# NEW PERSPECTIVES ON LIMIT LINEAR SERIES

BRIAN OSSERMAN

ABSTRACT. Maps to projective spaces correspond to linear series. To better study linear series on smooth curves, Eisenbud and Harris developed their theory of limit linear series for reducible curves of "compact type." We discuss a new equivalent description of classical Eisenbud-Harris limit linear series which generalizes to the higher-rank case, agreeing with the earlier construction of Teixidor i Bigas, and also to the case of curves not of compact type, leading to a new definition.

## 1. LINEAR SERIES

Linear series arise naturally from the study of maps of varieties into projective space. Let $X$ be a smooth proper curve over a field $k$. Then we define:

**Definition 1.1.** A $\mathfrak{g}^r_d$ on $X$ is a pair $(\mathscr{L}, V)$ where $\mathscr{L}$ is a line bundle of degree $d$ on $X$, and $V$ is an $(r+1)$-dimensional subspace of $H^0(X, \mathscr{L})$.

The idea of ramification will also be important:

**Definition 1.2.** Given a linear series $(\mathscr{L}, V)$ and a point $P \in X$, the **vanishing sequence** $a_0(P) < \cdots < a_r(P)$ of $(\mathscr{L}, V)$ at $P$ is defined to be the increasing sequence of orders of vanishing at $P$ of sections of $V$. The non-decreasing **ramification sequence** $\alpha_0(P), \ldots, \alpha_r(P)$ is defined by $\alpha_i(P) = a_i(P) - i$.

**Example 1.3.** A few basic cases:
   (1) The pair $(\mathscr{L}, V)$ corresponds to a map to $\mathbb{P}^r$ when $a_0(P) = 0$ for all $P$. In this case, we say $(\mathscr{L}, V)$ is **basepoint-free**.
   (2) If $r = 1$, and $(\mathscr{L}, V)$ is basepoint-free, then $a_1(P)$ (or $\alpha_1(P) := a_1(P) - 1$, depending on convention) corresponds to the usual ramification index at $P$ of the associated map of curves.
   (3) If $r = 2$ and $(\mathscr{L}, V)$ defines a birational map onto its image in $\mathbb{P}^2$, then a point with $a_1(P) > 1$ corresponds to a cusp-type singularity in the image of $X$, while a point with $a_1(P) = 1$ but $a_2(P) > 2$ corresponds to an inflection point.
   (4) If $r = g - 1$, and $d = 2g - 2$, the only $\mathfrak{g}^r_d$ on $X$ is the canonical linear series $(\Omega^1_X, H^0(X, \Omega^1_X))$, and ramification points correspond precisely to the Weierstrass points of $X$.

The following question is then quite natural:

**Question 1.4.** Given a curve $X$, and $r, d$, does $X$ have a $\mathfrak{g}^r_d$? If so, what is the dimension of the space of $\mathfrak{g}^r_d$s? What about if we also impose ramification conditions?

This question is answered for general curves by the famous Brill-Noether theorem, and its generalization due to Eisenbud and Harris:

**Theorem 1.5.** *Given $g, r, d, n$, and $n$ nondecreasing sequences $\alpha^j := \alpha^j_0 \leqslant \ldots \leqslant \alpha^j_r$ of nonnegative integers bounded by $d - r$, let*

$$\rho = (r+1)(d-r) - rg - \sum_{j=1}^{n} \sum_{i=0}^{r} \alpha^j_i.$$

*Suppose that (at least) one of the following holds:*
   (1) *char $k = 0$ or $d < \text{char } k$;*
   (2) *$n \leqslant 2$.*

*If $X$ together with $P_1, \ldots, P_n$ is a general (smooth) proper $n$-marked curve of genus $g$, the space of $\mathfrak{g}^r_d$s on $X$ with ramification sequence at least $\alpha^j$ at every $P_j$ has dimension exactly $\rho$ if it is nonempty.*

For $n > 0$, nonemptiness doesn't always hold when $\rho \geqslant 0$, but there is an explicit criterion for nonemptiness in terms of Schubert calculus.

## 2. Limit linear series

Theorem 1.5 was proved via degeneration techniques, using the Eisenbud-Harris theory of limit linear series. We describe their theory and briefly sketch this proof. As further motivation, we mention in passing that Eisenbud and Harris gave a range of other applications, including proving the existence of a range of types of Weierstrass points, and proving that the moduli space of curves of genus at least 24 is of general type. For the latter argument, the key point is the explicit computation of the classes in the Picard group of $\mathscr{M}_g$ of some effective divisors defined in terms of Brill-Noether theory.

As the name suggests, the theory of limit linear series attempts to understand what happens to a linear series as a smooth curve degenerates to a singular one. Eisenbud and Harris considered the case of curves of compact type, of which we will focus on the simplest case: a curve $X_0$ consisting of two smooth components $Y$ and $Z$ glued at a single simple node $Q$. Their definition, which we will motivate further shortly, is as follows:

**Definition 2.1.** Let $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ be $\mathfrak{g}_d^r$s on $Y$ and $Z$ respectively. Then they constitute an (Eisenbud-Harris) **limit** $\mathfrak{g}_d^r$ on $X_0$ if they satisfy the following condition:

$$(2.1.1) \qquad\qquad a_i^Y + a_{r-i}^Z \geqslant d, \quad \forall i : 0 \leqslant i \leqslant r,$$

where $a_i^Y$, $a_i^Z$ denote the vanishing sequences at $P$ of $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ respectively. If further (2.1.1) is an equality for all $i$, we say the limit $\mathfrak{g}_d^r$ is **refined**.

The same definition works for more complicated curves of compact type, working with a collection of $\mathfrak{g}_d^r$s for each component, with the condition (2.1.1) imposed independently at each node.

Eisenbud and Harris proved the following:

**Theorem 2.2** (Eisenbud-Harris). *If we have a $\mathfrak{g}_d^r$ generically on a one-parameter family with special fiber $X_0$, then we obtain an induced limit $\mathfrak{g}_d^r$ on $X_0$. In characteristic 0, after base change and blowup (introducing new rational components on $X_0$), we may further assume that the resulting limit $\mathfrak{g}_d^r$ is refined.*

*Conversely, if the space of refined limit $\mathfrak{g}_d^r$s on $X_0$ has the expected dimension $\rho$, then every refined limit $\mathfrak{g}_d^r$ arises as the limit of $\mathfrak{g}_d^r$s in any such one-parameter family.*

This last result is proved by constructing a single scheme parametrizing $\mathfrak{g}_d^r$s on $X_\eta$ and refined limit $\mathfrak{g}_d^r$s on $X_0$. One then carries out a dimension count to prove the desired statement.

Given these results, to study limits of linear series on $X_\eta$, it more or less suffices to study linear series on $Y$ and $Z$ (which can be chosen to have smaller genus), at the price of considering also imposed ramification conditions. Thus, the context of Theorem 1.5 is the natural setting to apply the theory.

The first case of Theorem 1.5 can be proved via a relatively straightforward inductive argument using limit linear series. This proof encapsulates the power of the Eisenbud-Harris approach: limit linear series on the reducible curve can be described in terms of independent linear series on each component. However, it also points to the weakness of the approach: it is not at all obvious how to construct a moduli space in families as discussed above. Even the construction of Eisenbud and Harris, which only contains refined limit $\mathfrak{g}_d^r$s on $X_0$, is difficult and technical, and it has remained open until now to produce a proper moduli space which contains not necessarily refined limit $\mathfrak{g}_d^r$s as well. The fact that the space constructed only contains refined limit $\mathfrak{g}_d^r$s means that we need an alternate construction to handle the second case of the theorem.

## 3. A new construction

We now discuss the idea behind the Eisenbud-Harris definition: imagine a family $\mathcal{X}$ of curves over the spectrum of a DVR, with regular total space, $X_0$ as the special fiber, and smooth generic fiber $X_\eta$; then we want to think about how a $\mathfrak{g}_d^r$ on $X_\eta$ extends to $X_0$. First, any line bundle $\mathscr{L}$ of degree $d$ on $X_\eta$ can be extended over $X_0$, but not uniquely: $Y$ and $Z$ are divisors on $\mathcal{X}$, and twisting by them doesn't change anything away from $X_0$. However, it turns out that this is the only ambiguity, and it follows that if we specify a pair of degrees $(i, d-i)$ for the restrictions to $Y$ and $Z$ there is a unique extension which we denote by $\mathscr{L}_{(i,d-i)}$. If we have a linear series $(\mathscr{L}, V)$ on $X_\eta$, then given a choice of the extension of $\mathscr{L}$, the

space $V$ of global sections extends uniquely. Thus, we find that we have an infinite family of extensions $(\mathscr{L}_{(i,d-i)}, V_{(i,d-i)})$ of $(\mathscr{L}, V)$ to $X_0$.

Eisenbud and Harris consider the extensions $(\mathscr{L}_{(d,0)}, V_{(d,0)})$ and $(\mathscr{L}_{(0,d)}, V_{(0,d)})$ and observe that in this case, no information is lost by restricting to $Y$ and $Z$ respectively. Thus, from a $\mathfrak{g}_d^r$ on $X_\eta$ defined away from $X_0$, one obtains a pair $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ of $\mathfrak{g}_d^r$s on $Y$ and $Z$ respectively. One then wants to determine what additional condition best captures that they arose as limits of a $\mathfrak{g}_d^r$ on $X_\eta$, and this is precisely what their condition on vanishing sequences accomplishes.

In my thesis, I introduced an alternate construction of limit linear series, which has recently been applied in joint work with Eduardo Esteves to fibers of Abel map for reducible curves, and in joint work with Montserrat Teixidor i Bigas to Brill-Noether theory for vector bundles of rank 2 with special determinant. Inspired by this construction, I have recently realized how to construct a proper space of (limit) linear series in families, so that fibers parametrize linear series for smooth curves, and (not necessarily refined) limit linear series for reducible curves. The first observation is that if we have a $\mathfrak{g}_d^r$ on the smooth generic fiber $X_\eta$ of $\mathcal{X}$ as above, and $(\mathscr{L}^Y, V^Y), (\mathscr{L}^Z, V^Z)$ is the resulting limit $\mathfrak{g}_d^r$, if we consider the $\mathfrak{g}_d^r$ on $X_0$ given by $(\mathscr{L}_{(i,d-i)}, V_{(i,d-i)})$ for some $i$ between 0 and $d$, the sections of $V_{(i,d-i)}$ are glued together from sections of $V^Y$ vanishing to order at least $d-i$ at the node $Q$, and sections of $V^Z$ vanishing to order at least $i$ at $Q$.

The next observation is that in fact the Eisenbud-Harris conditions (2.1.1) are equivalent to the condition that for every nonnegative degree pair $(i, d-i)$, it is possible to construct an $(r+1)$-dimensional space of global sections of $\mathscr{L}_{(i,d-i)}$ by gluing together sections from $V^Y$ and $V^Z$. This is more or less expressible as a determinantal condition (there are subtleties because the pushforward need not commute with base change), and can be stated equivalently in terms of $V_{(d,0)}$ and $V_{(0,d)}$. The latter version can easily be applied in families for which some fibers are singular and some are smooth.

From this point of view, we see that one may think of Eisenbud-Harris limit linear series spaces as something of a hybrid between classical $G_d^r$ and $W_d^r$ spaces. These moduli spaces can be constructed by using a generalization of determinantal loci to the case of pushforwards of coherent sheaves. More substantively, ideally one would be able to carry out the Eisenbud-Harris dimension count (thus obtaining the resulting smoothing theorem) for the compactified space. This is not so easy, but should be doable. However, it is not necessary for most arguments: for instance, for the non-existence portion of the Brill-Noether theorem it is enough to know that the space is proper, while for the existence portion, it is enough to have the dimension lower bound on the refined locus, which certainly holds.

## 4. CURVES NOT OF COMPACT TYPE

Although the above-discussed construction was first developed in the context of higher-rank limit linear series, in this talk we focus on a different application: towards developing a theory of limit linear series for curves not of compact type.

Developing a robust theory of limit linear series for (reducible) curves not of compact type has been a long-standing open problem. Aside from the intrinsic interest in extending the moduli space of limit linear series over all of $\overline{\mathcal{M}}_g$, one hopes that such a theory would help compute cohomology classes of effective cycles on $\overline{\mathcal{M}}_g$ having codimension higher than 1. Also, the flexibility to degenerate to curves not of compact type may prove useful in dealing with remaining open questions on linear series on curves, such as the maximal rank conjecture. In any event, the first test of a robust theory would be to provide a new proof of the Brill-Noether theorem via degeneration to a reducible curve not of compact type.

I have recently started to work on developing a theory of limit linear series for curves not of compact type. For concreteness, we will focus on the case that $X_0$ is a "banana curve," with components $Y$ and $Z$ glued at nodes $Q_1$ and $Q_2$. Complications include that line bundles are no longer determined by their restrictions to components, that the effect of twisting by $Y$ or $Z$ depends on the family $\mathcal{X}$, and that the moduli space of line bundles is no longer proper. However, these issues appear surmountable, so we will instead focus on the question of what is the right generalization of the Eisenbud-Harris conditions for limit linear series. As in the case of a single node, given a family of curves $\mathcal{X}$ degenerating to $X_0$, there are infinitely many extensions to $X_0$ of line bundles on $X_\eta$. Again, an extension is uniquely determined by its degrees on $Y$ and $Z$, but not every degree can be achieved: the parity is predetermined. Again for simplicity, let's consider the case that $d$ is even and $(d, 0)$ (equivalently) $(0, d)$ can be achieved as degrees on $Y$ and $Z$. Then as in the

Eisenbud-Harris case, from a family of $\mathfrak{g}_d^r$s on smooth fibers we obtain a pair of $\mathfrak{g}_d^r$s $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ on $Y$ and $Z$ respectively. The natural question is what additional conditions this pair will satisfy, analogous to (2.1.1).

Here, the philosophy espoused above has been useful: the conditions should ensure that for any intermediate degree pair $(i, d-i)$ (with $i$ even), we should have an $(r+1)$-dimensional space of sections of the corresponding extended line bundle, obtained by gluing together sections of $V^Y$ and $V^Z$. Following through this idea, we are led to the following sequence giving orders of simultaneous vanishing at a collection of points:

**Definition 4.1.** Given points $P_1, \ldots, P_m$ on a smooth curve $X$, and $(\mathscr{L}, V)$ a $\mathfrak{g}_d^r$ on $X$, the **multivanishing sequence** $a_0 \leqslant a_1 \leqslant \ldots a_r$ of $(\mathscr{L}, V)$ at the $P_i$ is the sequence in which the number of times $mj$ appears is equal to the dimension of

$$V(-j(P_1 + \cdots + P_m))/V(-(j+1)(P_1 + \cdots + P_m)),$$

where for an effective divisor $D$, $V(-D)$ denotes $V \cap \Gamma(\mathscr{L}(-D))$.

Thus, the sequence is allowed to have up to $m$ repetitions of any given number. The $m = 1$ case is simply the usual vanishing sequence at a point. It then turns out that the correct generalization of the Eisenbud-Harris condition (2.1.1) to the case of the banana curve is (still in the case that $d$ is even and $(d, 0)$ can be achieved as an extension of $\mathscr{L}$) given by

$$(4.1.1) \qquad\qquad a_i^Y + a_{r-i}^Z \geqslant d, \quad \forall i : 0 \leqslant i \leqslant r,$$

where the $a_i^Y$ and $a_i^Z$ are now the multivanishing sequences of $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ at $P_1$ and $P_2$. However, this condition is not enough: in indices where (4.1.1) achieves equality and a particular value shows up without repetition in the vanishing sequence, we also need to impose a gluing condition on the sections with this vanishing. Imposing both (4.1.1) and the gluing condition appears to produce the correct expected dimension, although some work remains to show that the gluing conditions are actually independent for sufficiently general curves.

The same approach works for two components glued at more than two nodes, and I am currently working on trying to prove that when two rational curves are glued to one another at $g + 1$ general points, the space of limit linear series as defined above has the expected dimension.