HAUSDORFF DIMENSION OF FRACTALS GENERATED FROM STEP-WISE ADJUSTABLE ITERATED FUNCTION SYSTEMS

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ABSTRACT. Iterated function systems have been powerful tools to generate fractals. However, the requirement of using the same maps at every iteration results in a fractal that may be too self-similar for certain applications. We present a construction in which the maps are allowed to be updated at each iteration in order to generate more general fractals without changing the computational complexity. We then provide bounds for the Hausdorff dimension of the fractals created from this generalized process.

1. Introduction

Fractals have been studied extensively as a way to describe how objects grow in nature. To this end, many advances have been made in approximating branching systems by using self-similar mathematical structures, especially by Iterated Function Systems (see [3] and references therein). Probabilistic models have also been used to capture the inherent randomness from an environmental influence on the given object [1],[2]. However, many objects from nature have structures which are neither strictly self-similar nor completely random. In this paper we seek a description of fractals which combine self-similarity and randomness by adjusting an iterated function system at each generating step. At each step, the functions in this modified version are allowed to be updated. Our approach produces more general outcomes than classical iterated function systems without changing the computational complexity.

We will start with a motivating example in section 2 derived from the classical Cantor set. Then we formulate the general setup in section 3, and give more examples from higher dimensions in section 4. We give estimates of the Hausdorff dimensions of the general construction in section 5, and finally in section 6 we apply the results to the specific examples from sections 2 and 4.

For completeness, we begin with some definitions and notations as described in [3]. In what follows, let $D$ be a closed subset of $\mathbb{R}^n$.

Definition 1.1. A mapping $S: D \rightarrow D$ is called a contraction on $D$ if there exists a number $r$ with $0 < r < 1$ such that $|S(x) - S(y)| \leq r|x - y|$ for all $x, y \in D$. If equality holds, then $S$ is a similarity with contraction ratio $r$.

Definition 1.2. A finite family of contractions $\{S_1, S_2, \ldots, S_m\}$ with $m \geq 2$ is called an iterated function system (IFS). A non-empty compact subset $F \subseteq D$ is
called the *attractor* for an IFS if

\[ F = \bigcup_{i=1}^{m} S_i(F). \]

An IFS \( \{S_1, S_2, \ldots, S_m\} \) is said to be a *self-similar system* if each \( S_i \) is a similarity.

**Definition 1.3.** A self-similar system \( \{S_1, S_2, \ldots, S_m\} \) satisfies the open set condition if there is a nonempty bounded open set \( V \) such that

\[ \bigcup_{i=1}^{m} S_i(V) \subset V \]

with this union disjoint.

**Theorem 1.4** (Moran’s Theorem). Let \( \{S_1, S_2, \ldots, S_m\} \) be a self-similar system with contraction ratios \( \{r_i\}_{i=1}^{m} \) that satisfies the open set condition. Then the Hausdorff dimension of the attractor \( F \) is given by the unique nonnegative real-valued solution, \( s \), of the equation

\[ \sum_{i=1}^{m} r_i^s = 1. \]

We call the above equation Moran’s equation.

2. **Motivating Example**

Recall that the middle third Cantor set \( C \) can be defined as the attractor of the IFS \( \{S_1, S_2\} \) on the set \( D = [0, 1] \) where \( S_1(x) = x/3 \) and \( S_2(x) = (x + 2)/3 \).

We start by showing the Cantor set can be made from a sequence of updatable IFS procedures. Let \( X = \{[a, b] : a, b \in \mathbb{R}\} \) be the collection of closed intervals, and let \( Y = [0, 1]^2 \subseteq \mathbb{R} \). For each \( k = (k^{(1)}, k^{(2)}) \in Y \), we consider the following two maps from \( X \) to \( X \) defined by :

\[
\begin{align*}
  f_k^{(1)} : & \quad X \to X \\
  & \quad [a, b] \mapsto [a, k^{(1)}(b-a) + a] \\
\end{align*}
\]

\[
\begin{align*}
  f_k^{(2)} : & \quad X \to X \\
  & \quad [a, b] \mapsto [k^{(2)}(a-b) + b, b] \\
\end{align*}
\]

Let \( \mathcal{F}^{(1)} = \{f_k^{(1)} : k \in Y\} \) and \( \mathcal{F}^{(2)} = \{f_k^{(2)} : k \in Y\} \) be families of such maps and

\[ \mathcal{F}^{(1)} \cup \mathcal{F}^{(2)}. \]

Take \( E_0 = [0, 1] \in X \) and \( \{k_\ell\}_{\ell=0}^{\infty} \) be any sequence in \( Y \). Then we define the following:
STEP-WISE ADJUSTABLE IFS

\[ E^{(0)} = E_0 \]
\[ E^{(1)} = f_{k_0}^{(1)}(E_0) \cup f_{k_0}^{(2)}(E_0) =: E_1 \cup E_2 \]
\[ E^{(2)} = f_{k_1}^{(1)}(E_1) \cup f_{k_1}^{(2)}(E_1) \cup f_{k_2}^{(1)}(E_2) \cup f_{k_2}^{(2)}(E_2) \]
\[ := E_3 \cup E_4 \cup E_5 \cup E_6 \]
\[ \vdots \]
\[ E^{(n)} = \bigcup_{i=2^{n-1}-1}^{2^n-2} \left( f_{k_i}^{(1)}(E_i) \cup f_{k_i}^{(2)}(E_i) \right) := \bigcup_{i=2^{n-1}-1}^{2^n-2} (E_{2i+1} \cup E_{2i+2}) = \bigcup_{\ell=2^{n-1}-1}^{2(2^n-1)} E_\ell. \]

Note that when \( k_\ell = (\frac{1}{3}, \frac{1}{3}) \) for all \( \ell \), \( E^{(n)} \) is the \( n \)-th generation of the Cantor set \( C \) and \( \lim_{n \to \infty} E^{(n)} = \bigcap_n E^{(n)} = C. \)

To allow for more general outcomes, we can update the linear functions \( f_{k_1}^{(1)} \) and \( f_{k_2}^{(2)} \) simply by changing the value of \( k \) at each stage of the construction. Note that changing the value of \( k \) does not change the computational complexity of the process. In other words, we can generate a family of Cantor-like sets by varying our choices of linear functions \( f_{k_1}^{(1)} \) and \( f_{k_2}^{(2)} \). To illustrate the flexibility of such a formulation, we now give some examples of Cantor-like sets and their constructions by choosing suitable sequences \( \{k_\ell\}_{\ell=0}^{\infty} \).

**Figure 1.** Comparison of classical Cantor set (blue) and new Cantor-like set (red)
Example 2.1. Let $E_0 = [0, 1]$ and $F$ be as in (2.1). By taking $k_\ell = \left( \frac{\ell+1}{4\ell+6}, \frac{2\ell+5}{8\ell+16} \right)$ for $\ell \geq 0$, the process described above generates a modified Cantor-like set. In Figure 1, we have plotted the usual Cantor set (in blue) below the new Cantor-like set (in red) to illustrate the comparison.

We can see that the new Cantor-like set has the same basic shape as the Cantor set, but is no longer strictly self-similar.

Example 2.2. Again let $E_0 = [0, 1]$ and $F$ be as in (2.1). Now for each $\ell \geq 0$, we take $k_\ell = (q_\ell, \frac{1}{2} - q_\ell)$ where $q_\ell$ is a random number between $\frac{1}{8}$ and $\frac{3}{8}$. Following the same process described above, we construct a random Cantor-like set, as shown in Figure 2. In this example, the total length of $E^{(n)}$ is chosen to be $(\frac{1}{2})^n$, while the scaling factors of the left subintervals at each stage are randomly chosen.

\[
\begin{array}{cccccccc}
  \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
  \begin{array}{cccc}
  \ldots & \ldots & \ldots & \ldots \\
  - & - & - & - \\
  - & - & - & - \\
  \end{array} & \\
  \begin{array}{cccc}
  \ldots & \ldots & \ldots & \ldots \\
  - & - & - & - \\
  \end{array} & \\
  \begin{array}{cccc}
  \ldots & \ldots & \ldots & \ldots \\
  - & - & - & - \\
  \end{array} & \\
  \begin{array}{cccc}
  \ldots & \ldots & \ldots & \ldots \\
  - & - & - & - \\
  \end{array} & \\
  \end{array}
\]

Figure 2. A randomly generated Cantor-like set

Example 2.3. Let $E_0 = [0, 1]$ and $F$ be as in (2.1). In this example, we create a sequence $\{k_\ell\}_{\ell=0}^\infty$ that results in a limit set with a given measure, e.g., $1/3$. Of course, the classic example of such a limiting set is the fat Cantor set. For a different approach, let $\sum_{n=0}^\infty a_n$ be any convergent series of positive terms with limit $L$. We consider a sequence $\{k_\ell\}_{\ell=0}^\infty$ defined in the following way.

Let $n \geq 1$ be the generation of the construction and for each $\ell$ with $2^{n-1} - 1 \leq \ell \leq 2^n - 2$, define $k_\ell = (b_n, b_n)$ where

\[
b_1 := \frac{3}{2}L - a_0 \quad \text{and} \quad b_n := \frac{3}{2}L - \frac{\sum_{i=0}^{n-1} a_i}{2 \left( \frac{3}{2}L - \sum_{i=0}^{n-2} a_i \right)} \quad \text{for} \quad n \geq 2.
\]
With this sequence $k_1, k_2, \ldots, k_n$, one can find that the length of each interval in the $n^{th}$ generation is

$$b_1 b_2 \cdots b_n = \frac{3L - \sum_{i=0}^{n-1} a_i}{2^n \cdot \frac{3}{2} L}.$$  

Thus, the total length of the $n^{th}$ generation is

$$\frac{3L - \sum_{i=0}^{n-1} a_i}{\frac{3}{2} L} = 1 - \frac{2}{3L} \sum_{i=0}^{n-1} a_i$$

which converges to $1/3$ as desired.

As an example, we take the convergent series $\sum_{n=0}^{\infty} \frac{1}{n!} = e$ and use it to create a fractal with measure $1/3$. The first few generations are shown in Figure 3.

![Figure 3. Fractal of measure $\frac{1}{3}$ created by using $\sum_{n=0}^{\infty} \frac{1}{n!} = e$](image)

3. General Setup

The examples described in section 2 motivate us to consider the following general setup.

Let $D$ be a closed subset of a metric space. Let $X$ be a collection of non-empty compact sets of $D$ (e.g., intervals, polygons, surfaces), $Y$ be a closed subset of $\mathbb{R}^d$ and $m \geq 2$ be a fixed natural number. For $i = 1, \ldots, m$, let

$$\mathcal{F}^{(i)} = \{ f_k^{(i)} : X \rightarrow X \mid k \in Y \}$$
be a collection of (e.g. linear, affine, contraction) transformations of elements in $X$ and
\[ \mathcal{F} = \bigcup_{i=1}^{m} \mathcal{F}^{(i)}. \]

Let $\{ k_\ell \}_{\ell=0}^\infty$ be a sequence in $Y$, $E_0 \in X$ be a starting set. For each $\ell = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, m$, we iteratively denote the set
\[ E_{m\ell+j} = f_{k_\ell}^{(j)}(E_\ell) \in X, \]
where $f_{k_\ell}^{(j)} \in \mathcal{F}^{(j)}$. For each $n \in \mathbb{N}$, set
\[ E^{(n)} = \bigcup_{\ell = G_m(n-1)+1}^{G_m(n)} E_\ell \subseteq D, \]
denoting the union of all subsets constructed in the $n^{th}$ generation, where $G_m(0) = 0$ and for $n \geq 1$, \begin{equation}
G_m(n) = m + m^2 + \cdots + m^n = \frac{m^{n+1} - m}{m - 1}. \tag{3.2} \end{equation}

As a result, given the families of transformations $\mathcal{F}^{(i)}$ on $X$, the sequence $\{ k_\ell \}_{\ell=0}^\infty$ in $Y$, and the starting set $E_0$ in $X$, we generate a sequence of subsets $\{ E^{(n)} \}_{n=0}^\infty$ in $D$.

If each $f \in \mathcal{F}$ satisfies the condition that
\[ f(K) \subseteq K, \quad \text{for every } K \in X, \]
then the sequence of subsets $\{ E^{(n)} \}_{n=1}^\infty$ constructed above is a nested sequence of compact sets in $D$. As a result, this sequence has a limit \begin{equation}
F = \bigcap_{n=1}^{\infty} E^{(n)}. \tag{3.3} \end{equation}

We call $F$ the limit set (or limit fractal) generated by the triple $(\mathcal{F}, \{ k_\ell \}_{\ell=0}^\infty, E_0)$. Typically this set $F$ is a fractal.

**Proposition 3.1.** The attractor of any IFS can be viewed as a limit set $F$ generated by some triple $(\mathcal{F}, \{ k_\ell \}_{\ell=0}^\infty, E_0)$.

**Proof.** Let $\{ S_1, S_2, \ldots, S_m \}$ be an IFS on $D$ and $X$ be the set of all non-empty compact subsets of $D$. Then each $S_i$ naturally determines a map on $X$ given by \begin{equation}
f^{(i)}(E) = \{ S_i(x) | x \in E \subseteq D \} \in X \tag{3.4} \end{equation}
for each $E \in X$. For each $i = 1, 2, \ldots, m$ and $k \in Y$, set $f_{k_\ell}^{(i)} = f^{(i)}$, and denote $\mathcal{F}^{(i)} = \{ f_{k_\ell}^{(i)} | k \in Y \}$. Then, for any sequence $\{ k_\ell \}_{\ell=0}^\infty$ in $Y$ and any non-empty compact subset $E_0 \subseteq D$, the limit set $F$ generated by the triple $(\mathcal{F}, \{ k_\ell \}_{\ell=0}^\infty, E_0)$ coincides with the attractor of the given IFS $\{ S_1, S_2, \ldots, S_m \}$.

Conversely, we also have the following proposition.

**Proposition 3.2.** Let $F$ be the limit set generated by the triple $(\mathcal{F}, \{ k_\ell \}_{\ell=0}^\infty, E_0)$. Suppose the following two conditions hold:

- Each $f_{k_\ell}^{(j)} = f^{(j)}$ for a fixed function $f^{(j)} \in \mathcal{F}^{(j)}$ and all $\ell$,
for each \( j = 1, 2, \ldots, m \), there exists a contraction \( S_j \) on \( D \) such that equation (3.4) holds for each \( E \in X \).

Then \( F \) is the attractor of the IFS \( \{ S_1, S_2, \ldots, S_m \} \).

In the above sense, our approach is a generalization of the standard IFS construction.

An important observation is that replacing \( \{ \ell k \}_{\ell=0}^{\infty} \) by another sequence will not change the computational complexity of the construction. Thus, generating the limit set \( F \) will have the same computational complexity as generating the attractor of a comparable IFS.

4. Examples

In this section we give a few constructions inspired from classical fractals, but in the language of this step-wise adjustable IFS procedure.

4.1. Cantor Set. In section 2 we showed three examples of a Cantor-like set made from this formulation.

4.2. Sierpinski Triangle. The Sierpinski triangle is another well known fractal. It can be made as the attractor of an IFS with three maps. If we consider a filled equilateral triangle \( \triangle ABC \) with unit side length embedded in \( \mathbb{R}^2 \), we can create the fractal by connecting the midpoints of each side and removing the middle triangle. We continue this process to each remaining triangle ad infinitum, and end up with the desired fractal.

Following the general setup, we take

(4.1) \[ X = \{(A, B, C)|A, B, C \in \mathbb{R}^2\} \]

representing the collection of all triangles \( \triangle ABC \) in \( \mathbb{R}^2 \). Let \( Y = [0, 1]^6 \subseteq \mathbb{R}^6 \) the six dimensional unit cube and \( m = 3 \). For each \( k = (k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}, k^{(5)}, k^{(6)}) \in Y \) and \( i = 1, 2, 3 \) we can define affine transformations \( f_k^{(i)} : X \to X \) as

\[
\begin{align*}
    f_k^{(1)}(A, B, C) &= (A, A + k^{(1)}(B - A), A + k^{(2)}(C - A)) \\
    f_k^{(2)}(A, B, C) &= (B + k^{(4)}(A - B), B, B + k^{(3)}(C - B)) \\
    f_k^{(3)}(A, B, C) &= (C + k^{(5)}(A - C), C + k^{(6)}(B - C), C)
\end{align*}
\]

for every \( (A, B, C) \in X \). Now set

(4.2) \[ \mathcal{F}^{(i)} = \left\{ f_k^{(i)}|k \in Y \right\} \quad \text{and} \quad \mathcal{F} = \bigcup_{i=1}^{3} \mathcal{F}^{(i)}. \]

Using the general setup, for any starting triangle \( E_0 = (A, B, C) \in X \), we can generate a sequence of sets that follows a similar construction to the Sierpinski triangle.

To create the normal Sierpinski triangle, we choose

(4.3) \[ E_0 = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 0 & 0 & \sqrt{3}/2 \end{bmatrix}, \]

the equilateral triangle of unit side length, and \( k_{\ell} \in Y \) to be the constant sequence \( k_{\ell} = k = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2) \) so that each generation maps a triangle to three triangles of half the side length with the desired translation.
Figure 4. First generation of disconnected and connected triangles

Of course, to prevent overlaps we can require that $k^{(1)} + k^{(4)} \leq 1, k^{(2)} + k^{(5)} \leq 1, k^{(3)} + k^{(6)} \leq 1$. When each of the inequalities are strict, the images of $f_k^{(i)}$ are three disconnected triangles, as illustrated in Figure (4a). When all equalities hold, the images are connected, as illustrated in Figure (4b).

In the case of the connected sets, the values of $k = (k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}, k^{(5)}, k^{(6)})$ are determined by $k^{(4)} = 1 - k^{(1)}$, $k^{(5)} = 1 - k^{(2)}$, $k^{(6)} = 1 - k^{(3)}$. In this sense, we may also view $k = (k^{(1)}, k^{(2)}, k^{(3)})$ as a vector in $[0, 1]^3 \subseteq \mathbb{R}^3$.

Now we give a few examples of this general form to illustrate the process.

**Example 4.1.** Let $X, \mathcal{F}$ and $E_0$ be as in (4.1, 4.2, 4.3). Let $k_\ell = (k^{(1)}_\ell, k^{(2)}_\ell, k^{(3)}_\ell)$ be a sequence in $[0, 1]^3$ with each $k^{(i)}_\ell$ a random number between given numbers $\lambda$ and $\Lambda$ for each $i = 1, 2, 3$. Then the output of the general setup results in images like Figure 5. Here, in Figure 5a, $\lambda = \frac{1}{4}$ and $\Lambda = \frac{3}{4}$; while in Figure 5b, $\lambda = 0.45$ and $\Lambda = 0.55$. Note that the sets are no longer self-similar.

Figure 5. Generation 6 of Random Sierpinski triangle

(A) Each $k^{(i)}_\ell$ is random in $[\frac{1}{4}, \frac{3}{4}]$. (B) Each $k^{(i)}_\ell$ is random in $[0.45, 0.55]$. 
Example 4.2. As in Example 4.1, but with $E_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the output of the general setup results in an image like Figure 6 for $\lambda = \frac{1}{4}$ and $\Lambda = \frac{3}{4}$.

Example 4.3. Let $X$, $\mathcal{F}$ and $E_0$ be as in (4.1), (4.2) and (4.3) respectively. For each $\ell$, let $k_\ell = (k^{(1)}_\ell, k^{(2)}_\ell, \cdots, k^{(6)}_\ell)$ where

\[
\begin{align*}
k^{(1)}_\ell &= \frac{1}{2} + \frac{a_\ell}{\sqrt{\ell + 1}}, & k^{(2)}_\ell &= 1 - k^{(1)}_\ell, \\
k^{(3)}_\ell &= \frac{1}{2} + \frac{b_\ell}{\sqrt{\ell + 1}}, & k^{(4)}_\ell &= 1 - k^{(3)}_\ell, \\
k^{(5)}_\ell &= \frac{1}{2} + \frac{c_\ell}{\ell + 1}, & k^{(6)}_\ell &= 1 - k^{(5)}_\ell.
\end{align*}
\]

for random numbers $a_\ell, b_\ell, c_\ell \in \left[-\frac{1}{3}, \frac{1}{3}\right]$. Then the seventh iteration of the general setup results in an image like Figure 7. Later, we will show that the Hausdorff dimension of the set $F$ is $\frac{\log(3)}{\log(2)}$. 

Figure 7. Generation 7 of a Sierpinski-type triangle with controlled dimension
4.3. Menger Sponge. Following the general setup again, we take

\[(4.4)\]

\[X = \{(O, A, B, C)|O, A, B, C \in \mathbb{R}^3\}\]

representing the collection of all rectangular prisms \((OABC)\) in \(\mathbb{R}^3\). Let

\[Y = \left\{ \left( k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}, k^{(5)}, k^{(6)} \right) \in [0, 1]^6 : k^{(1)} \leq k^{(2)}, k^{(3)} \leq k^{(4)}, k^{(5)} \leq k^{(6)} \right\}, \]

and \(m = 20\). For each \(k \in Y\) and \(i = 1, 2, \ldots, 20\), we can define affine transformations \(f^{(i)}_k : X \to X\) as follows.

For any \(k = (k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}, k^{(5)}, k^{(6)}) \in Y\), define

\[T = \begin{bmatrix} 0 & k^{(1)} & k^{(2)} & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & k^{(3)} & k^{(4)} & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & k^{(5)} & k^{(6)} & 1 \end{bmatrix}.\]

Let

\[I = \{(a, b, c)|1 \leq a, b, c \leq 3\text{ with } a, b, c \in \mathbb{Z}, \text{ and no two of } a, b, c \text{ equal to } 2\}.\]

For each \((a, b, c) \in I\) and \(k \in Y\), define

\[M_k(a, b, c) = \begin{bmatrix} 1 - (T(a) + R(b) + S(c)) & T(a) & R(b) & S(c) \\ 1 - (T(a + 1) + R(b) + S(c)) & T(a + 1) & R(b) & S(c) \\ 1 - (T(a) + R(b + 1) + S(c)) & T(a) & R(b + 1) & S(c) \\ 1 - (T(a) + R(b) + S(c + 1)) & T(a) & R(b) & S(c + 1) \end{bmatrix}.\]

Note that the set \(I\) contains 20 elements, so we can express it as

\[I = \{(a_i, b_i, c_i)|1 \leq i \leq 20\}.\]

For each \(k \in Y\) and \(1 \leq i \leq 20\), we consider the affine transformation \(f^{(i)}_k : X \to X\) given by

\[(4.5)\]

\[f^{(i)}_k(O, A, B, C) = M_k(a_i, b_i, c_i) \begin{bmatrix} O \\ A \\ B \\ C \end{bmatrix}\]

for every \((O, A, B, C) \in X\). Then set

\[(4.6)\]

\[F^{(i)} = \left\{ f^{(i)}_k | k \in Y \right\} \text{ and } F = \bigcup_{i=1}^{20} F^{(i)}.\]

Using this, for any starting rectangular prism \(E_0 = (O, A, B, C) \in X\), we can generate a sequence of sets that follows a similar construction to the Menger Sponge.

**Example 4.4.** Let \(X\) and \(F\) be as in (4.4) and (4.6). Let

\[(4.7)\]

\[E_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\]

be the cube of unit side length and choose \(k_\ell \in Y\) to be the constant sequence \(k_\ell = k = (1/3, 2/3, 1/3, 2/3, 1/3, 2/3)\). Then the limiting set generated by the triple \((F, \{k_\ell\}_{\ell=0}^\infty, E_0)\) is the classical Menger sponge.
Example 4.5. Let $X$ and $F$ be as in (4.4), (4.6). Let

$$E_0 = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$ 

Let $(k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}, k^{(5)}, k^{(6)}) \in Y$ where each $k^{(i)}$ is a random number in $[0, 1]$, but still satisfying the condition $k^{(1)} \leq k^{(2)}, k^{(3)} \leq k^{(4)}, k^{(5)} \leq k^{(6)}$. Then the first generation $E^{(1)}$ of the general setup results in a set like Figure 8.

![Figure 8](image)

**Figure 8.** First generation of a randomly generated Menger sponge

Example 4.6. Let $X$, $F$ and $E_0$ be as in (4.4), (4.6) and (4.7) respectively. Let $k_\ell = (k^{(1)}_\ell, k^{(2)}_\ell, k^{(3)}_\ell, k^{(4)}_\ell, k^{(5)}_\ell, k^{(6)}_\ell) \in Y$ with each $k^{(2j-1)}_\ell$ a random number between given parameters $\lambda$ and $\Lambda$ and $k^{(2j)}_\ell = 1 - k^{(2j-1)}_\ell$ for each $j = 1, 2, 3$. Then the third iteration of the general setup results in images like Figure 9. Here, in Figure 9a the parameters $\lambda = 0$ and $\Lambda = \frac{1}{2}$, while in Figure 9b the parameters $\lambda = 0.32$ and $\Lambda = 0.35$.

Example 4.7. Let $X$, $F$ and $E_0$ be as in (4.4), (4.6) and (4.7) respectively. For each $\ell \geq 0$, let $k_\ell = (k^{(1)}_\ell, k^{(2)}_\ell, \ldots, k^{(6)}_\ell)$ where

$$k^{(1)}_\ell = \frac{1}{3} + \frac{(-1)^\ell}{12(\ell + 1)}, \quad k^{(2)}_\ell = 1 - k^{(1)}_\ell,$$

$$k^{(3)}_\ell = \frac{1}{3} - \frac{(-1)^\ell}{6(\ell + 1)}, \quad k^{(4)}_\ell = 1 - k^{(3)}_\ell,$$

$$k^{(5)}_\ell = \frac{1}{3} + \frac{(-1)^\ell}{18(\ell + 1)}, \quad k^{(6)}_\ell = 1 - k^{(5)}_\ell.$$ 

Then the third iteration of the general setup results in an image like Figure 10.

5. Hausdorff Dimension of the Limit Sets

In this section we investigate the Hausdorff dimension $\dim_H(F)$ of the fractals $F$ obtained from the process described in Section 3. To start, we determine an upper bound for the dimension of the limit set $F$ by considering the step-wise relative ratios between the diameters of sets.
Figure 9. Generation 3 of random Menger sponge

Figure 10. Generation 3 of random Menger sponge with controlled dimension
Proposition 5.1. Let $F$ be the limit set generated by the triple $(\mathcal{F}, \{k_\ell\}_{\ell=0}^\infty, E_0)$. If for some $s > 0$, there exists a number $0 < c < 1$ such that

\begin{equation}
\sum_{j=1}^m \operatorname{diam}(E_{m\ell+j})^s \leq c \cdot \operatorname{diam}(E_\ell)^s
\end{equation}

for all $\ell$, then $\dim_H(F) \leq s$.

Proof. We prove by using mathematical induction that for $n = 1, 2, \cdots$,

\begin{equation}
\mathcal{G}_{m}(n) \sum_{\ell=\mathcal{G}_{m}(n)+1}^{\mathcal{G}_{m}(n+1)} \operatorname{diam}(E_\ell)^s \leq c^n \operatorname{diam}(E_0)^s,
\end{equation}

where $\mathcal{G}_{m}(n)$ is given by (3.2). When $n = 1$, (5.2) follows from (5.1) with $\ell = 0$. Now assume (5.2) is true for some $n \geq 1$. Then,

\begin{align*}
\mathcal{G}_{m}(n+1) \sum_{\ell=\mathcal{G}_{m}(n+1)+1}^{\mathcal{G}_{m}(n+2)} \operatorname{diam}(E_\ell)^s & \leq c \sum_{\ell=\mathcal{G}_{m}(n)+1}^{\mathcal{G}_{m}(n+1)} \left( \sum_{j=1}^m \operatorname{diam}(E_{m\ell+j})^s \right) \\
& \leq c^n \operatorname{diam}(E_0)^s
\end{align*}

as desired. By induction principle, (5.2) holds for all $n = 1, 2, \cdots$.

For each $n$, set

$$
\delta_n = \max \{ \operatorname{diam}(E_\ell) : \mathcal{G}_{m}(n-1) + 1 \leq \ell \leq \mathcal{G}_{m}(n) \}.
$$

Then, by (5.2), $\delta_n \leq c^{n+1/s} \operatorname{diam}(E_0)$, which implies that $\lim_{n \to \infty} \delta_n = 0$. Moreover,

$$
\mathcal{H}^s_{\delta_n}(F) \leq \mathcal{H}^s_{\delta_n}(E^{(n)}) \leq \sum_{\ell=\mathcal{G}_{m}(n)+1}^{\mathcal{G}_{m}(n+1)} \alpha(s) \left( \frac{\operatorname{diam}(E_\ell)}{2} \right)^s \leq c^n \alpha(s) \left( \frac{\operatorname{diam}(E_0)}{2} \right)^s.
$$

Thus, $\mathcal{H}^s(F) = \lim_{n \to \infty} \mathcal{H}^s_{\delta_n}(F) = 0$, and hence $\dim_H(F) \leq s$. \qed

Conversely, a lower bound on the dimension of the limit set $F$ can also be obtained as follows.

Proposition 5.2. Suppose the ambient space is Euclidean space $\mathbb{R}^N$, and let $F$ be the limit set generated by the triple $(\mathcal{F}, \{k_\ell\}_{\ell=0}^\infty, E_0)$. If for some $s > 0$,

\begin{equation}
\sum_{j=1}^m \operatorname{diam}(E_{m\ell+j})^s \geq \operatorname{diam}(E_\ell)^s
\end{equation}

for all $\ell$, then $\dim_H(F) \geq s$.

Proof. We first show that under condition (5.3), there exists a probability measure $\mu$ concentrated on $F$ such that for each Borel subset $C$,

\begin{equation}
\mu(C) \leq \left( \frac{\operatorname{diam}(C)}{\operatorname{diam}(E_0)} \right)^s.
\end{equation}
Indeed, let $\mathcal{E}$ be the collection of all subsets $E_\ell$ for all $\ell$. Let $\mu(E_0) = 1$, and for each $\ell \geq 0$ and $i = 1, \cdots, m$, we inductively set

$$\mu(E_{m\ell+i}) = \frac{\text{diam}(E_{m\ell+i})^s}{\sum_{j=1}^{m}\text{diam}(E_{m\ell+j})^s} \mu(E_\ell).$$

Then by Proposition 1.7 in [3], $\mu$ can be uniquely extended to a probability measure on $\mathbb{R}^N$. The support of $\mu$ is concentrated on $F = \bigcap_{i=1}^{\infty} \bar{E}^{(i)}$. For any Borel set $A$, the value

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu(U_i) : A \cap F \subset \bigcup_{i=1}^{\infty} U_i \text{ and } U_i \in \mathcal{E} \right\}.$$ 

To prove (5.4), it is sufficient to prove this for each $E_\ell$. We proceed by using induction on $n$ when $\ell \in [\mathcal{G}_m(n-1)+1, \mathcal{G}_m(n)]$. It is clear for $n = 0$. Now assume that (5.4) holds for each $E_\ell$ with $\ell \in [\mathcal{G}_m(n-1)+1, \mathcal{G}_m(n)]$ for some $n \in \mathbb{N}$. Then by induction assumption and (5.3), for each $i = 1, \cdots, m$,

$$\mu(E_{m\ell+i}) = \frac{\text{diam}(E_{m\ell+i})^s}{\sum_{j=1}^{m}\text{diam}(E_{m\ell+j})^s} \mu(E_\ell)$$

$$\leq \frac{\text{diam}(E_{m\ell+i})^s}{\sum_{j=1}^{m}\text{diam}(E_{m\ell+j})^s} \left( \frac{\text{diam}(E_\ell)}{\text{diam}(E_0)} \right)^s$$

$$\leq \left( \frac{\text{diam}(E_{m\ell+i})}{\text{diam}(E_0)} \right)^s.$$ 

This proves inequality (5.4).

Now, for any $\delta > 0$, let $\{C_i\}$ be any collection of closed balls with $\text{diam}(C_i) \leq \delta$ and $F \subset \bigcup_i C_i$. Then, by (5.4),

$$\sum_i \alpha(s) \left( \frac{\text{diam}(C_i)}{2} \right)^s \geq \alpha(s) \left( \frac{\text{diam}(E_0)}{2} \right)^s \sum_i \mu(C_i) \geq c \mu \left( \bigcup_i C_i \right) \geq c \mu(F) \geq c,$$

where $c = \alpha(s) \left( \frac{\text{diam}(E_0)}{2} \right)^s$. Thus, $\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F) \geq c > 0$, and hence $\dim H(F) \geq s$.

As indicated in the previous two propositions, the relative ratio between the diameters of the sets plays an important role in the calculation of the dimension of the limit set. Therefore, we introduce the following definition.

**Definition 5.3.** For any $f \in \mathcal{F}$, define

$$U(f) = \sup_{K \in \mathcal{X}} \frac{\text{diam}(f(K))}{\text{diam}(K)}, \text{ and } L(f) = \inf_{K \in \mathcal{X}} \frac{\text{diam}(f(K))}{\text{diam}(K)}.$$ 

Note that, for each $K \in \mathcal{X}$,

$$L(f) \cdot \text{diam}(K) \leq \text{diam}(f(K)) \leq U(f) \cdot \text{diam}(K).$$

For any $k \in \mathcal{Y}$, define

$$U_k = \left( U(f^{(1)}_k), \cdots, U(f^{(m)}_k) \right) \in \mathbb{R}^m,$$

and

$$L_k = \left( L(f^{(1)}_k), \cdots, L(f^{(m)}_k) \right) \in \mathbb{R}^m,$$

where $f^{(i)}_k \in \mathcal{F}^{(i)}$. 

Also, for each $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $s > 0$, denote

$$||x||_s = \left( \sum_{i=1}^{m} |x_i|^s \right)^{\frac{1}{s}}.$$

Using these notations, the previous two propositions motivate our main theorem.

**Theorem 5.4.** Let $F$ be the limit set generated by the triple $(\mathcal{F}, \{k_\ell\}_{\ell=0}^\infty, E_0)$, and $s > 0$.

(a) If

$$\inf_{\ell} \{||L_{k_\ell}||_s\} \geq 1,$$

then $\dim_H(F) \geq s$.

(b) If

$$\sup_{\ell} \{||U_{k_\ell}||_s\} < 1,$$

then $\dim_H(F) \leq s$.

**Proof.** (a) For all $\ell$,

$$\sum_{j=1}^{m} \operatorname{diam}(E_{m\ell+j})^s = \sum_{j=1}^{m} \operatorname{diam}\left( f^{(j)}(E_\ell) \right)^s \geq \sum_{j=1}^{m} \left( \lambda(f^{(j)}_{k_\ell}) \right)^s \operatorname{diam}(E_\ell)^s \geq \operatorname{diam}(E_\ell)^s.$$

Thus, by Proposition 5.2, $\dim_H(F) \geq s$.

(b) Similarly, for all $\ell$,

$$\sum_{j=1}^{m} \operatorname{diam}(E_{m\ell+j})^s \leq \sum_{j=1}^{m} \left( U(f^{(j)}_{k_\ell}) \right)^s \operatorname{diam}(E_\ell)^s \leq c \cdot \operatorname{diam}(E_\ell)^s,$$

where

$$c := \sup_{\ell} \{\langle U_{k_\ell} \rangle_s \}^s < 1.$$

By Proposition 5.1, $\dim_H(F) \leq s$. \hfill \Box

When both $\{||L_{k_\ell}||_s\}_{\ell=1}^\infty$ and $\{||U_{k_\ell}||_s\}_{\ell=1}^\infty$ are convergent sequences, the following corollary enables us to quickly estimate the dimension of $F$.

**Corollary 5.5.** Let $F$ be the limit set generated by the triple $(\mathcal{F}, \{k_\ell\}_{\ell=0}^\infty, E_0)$. Then,

$$s_* \leq \dim_H(F) \leq s^*,$$

where

$$s_* = \sup \left\{ s : \lim_{\ell \to \infty} \inf \{||L_{k_\ell}||_s\} > 1 \right\}, \text{ and } s^* = \inf \left\{ s : \lim_{\ell \to \infty} \sup \{||U_{k_\ell}||_s\} < 1 \right\}.$$

**Proof.** For any $0 < s < s_*$, by the definition of $s_*$,

$$\lim_{\ell \to \infty} \inf \{||L_{k_\ell}||_s\} > 1.$$

Thus, when $\ell_* \in \mathbb{N}$ is large enough,

$$\inf_{\ell \geq \ell_*} \{||L_{k_\ell}||_s\} \geq 1,$$

i.e. $\inf_{\ell \geq 0} \{||L_{k_{\ell_*+\ell}}||_s\} \geq 1$. 

STEP-WISE ADJUSTABLE IFS
Since $F \cap E_{\ell}$ is the set generated by the triple $(F, \{k_{\ell_i}+\ell\}_{i=0}^{\infty}, E_{\ell})$, by Theorem 5.4, it follows that $\dim_H(F \cap E_{\ell}) \geq s$ for any $\ell$, large enough. This implies that $\dim_H(F) \geq s$ for any $s < s_*$ and hence $\dim_H(F) \geq s_*$. 

Similarly, we also have $\dim_H(F) \leq s^*$. □

In the following corollaries, we will see that bounds of the dimension of $F$ can also be obtained from corresponding bounds on $L_1 \leq T$. LAZARUS, Q. XIA

Corollary 5.6. Let $t = (t_1, \ldots, t_m)$ and $r = (r_1, \ldots, r_m)$ be two points in $(0, 1)^m \subset \mathbb{R}^m$. Let $s_*$ and $s^*$ be the solutions to $||t||_s = 1$, and $||r||_{s^*} = 1$ respectively, i.e.

$t_1^s + t_2^s + \cdots + t_m^s = 1$, and $r_1^{s^*} + r_2^{s^*} + \cdots + r_m^{s^*} = 1$.

(a) If $L_{k_\ell} \geq t$ for all $\ell$, then $\dim_H(F) \geq s_*$. 
(b) If $U_{k_\ell} \leq r$ for all $\ell$, then $\dim_H(F) \leq s^*$. 
(c) If $L_{k_\ell} = r = U_{k_\ell}$ for all $\ell$, then $\dim_H(F) = s^*$.

Proof. (a) Let $0 < s < s_*$. Then,

$$\inf_{\ell} \{|L_{k_\ell}|_s\} \geq ||t||_s \geq ||t||_{s_*} = 1.$$ 

Thus, by Theorem 5.4, $\dim_H(F) \geq s$ for any $s < s_*$, and hence $\dim_H(F) \geq s_*$. 

(b) Similarly, let $0 < s^* < s$. Then,

$$\sup_{\ell} \{|U_{k_\ell}|_s\} \leq ||r||_s \leq ||r||_{s^*} = 1.$$ 

Thus, by Theorem 5.4, $\dim_H(F) \leq s$ for any $s > s^*$, and hence $\dim_H(F) \leq s^*$.

(c) follows from (a) and (b). □

A special case of Corollary 5.6 gives the following explicit formulas for the bounds on the dimension of $F$.

Corollary 5.7. Let $F$ be the limit set generated by the triple $(F, \{k_{\ell_i}\}_{i=0}^{\infty}, E_0)$. Let $t = (t, \ldots, t)$ and $r = (r, \ldots, r)$, for some $0 < t, r < 1$.

(a) If $L_{k_\ell} \geq t$ for all $\ell$, then $\dim_H(F) \geq \frac{\log m}{-\log t}$.
(b) If $U_{k_\ell} \leq r$ for all $\ell$, then $\dim_H(F) \leq \frac{\log m}{-\log r}$.
(c) If $L_{k_\ell} = r = U_{k_\ell}$ for all $\ell$, then $\dim_H(F) = \frac{\log m}{-\log r}$.

Other types of bounds on $L_{k_\ell}$ and $U_{k_\ell}$ can also be used to provide bounds on $\dim_H(F)$, as indicated by the following result.

Corollary 5.8. Let $F$ be the limit set generated by the triple $(F, \{k_{\ell_i}\}_{i=0}^{\infty}, E_0)$.

(a) If $w := \inf_{\ell} \{|L_{k_\ell}|_1\} \geq 1$,

then $\dim_H(F) \geq \frac{\log(m)}{-\log(w)}$.

(b) If $u := \sup_{\ell} \{|U_{k_\ell}|_1\} < 1$,

then $\dim_H(F) \leq \frac{\log(m)}{-\log(u)}$. 

Proof. (a). In this case, for \( s = \frac{\log(m)}{\log(m) - \log(w)} \geq 1 \), we have

\[
\sum_{j=1}^{m} \left( \frac{L(f_{k \ell}^{(j)})}{m} \right)^s \geq \left( \sum_{j=1}^{m} \frac{L(f_{k \ell}^{(j)})}{m} \right)^s \geq \left( \frac{w}{m} \right)^s
\]

for each \( \ell \). Thus,

\[
\inf_{\ell} \{||L_{k \ell}||_s\} \geq m^s \frac{w}{m} = 1,
\]

then by Theorem 5.4, \( \dim_H(F) \geq s \).

(b). In this case, for any \( 1 \geq s > \frac{\log(m)}{\log(m) - \log(u)} \), we have

\[
\sum_{j=1}^{m} \left( \frac{U(f_{k \ell}^{(j)})}{m} \right)^s \leq \left( \sum_{j=1}^{m} \frac{U(f_{k \ell}^{(j)})}{m} \right)^s \leq \left( \frac{u}{m} \right)^s
\]

for each \( \ell \). Thus,

\[
\sup_{\ell} \{||U_{k \ell}||_s\} \leq m^s \frac{u}{m} < 1.
\]

By Theorem 5.4 \( \dim_H(F) \leq s \). Hence, \( \dim_H(F) \leq \frac{\log(m)}{\log(m) - \log(u)} \). \( \square \)

Note that this corollary generally provides better bounds on \( \dim_H(F) \) than those obtained from directly applying Theorem 5.4.

6. Examples of estimating dimensions

In the end, we estimate the Hausdorff dimensions of a few limit fractals achieved in Section 2 and Section 4 via results obtained in Section 5.

Example 6.1. We first consider again limit fractals constructed in the motivating examples in Section 2. In this case, \( X = \{[a, b] : a, b \in \mathbb{R}\} \) and \( m = 2 \). For each \( k = (k^{(1)}, k^{(2)}) \) and \( f_{k}^{(i)} : X \to X \) described there, one can clearly see that

\[
diam(f_{k}^{(i)}([a, b])) = k^{(i)} \cdot diam([a, b]).
\]

Thus, \( L(f_{k}^{(i)}) = k^{(i)} = U(f_{k}^{(i)}) \), and hence \( L_{k} = k = U_{k} \).

- In Example 2.1 \( k_{\ell} = \left( \frac{\ell + 1}{4\ell + 5}, \frac{2\ell + 5}{8\ell + 16} \right) \) for each \( \ell \geq 0 \). We apply Corollary 5.5 here. Note that

\[
\lim_{\ell \to \infty} ||L_{k_{\ell}}||_s = \lim_{\ell \to \infty} ||k_{\ell}||_s = \frac{2^3}{4}.
\]

So,

\[
s_* = \sup_s \{\liminf_{\ell \to \infty} ||L_{k_{\ell}}||_s > 1\} = \sup_s \left\{ \frac{2^\frac{s}{4}}{4} > 1 \right\} = \frac{1}{2}.
\]

Similarly, we also have \( s^* = \frac{1}{2} \). By (5.6), \( \dim_H(F) = \frac{1}{2} \).

- In Example 2.2 \( k_{\ell} = (q_{\ell}, \frac{1}{2} - q_{\ell}) \) where \( q_{\ell} \) is a random number in \( [\frac{1}{8}, \frac{3}{8}] \). So,

\[
\left( \frac{1}{8}, \frac{1}{8} \right) \leq L_{k_{\ell}} = k_{\ell} = U_{k_{\ell}} \leq \left( \frac{3}{8}, \frac{3}{8} \right).
\]
By Corollary 5.7
\[
\frac{\log(2)}{-\log(1/8)} \leq \dim_H(F) \leq \frac{\log(2)}{-\log(3/8)}.
\]
That is,
\[
\frac{1}{3} \leq \dim_H(F) \leq \frac{\log(2)}{\log(8/3)} \approx 0.7067.
\]

**Example 6.2.** In this example, we consider variations of Sierpinski Triangle introduced in subsection 4.2. Here, \( X \) is given by (4.1), and \( m = 3 \). For each \( k = (k^{(1)}, k^{(2)}, \ldots, k^{(6)}) \in Y \) and \( 1 \leq i \leq 3 \),
\[
U \left( f^{(i)}_k \right) = \sup_{(A,B,C) \in X} \frac{\text{diam} \left( f^{(i)}_k(A,B,C) \right)}{\text{diam} \left( (A,B,C) \right)} = \max \left\{ k^{(2i-1)}, k^{(2i)} \right\},
\]
and
\[
L \left( f^{(i)}_k \right) = \inf_{(A,B,C) \in X} \frac{\text{diam} \left( f^{(i)}_k(A,B,C) \right)}{\text{diam} \left( (A,B,C) \right)} = \min \left\{ k^{(2i-1)}, k^{(2i)} \right\}.
\]
(a) If \( \lambda \leq k^{(j)} \leq \Lambda < 1 \) for all \( j = 1, \ldots, 6 \), then
\[
U_k \leq r := (r, \ldots, r) \text{ and } L_k \geq s := (s, \ldots, s),
\]
where \( r = \max\{1 - \lambda, \Lambda\} \) and \( s = \min\{1 - \lambda, \Lambda\} \).

In Example 4.1(b), we pick \( \lambda = 0.45 \) and \( \Lambda = 0.55 \). The output is illustrated in Figure 5(b). By Corollary 5.7
\[
\frac{\log(m)}{-\log(s)} \leq \dim_H(F) \leq \frac{\log(m)}{-\log(r)},
\]
where \( m = 3 \), \( r = 0.55 \) and \( s = 0.45 \). That is,
\[
1.3758 \leq \dim_H(F) \leq 1.8377.
\]

(b) We now apply Corollary 5.5 to Example 4.3. In this example, one can calculate that
\[
\lim_{t \to \infty} \left( ||U_k||_t \right)^t = \frac{3}{2s} = \lim_{t \to \infty} \left( ||L_k||_t \right)^t.
\]
Thus, by Corollary 5.5, \( \dim_H(F) = \frac{\log(3)}{\log(2)} \).

**Example 6.3.** In this example, we consider variations of Menger Sponge introduced in subsection 4.3. Here, \( X \) is given by (4.4), and \( m = 20 \). For each \( k = (k^{(1)}, k^{(2)}, \ldots, k^{(6)}) \in Y \) and \( 1 \leq i \leq 20 \),
\[
U \left( f^{(i)}_k \right) = \sup_{(O,A,B,C) \in X} \frac{\text{diam} \left( f^{(i)}_k(O,A,B,C) \right)}{\text{diam} \left( (O,A,B,C) \right)} = \sup_{(O,A,B,C) \in X} \frac{\text{diam} \left( M_k(a_i,b_i,c_i)(O,A,B,C) \right)}{\text{diam} \left( (O,A,B,C) \right)} = \max\{T(a_{i+1}) - T(a_i), R(b_{i+1}) - R(b_i), S(c_{i+1}) - S(c_i)\}.
\]
Similarly,
\[
L \left( f^{(i)}_k \right) = \min\{T(a_{i+1}) - T(a_i), R(b_{i+1}) - R(b_i), S(c_{i+1}) - S(c_i)\}.
\]
When $k^{(2j)} = 1 - k^{(2j-1)}$ for each $j = 1, 2, 3$, it is easy to check that
\[
\sum_{i=1}^{20} U(f_k^{(i)})^s = \sum_{i=1}^{20} \max \{ T(a_{i+1}) - T(a_i), R(b_{i+1}) - R(b_i), S(c_{i+1}) - S(c_i) \}^s
\]
\[
= 8 \max \{ k^{(1)}, k^{(3)}, k^{(5)} \}^s + 4 \max \{ 1 - 2k^{(1)}, k^{(3)}, k^{(5)} \}^s
\]
\[
+ 4 \max \{ k^{(1)}, 1 - 2k^{(3)}, k^{(5)} \}^s + 4 \max \{ k^{(1)}, k^{(3)}, 1 - 2k^{(5)} \}^s.
\]

(a) We now calculate the dimension of the limit fractal $F$ illustrated by Figure 9b in Example 4.6. Note that when $\lambda \leq k^{(2j-1)} \leq \Lambda$ for each $j = 1, 2, 3$, it follows that
\[
\left( \left\| U_k^s \right\| \right)^s = \sum_{i=1}^{20} U(f_k^{(i)})^s \leq 8\Lambda^s + 12 \max \{ 1 - 2\lambda, \Lambda \}^s.
\]
Similarly,
\[
\left( \left\| L_k^s \right\| \right)^s \geq 8\lambda^s + 12 \min \{ 1 - 2\Lambda, \lambda \}^s.
\]
For example, when $\lambda = 0.32$ and $\Lambda = 0.35$, for any $s > 2.901$, 
\[
\left( \left\| U_k^s \right\| \right)^s \leq 8\Lambda^s + 12 \max \{ 1 - 2\lambda, \Lambda \}^s \leq 8 \times 0.35^s + 12 \times 0.36^s
\]
\[
< 8 \times 0.35^{2.901} + 12 \times 0.36^{2.901} \approx 1.000.
\]
By Theorem 5.4, $\dim_H(F) \leq 2.901$. Similarly,
\[
\left( \left\| L_k^s \right\| \right)^s \geq 8\lambda^s + 12 \min \{ 1 - 2\Lambda, \lambda \}^s \geq 8 \times 0.32^s + 12 \times 0.3^s \geq 8 \times 0.32^{2.546} + 12 \times 0.3^{2.546} \approx 1.000.
\]
By Theorem 5.4 again, $\dim_H(F) \geq 2.546$. As a result, 
\[
2.546 \leq \dim_H(F) \leq 2.901.
\]
(b) We now apply Corollary 5.5 to Example 6.1. In this example, one can calculate that
\[
\lim_{\ell \to \infty} \left( \left\| U_{k_\ell}^s \right\| \right)^s = \frac{20}{3^s} = \lim_{\ell \to \infty} \left( \left\| L_{k_\ell}^s \right\| \right)^s.
\]
Thus, by Corollary 5.5, $\dim_H(F) = \frac{\log(20)}{\log(3)} \approx 2.7268$.

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