The definition of optimal transport paths

Let X be a compact convex subset of a Euclidean space $\mathbb{R}^m$. Suppose

$$\mathbf{a} = \sum_{i=1}^{k} a_i \delta_{x_i} \text{ and } \mathbf{b} = \sum_{j=1}^{l} b_j \delta_{y_j} \tag{0.1}$$

are two atomic probability measures on X. A *transport path* from a to b is a weighted directed graph $\mathbf{G}$ consists of a vertex set $V(\mathbf{G})$, a directed edge set $E(\mathbf{G})$ and a weight function

$$w : E(\mathbf{G}) \to (0, +\infty)$$

such that $\{x_1, x_2, \cdots, x_k\} \cup \{y_1, y_2, \cdots, y_l\} \subset V(\mathbf{G})$ and for any vertex $v \in V(\mathbf{G})$,

$$\sum_{\substack{e \in E(\mathbf{G}) \\ e^- = v}} w(e) = \sum_{\substack{e \in E(\mathbf{G}) \\ e^+ = v}} w(e) + \begin{cases} a_i, & \text{if } v = x_i \text{ for some } i = 1, \cdots, k \\ -b_j, & \text{if } v = y_j \text{ for some } j = 1, \cdots, l \\ 0, & \text{otherwise} \end{cases} \tag{0.2}$$

where e^- and e^+denotes the starting and ending endpoints of each edge $e \in E(\mathbf{G})$.

The balance equation (0.2) simply means that the total mass flows into v equals to the total mass flows out of v. When $\mathbf{G}$ is viewed as a polyhedral chain or current, (0.2) can be simply expressed as $\partial \mathbf{G} = \mathbf{b} - \mathbf{a}$. Also, when $\mathbf{G}$ is viewed as a vector valued measure, the balance equation is simply $\mathbf{div}(\mathbf{G}) = \mathbf{a} - \mathbf{b}$ in the sense of distributions.

Let $\alpha \leq 1$ be a parameter. The $\mathbf{M}_\alpha$ **cost function** on a transport path $\mathbf{G}$ is defined by

$$\mathbf{M}_\alpha(\mathbf{G}) \equiv \sum_{e \in E(\mathbf{G})} [w(e)]^\alpha \, length(e)$$

for any transport path $\mathbf{G}$ from $\mathbf{a}$ to $\mathbf{b}$. Any $\mathbf{M}_\alpha$ minimizer in the family of all transport paths from $\mathbf{a}$ to $\mathbf{b}$ is called **an optimal transport path**.

Now, we can talk about transport paths between general probability measures. Let μ^+, μ^- be any two probability measures on X. Extending the above definition, we say a vector measure $\mathbf{T}$ on X is **a transport path from** μ^+ **to** μ^- if there exist two sequences $\{\mathbf{a}_i\}$,$\{\mathbf{b}_i\}$ of atomic probability measures on X with a corresponding sequence of transport paths $\mathbf{G}_i$ from $\mathbf{a}_i$ to $\mathbf{b}_i$ such that

$$\mathbf{a}_i \rightharpoonup \mu^+, \ \mathbf{b}_i \rightharpoonup \mu^-, \mathbf{G}_i \rightharpoonup \mathbf{T}$$

weakly as probability measures and vector measures. Note that for any such $\mathbf{T}$, $div(\mathbf{T}) = \mu^+ - \mu^-$ in the sense of distributions. Also, given any $\alpha \in [0,1]$, for any transport path $\mathbf{T}$ from μ^+ to μ^-, we define its $\mathbf{M}_\alpha$ cost to be

$$\mathbf{M}_\alpha(\mathbf{T}) := \inf \lim \inf_{i \to \infty} \mathbf{M}_\alpha(\mathbf{G}_i),$$

where the infimum is over the set of all possible approximating graph sequence $\{\mathbf{a}_i, \mathbf{b}_i, \mathbf{G}_i\}$ of $\mathbf{T}$.