}}(C)$ has dimension at least $\rho_{\mathcal{L}}$;
(2) if C is generic, $G^{k,s}_{2,\mathcal{L}}(C)$ is smooth of dimension $\rho_{\mathcal{L}}$;
(3) $\rho_{\mathcal{L}}$ should depend on an explicit stratification of $\mathrm{Pic}^d(C)$, and on a generic curve nonemptiness of $G^{k,s}_{2,\mathcal{L}}(C)$ should depend only on which stratum $\mathcal{L}$ lies in.

A candidate for the stratification of $\mathrm{Pic}^d(C)$ would be given by the sequence of minimal $\delta_i \geq 0$ such that there exists an effective divisor Δ_i with $h^1(C, \mathcal{L}(-\Delta_i)) \geq i$. For the moment, there is little to reason to believe this is true quite as stated (for instance, it will likely be necessary to restrict to some sort of stable loci even to get the lower bound on dimensions), but our techniques do hold out promise of further generalization in this direction.